Poincaré Duality

Exercise 1.
Show that removing a point from a manifold of dimension 2 or more doesn't affect orientability.

Exercise 2.
Let M be a topological manifold of dimension n.
We define the *orientation bundle* OM of M by

$$OM = \{(x, \mu) \mid x \in M, \mu \text{ is a generator of } \mathrm{H}_n(M, M \setminus \{x\})\}.$$

1. Show that there is a unique topological manifold structure on OM such that the first projection $OM \to M$ is a double cover. Show that if M is differentiable, so is OM and that the first projection $OM \to M$ is smooth.
2. Show that the manifold OM is orientable.
3. Show that M is orientable if and only if OM is not connected and that in this case, the manifold OM is homeomorphic to a disjoint union of two copies of M.
4. Deduce that simply connected manifolds (or more generally, manifolds whose fundamental group has no subgroup of index 2) are orientable.

Exercise 3. Let M be an orientable manifold of dimension n endowed with an orientation $(\mu_x)_{x\in M}$. Let f be a homeomorphism of M.

1. Show that the family $(f_*(\mu_x))_{x\in M}$ is still an orientation of M, *i.e.*

$$(f_*(\mu_x))_{x\in M} \in \{(\mu_x)_{x\in M}; (-\mu_x)_{x\in M}\}.$$

   If $(f_*(\mu_x))_{x\in M} = (\mu_x)_{x\in M}$, we say that f *is orientation-preserving.*
2. Show that if M is compact, a map f is orientation-preserving if and only if $f_*([M]) = [M]$. Deduce that the antipodal map $S^n \to S^n$ is orientation-preserving when the map n is odd.
3. Show that if G is a discrete group which acts properly and totally discontinuously on M by orientation-perserving homeomorphisms, the quotient M/G is orientable. Deduce a new proof that $\mathbb{R}\mathrm{P}^{2n+1}$ is orientable.

Exercise 4.
Describe a fundamental class of Σ_g and of S^n. Deduce a fundamental class of $\mathbb{R}\mathrm{P}^{2n+1}$ by using the last exercise.

Exercise 5. (Homology of orientable compact 3-manifolds).

1. Let M be a simply connected orientable compact 3-manifold. Compute the cohomology groups of M.
2. Let M be an orientable compact 3-manifold. Write

$$\mathrm{H}_1(M) = \mathbb{Z}^r \oplus F$$

   with F a finite group. Compute the homology groups of M with integral coefficients.

Exercise 6.
Let M be a compact connected orientable n-manifold. Show that $\mathrm{H}_{n-1}(M)$ is torsion-free.

Exercise 7.

Let M be a non-orientable compact n-manifold. Show that

$$\mathrm{H}_n(M;\mathbb{Z}/m\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } m \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Deduce that the torsion subgroup of $\mathrm{H}_{n-1}(M)$ is $\mathbb{Z}/2\mathbb{Z}$.

Exercise 8.
Let M be a non-orientable compact 3-manifold. Write

$$\mathrm{H}_1(M) = \mathbb{Z}^r \oplus F$$

with F a finite group.

1. Compute the homology groups with coefficients $\mathbb{Z}/2\mathbb{Z}$ of M.
2. Deduce that $\mathrm{H}_2(M) = \mathbb{Z}^{r-1} \oplus \mathbb{Z}/2\mathbb{Z}$ (in particular $r > 1$).
3. Deduce that the fundamental group of M is infinite.

Exercise 9.
Let X be a compact orientable n-manifold. Show that we have isomorphisms

$$\mathrm{H}_i(X,\mathbb{Q}) \simeq \mathrm{H}^i(X,\mathbb{Q}) \simeq \mathrm{H}_{n-i}(X,\mathbb{Q}) \simeq \mathrm{H}^{n-i}(X,\mathbb{Q})$$

induced by the choice of a fundamental class.

Exercise 10. Let X be a compact orientable n-manifold.

Show that by choosing of a fundamental class, the cup product induces a map called the *intersection product*

$$\mathrm{H}_k(X,\mathbb{Z}) \otimes \mathrm{H}_{n-k}(X,\mathbb{Z}) \to \mathbb{Z}.$$

Let M be a compact sub-k-manifold of X. We still denote by $[M]$ the image of the fundamental class of M in $\mathrm{H}_k(X,\mathbb{Z})$.

If M and N are submanifolds of X, of respective dimensions k and $n-k$. We call *intersection number* of M and N the image of $[M] \otimes [N]$ by the intersection product.

Compute the intersection number of two circles which meet transversely in a torus. We can in fact show that if X is a smooth manifold, the intersection number of two submanifolds which intersect "transversely" is the number of intersection points of the two submanifolds counted with some multiplicity ± 1.