ENS de Lyon — Master 1 – Algebraic topology
TD 5 — 2022-2023

More on Mayer-Vietoris; Degree and applications; CW-complexes

Exercise 1. Computations of homology groups
Consider the following topological spaces:

(a) The 2-dimensional torus.

(b) The Klein bottle.

(c) The orientable surface of genus 2.

Compute their homology groups using

1. the Mayer-Vietoris exact sequence.
2. the cellular homology theorem.

Generalize the result of (c) to the orientable surface of genus g using cellular homology.

Exercise 2. Cellular homology is not always the best choice!
Compute the homology groups of the complement of a circle inside the 3-dimensional space.

Exercise 3. On the topology of CW complexes

1. Show that a subset of a CW complex is open (resp. closed) if and only if its intersection with the interior of any cell C is open (resp. closed) in the interior of C.
2. Show that a CW complex is compact if and only if it has finitely many cells.
3. Show that a CW complex is path connected if and only if its 1-skeleton is path connected.
4. Endow the product of two CW complexes with a CW complex structure.

Exercise 4. Homology of the surface of genus g using Mayer-Vietoris

1. Let X_g be the complement in the 2-sphere of $2g$ disjoint open discs $D_1, \ldots, D_{2g}$. Compute the homology groups of X_g.
2. Compute the map in homology induced by the inclusion:
$$\bigsqcup_{i=1}^{2g} \partial D_i \to X_g.$$
3. Deduce the homology groups of the surface of genus g.

Exercise 5. Fundamental class of the sphere and higher Hurewicz maps
Let $n \geqslant 1$ be an integer. A *fundamental class* of the sphere S^n is a generator of the group $H_n(S^n)$.

1. Let (U, V) be a topological cover of a space X. Give a method to compute the connecting map:
$$\partial : H_n(X) \to H_{n-1}(U \cap V)$$
of the Mayer-Vietoris exact sequence.
2. Using the first question, find a fundamental class of the sphere S^n.
3. Let (X, x) be a pointed topological space, construct a group homomorphism
$$\pi_n(X, x) \to H_n(X)$$
Hint: map $f : S^n \to X$ to the pushforward by f of a fundamental class of S^n.

4. Show that this map is surjective when $X = S^n$.

Exercise 6. The Perron-Frobenius theorem

Use Brower's fixed point theorem to show that a square matrix with nonnegative entries has a real nonnegative eigenvalue.

Exercise 7. Homology of a product of spheres Let $m, n \geqslant 1$ be integers. Compute the homology groups of $S^n \times S^m$ using

1. the Mayer-Vietoris exact sequence.
2. the cellular homology theorem.