Homology with coefficients, Cohomology

Exercice 1.
Let M be an abelian group. Let n and g be non-negative integers. Compute the (co)homology groups of the following spaces with coefficients in M.

1. $\mathbb{C}P^n$.
2. $\mathbb{R}P^n$.
3. The orientable surface Σ_g of genus g.
4. The non-orientable surface Σ'_g of genus g.

You can use the following facts without proving them.

- If X is a CW complex,the complex $C_*^{\mathrm{CW}}(X) \otimes_{\mathbb{Z}} M$ computes the homology groups of X with coefficients in M.
- If X is a CW complex, the dual complex of the complex $C_*^{\mathrm{CW}}(X)$ computes the cohomology groups of X with coefficients in M.

Exercice 2. Let n be a positive integer. Let $f : S^n \to S^n$ be a map of degree d. Show that the map induced by f in cohomology is the multiplication by d. You can use the Mayer-Vietoris exact sequence and proceed by induction.

Exercice 3.(*) Let X be a finite connected graph of genus g (*i.e.* g is the rank of the abelian group $H_1(X;\mathbb{Z})$) with no vertex of degree 1.

1. Show that $H_1(X;\mathbb{Z})$ is free of rank g. Its automorphism group is therefore $GL_g(\mathbb{Z})$.
2. Let G be a finite group of homeomorphisms of X. Show that the map
$$\Psi : \begin{cases} G & \to & GL_g(\mathbb{Z}) \\ f & \mapsto & H_1(f) \end{cases}$$
is one to one. You may start with the case where X is a wedge sum of circles.
3. Show that this result still holds when $\mathbb{Z}$ is replaced by $\mathbb{Z}/m\mathbb{Z}$ with $m > 2$. What happens if $m = 2$?

Exercice 4.
Let X be a topological space. Deduce from the short exact sequence of complexes
$$0 \to C_*(X) \overset{\times n}{\to} C_*(X) \to C_*(X;\mathbb{Z}/n\mathbb{Z}) \to 0$$
a short exact sequence of abelian groups
$$0 \to H_i(X)/n \to H_i(X,\mathbb{Z}/n\mathbb{Z}) \to n-\mathrm{torsion}(H_{i-1}(X)) \to 0.$$