Cellular Homology

Exercise 1. On the topology of CW complexes

1. Show that a subset of a CW complex is open (resp. closed) if and only if its intersection with the interior of any cell C is open (resp. closed) in the interior of C.
2. Show that a CW complex is compact if and only if it has finitely many cells.
3. Show that a CW complex is path connected if and only if its 1-skeleton is path connected.
4. Endow the product of two CW complexes with a CW complex structure.

Exercise 2. Computations of homology groups of classical spaces

Let g be a positive integer. Compute the homology groups of the follwing topological spaces:

(a) The 3-dimensional torus.

(b) The non-orientable surface Σ_g of genus g

(c) The orientable surface Σ'_g of genus g.

(d) The real projective space.

Exercise 3. Homology of singular spaces

Compute the homology groups of the following topological spaces.

1. The space obtained from the sphere S^2 by collapsing n points.
2. The space obtained from the sphere S^2 by identifying any point of the equator with its antipode.
3. The space obtained from the sphere S^3 by identifying any point of the equator with its antipode.

Exercise 4. Moore Spaces

Let M be an abelian group and let $n \geqslant 1$ be an integer. A Moore space with respect to (M, n) is a topological space X such that

$$\widetilde{H}_i(X) \cong \begin{cases} M & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}.$$

1. Construct a Moore space with respect to $(\mathbb{Z}/m\mathbb{Z}, n)$ for any integer m and any positive integer n.
2. Construct a Moore space with respect to (M, n) for any finitely generated abelian group M and any positive integer n.
3. Let $(M_n)_{n\geqslant 1}$ be a sequence of abelian groups. Construct a path-connected topological space X such that for all $n \geqslant 1$, we have $H_n(X) = M_n$.

Exercise 5. Product of spheres

Let m and n be positive integers. Compute the homology groups of the product of spheres $S^n \times S^m$ using cellular homology.

Exercise 6. Euler characteristic

1. Let $Y \to X$ be a finite cover of degree n. Show that $\chi(Y) = n\chi(X)$.
2. Let g be a positive integer.

   (a) Compute the Euler characteristic of the surfaces Σ_g and Σ'_g.

(b) Let h be positive integer. Find a necessary condition to the existence of a cover $\Sigma_h \to \Sigma_g$ in terms of g and h. Show that this condition is in fact sufficient.

3. Let X be a CW complex which is the union of two sub-complexes A and B. Show that

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

4. Compute the Euler characteristic of a product of CW complexes.

Exercise 7. Homology of Poincaré's hypercubic variety

Let V be the topological space obtained from the cube $C = [0,1]^3$ by gluing opposite faces after turning them of an angle of $\frac{\pi}{2}$. More precisely, we set

$$V = C/\sim$$

with $(0,y,z) \sim (1,-z,y)$, $(x,0,z) \sim (z,1,-x)$ and $(x,y,0) \sim (-y,x,1)$.

Compute the homology groups of V.