Universal Coefficients Theorem and the functors Tor and Ext

Exercice 1.

Let M be an abelian group. Let n and g be non-negative integers. By using the universal coefficients theorem, compute the (co)homology groups of the following spaces with coefficients in M.

1. $\mathbb{C}P^n$.
2. $\mathbb{R}P^n$.
3. The orientable surface Σ_g of genus g.
4. The non-orientable surface Σ'_g of genus g.

Exercice 2.

Let M be an abelian group. Compute $\mathrm{Tor}(M, \mathbb{Q}/\mathbb{Z})$.

Exercice 3.

1. Let
$$0 \to M' \to M \to M'' \to 0$$
be an exact sequence of abelian group and N be an abelian group. Prove that we have an exact sequence:
$$0 \to \mathrm{Hom}(N, M') \to \mathrm{Hom}(N, M) \to \mathrm{Hom}(N, M'') \to \mathrm{Ext}(N, M') \to \mathrm{Ext}(N, M) \to \mathrm{Ext}(N, M'') \to 0.$$
2. Deduce the value of $\mathrm{Ext}(N, \mathbb{Q}/\mathbb{Z})$ when N is an abelian group of finite type.
3. Let N be a torsion abelian group. Show that $\mathrm{Ext}(N, \mathbb{Z}) = \mathrm{Hom}(N, \mathbb{Q}/\mathbb{Z})$. You can use the fact (see Exercise 5) that $\mathrm{Ext}(N, \mathbb{Q}) = 0$ for any abelian group N.

Exercice 4.

1. Recall the definition of a projective $\mathbb{Z}$-module and the equivalent characterizations of this notion.
2. A *projective resolution* of an abelian group M is a long exact sequence
$$\cdots \to P_1 \to P_0 \to M \to 0$$
such that every P_i is projective. Let N be an abelian group. Show that the homology groups of the complex
$$\cdots \to P_1 \otimes N \to P_0 \otimes N \to M \otimes N \to 0$$
is independent from the projective resolution. Deduce that $\mathrm{Tor}(N, M)$ can be computed by using any projective resolution of M (instead of a free resolution).

   There is a similar result for the functor Ext.

Exercice 5. An abelian group I is said to be *injective* when the functor $\mathrm{Hom}(-, I)$ is exact.

1. Show that I is injective if and only if for any one-to-one morphism of abelian groups $N \to M$ and any morphism $f: N \to I$, there is a morphism $M \to I$ which extends f.
2. Show that if I is injective, the multiplication by any non-zero integer is a surjective map $I \to I$ (we say that I is *divisible*).
3. Show that a divisible abelian group is injective (*Indication*: use Zorn's lemma and the fact that the abelian group I is divisible if and only if the criterion of the first question holds for the inclusion map of any ideal of $\mathbb{Z}$ into $\mathbb{Z}$).
4. Deduce that the abelian groups $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

5. An injective resolution of an abelian group M is an exact sequence

$$0 \to M \to I_0 \to I_1 \to \cdots$$

such that every I_i is injective. Show that if N is an abelian group, the homology groups of the complex

$$0 \to \mathrm{Hom}(N, M) \to \mathrm{Hom}(N, I_0) \to \mathrm{Hom}(N, I_1) \to \cdots$$

is independent from the resolution.

You can use the following (hard) theorem which is due to Eilenberg and Cartan: if

$$0 \to M \to I_0 \to I_1 \to \cdots$$

is an injective resolution of M, and if we denote by C^* the complex

$$\mathrm{Hom}(N, I_0) \to \mathrm{Hom}(N, I_1) \to \cdots$$

then,

$$H^i(C^*) = \begin{cases} \mathrm{Hom}(N, M) & \text{if } i = 0 \\ \mathrm{Ext}(N, M) & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

6. Deduce that if I is injective, we have $\mathrm{Ext}(N, I) = 0$.

7. Deduce that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of abelian group and if N is an abelian group, we have a long exact sequence:

$$0 \to \mathrm{Hom}(M'', N) \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N) \to \mathrm{Ext}(M'', N) \to \mathrm{Ext}(M, N) \to \mathrm{Ext}(M', N) \to 0.$$