Cup-product

Exercise 1.

Let R be a commutative ring and let X be a path-connected topological space. Describe the cup-product

$$\smile : \mathrm{H}^0(X,R) \times \mathrm{H}^i(X,R) \to \mathrm{H}^i(X,R).$$

Exercise 2.

Let R be a commutative ring. Compute the ring $\mathrm{H}^*(S^n,R)$ (with multiplication the cup-product).

Exercise 3.

A Δ-*complex structure* on a topological space X is a family of maps $\sigma_\alpha \colon \Delta^n \to X$, with n depending on the index α such that

- For any index α, the restriction of the map σ_α to the interior of Δ^n is injective.
- For any index α, each restriction of σ_α to one of the faces of Δ^n is one of the maps $\sigma_\beta \colon \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.
- A subset U of X is open if and only if for any index α, the subset $\sigma_\alpha^{-1}(U)$ is open in Δ^n.

This notion generalizes the notion of simplicial complex that we introduced in exercise sheet 7.

As in sheet 7, we can show the following facts (that we will use without proofs):

(i) Show that Δ-complexes are CW complexes.

(ii) Let X be a Δ-complex. Let $C_*^{\mathrm{simp}}(X)$ be the associated cellular complex. The complex $C_*^{\mathrm{simp}}(X)$ is a sub-complex of the singular complex $C_*(X)$.

1. Denote by $C^*_{\mathrm{simp}}(X,R)$ the complex $\mathrm{Hom}_{\mathbb{Z}}(C_*^{CW}(X),R)$ describe this complex explicitly.
2. Show that $C^*_{\mathrm{simp}}(X,R)$ is a sub-complex of the singular complex $C^*(X,R)$.
3. Deduce that to compute the cup-product:

$$\mathrm{H}^n(X,R) \times \mathrm{H}^m(X,R) \to \mathrm{H}^{n+m}(X,R)$$

it suffices to describe the map

$$C^n_{\mathrm{simp}}(X,R) \times C^m_{\mathrm{simp}}(X,R) \to C^{n+m}_{\mathrm{simp}}(X,R).$$

Exercise 4.

Let g be a positive integer. The goal of this exercise is to understand the cup-product over the cohomology with integral coefficients of the orientable surface of genus g (denoted by Σ_g).

1. Recall the value of the cohomology groups of Σ_g with integral coefficients.
2. Explain why the only "interesting" product is the product

$$\smile : \mathrm{H}^1(\Sigma_g,\mathbb{Z}) \times \mathrm{H}^1(\Sigma_g,\mathbb{Z}) \to \mathrm{H}^2(\Sigma_g,\mathbb{Z}).$$

3. Recall that the surface of genus g can be described as a $4g$-gone $a_1, b_1, a_1^{-1}, b_1^{-1}, \dots a_g, b_g, a_g^{-1}, b_g^{-1}$. Show that we can endow Σ_g with a Δ-complex structure by considering the $4g$ triangles with vertices the center of the $4g$-gone and the two vertices of an edge.
4. Construct cocycles ϕ_i and ψ_i in $C^1_{\mathrm{simp}}(\Sigma_g,\mathbb{Z})$ such that $\phi_i(a_j) = \delta_{i,j}$, $\phi_i(b_j) = 0$, $\psi_i(a_j) = 0$ et $\psi_i(b_j) = \delta_{i,j}$.
5. Deduce a description of the cup-product over the cohomology of Σ_g with integral coefficients.

Exercise 5.

Apply the techniques of Exericise 4 to compute the cup-product over the cohomology of the non-orientable surface of genus g with coefficients $\mathbb{Z}/2\mathbb{Z}$.