Simplicial Homology

An *abstract simplicial complex* is a pair of sets $K = (V, \Sigma)$ where Σ is a set of finite subsets of V such that

- $\forall \sigma \in \Sigma, \forall \tau \subseteq \sigma, \tau \in \Sigma$.
- Singletons lie in Σ.

The elements of V are called the *vertices of* K and the elements of Σ are called the *faces of* K.

Let K be an abstract simplicial complex. Its *geometric realization* $|K|$ is defined as follows:

- If $\sigma \in \Sigma$, denote by
$$\Delta_\sigma = \left\{ \sum_{v \in \sigma} a_v[v] \mid a_v \in [0,1], \sum_{v \in \sigma} a_v = 1 \right\} \subseteq [0,1]^\sigma$$
The set Δ_σ is endowed with the topology induced by the usual topology of $[0,1]^\sigma$, which gives an homeomorphism $\Delta_\sigma \approxeq \Delta^{|\sigma|-1}$.
- We define an equivalence relation $\sim$ on the disjoint union of the Δ_σ: if $\tau \subseteq \sigma$ and $\sum_{v \in \sigma} a_v[v] \in \Delta_\sigma$ is such that $a_v = 0$ if $v \notin \tau$, then, we set
$$\sum_{v \in \sigma} a_v[v] \sim \sum_{v \in \tau} a_v[v].$$
- The geometric realization of K is then defined by $|K| := \bigsqcup_{\sigma \in \Sigma} \Delta_\sigma / \sim$.

A *simplicial complex* is a topological space which is homeomorphic to the geometric realization of an abstract simplicial complex.

1. Show that a graph is a simplicial complex.
2. Let g be a positive integer. Show that the surface Σ_g of genus g is a simplicial complex.
3. Show that simplicial complexes are CW complexes.
4. Let X be a simplicial complex which is homeomorphic to the geometric realization of an abstract simplicial complex (V, Σ). Let $C_*^{\text{simp}}(X)$ be the associated cellular complex. Describe this complex explicitly in terms of Σ and show that it is a sub-complex of the singular complex $C_*(X)$.
5. Let g be a positive integer. Compute the homology of Σ_g.
6. Compute the homology of the torus of dimension 3 by using simplicial homology.