Exercices supplémentaires - Equations différentielles

Exercice 1

1. $y'(t) + y(t) = 2t - 1$.

$y_0(t) = ke^{-t}$

$y_1(t) = at + b \quad y'_1(t) = a$

$a + at + b = 2t - 1 \quad \begin{cases} a = 2 \\ a + b = -1 \end{cases} \quad \begin{matrix} a = 2 \\ b = -3 \end{matrix}$

$$\underline{y(t) = ke^{-t} + 2t - 3.}$$

2. $y'(t) - 3y(t) = t^2 + 2$

$y_0(t) = ke^{3t}$

$y_1(t) = at^2 + bt + c \quad y'_1(t) = 2at + b$

$2at + b - 3(at^2 + bt + c) = t^2 + 2$.

$-3at^2 + t(2a - 3b) + b - 3c = t^2 + 2$

$\begin{cases} -3a = 1 \\ 2a - 3b = 0 \\ b - 3c = 2 \end{cases} \quad \begin{aligned} a &= -1/3 \\ b &= \frac{2a}{3} = -\frac{2}{9} \\ c &= \frac{b-2}{3} = \frac{1}{3}\left(-\frac{2}{9} - 2\right) = -\frac{20}{27} \end{aligned}$

$y_1(t) = -\frac{1}{3}t^2 - \frac{2}{9}t - \frac{20}{27}$

$$\underline{y(t) = ke^{3t} - \left(\frac{1}{3}t^2 + \frac{2}{9}t + \frac{20}{27}\right)}$$

3. $y'(t) - y(t) = e^{3t}$

$y_0(t) = ke^{t} \quad y_1(t) = ae^{3t}$

$3ae^{3t} - ae^{3t} = e^{3t} \quad 2a = 1 \quad a = \frac{1}{2}$.

$$\underline{y(t) = ke^{t} + \frac{1}{2}e^{3t}}$$

4. $y'(t) - 2t\,y(t) = 0 \quad \underline{y(t) = ke^{t^2}}$

5. $y'(t) + (\ln t)y(t) = 0 \quad \underline{y(t) = ke^{-(t\ln t - t)}}$ (difficile, on vous aurait donné une indication)

exercice 2.

1) $y''(t) - 2y'(t) - 3y(t) = t$.

$r^2 - 2r - 3 = 0$ $\quad \Delta = 4 - 4(-3) = 16$ $\quad r_1 = \frac{2-4}{2} = -1$

$r_2 = \frac{2+4}{2} = 3$.

$y_0(t) = Ae^{-t} + Be^{3t}$

$y_1(t) = at + b$ (0 n'est pas racine de $r^2 - 2r - 3 = 0$)

$-2a - 3at - 3b = t$ $\quad -3a = 1$ $\quad a = -\frac{1}{3}$ $\quad b = -\frac{2}{3}a = \frac{2}{9}$

$-2a - 3b = 0$

$y_1(t) = -\frac{1}{3}t + \frac{2}{9}$ $\quad \underline{y(t) = Ae^{-t} + Be^{3t} - \frac{1}{3}t + \frac{2}{9}}$

2) $y''(t) - 2y'(t) - 3y(t) = te^{2t}$. $\quad$ de ($r^2 - 2r - 3 = 0$)

$y_0(t) = Ae^{-t} + Be^{3t}$ $\quad$ 2 n'est pas racine, on cherche une solution

de la forme $y_1(t) = (at+b)e^{2t}$

$y'_1(t) = ae^{2t} + 2(at+b)e^{2t} = e^{2t}(2at + a + 2b)$

$y''_1(t) = 2ae^{2t} + e^{2t}(4at + 2a + 4b) = e^{2t}(4at + 4a + 4b)$

$e^{2t}(4at + 4a + 4b) - 2e^{2t}(2at + a + 2b) - 3(at+b)e^{2t} = te^{2t}$

$\cancel{4at} + 4a + \cancel{4b} - \cancel{4at} - 2a - \cancel{4b} - 3at - 3b = t$

$-3a = 1$ $\quad a = -1/3$

$2a - 3b = 0$ $\quad b = \frac{2a}{3} = -\frac{2}{9}$ $\quad y_1(t) = \left(-\frac{1}{3}t - \frac{2}{9}\right)e^{2t}$

$\underline{y(t) = Ae^{-t} + Be^{3t} - \left(\frac{1}{3}t + \frac{2}{9}\right)e^{2t}}$

3) $y''(t) - 2y'(t) + y(t) = 2t + 1$

$r^2 - 2r + 1 = 0$ $\quad \Delta = 4 - 4 = 0$ $\quad r_0 = \frac{2}{2} = 1$

$y_0(t) = (A + Bt)e^{t}$. 0 n'est pas racine de $r^2 - 2r + 1 = 0$, donc on cherche y_1 sous la forme :

$y_1(t) = at + b$ $\quad y'_1(t) = a$

$-2a + at + b = 2t + 1$ $\quad a = 2$ $\quad a = 2$

$-2a + b = 1$ $\quad b = 1 + 2a = 5$

$y_1(t) = 2t + 5$ $\quad \underline{y(t) = (A + Bt)e^{t} + 2t + 5}$

$y''(t) - 2y'(t) + y(t) = e^{2t}$

$y_0(t) = (A+Bt)e^t$ $\quad y_1(t) = ae^{2t}$ (2 n'est pas racine de $r^2-2r+1=0$)

$y'_1(t) = 2a\,e^{2t}$ $\quad y''_1(t) = 4ae^{2t}$

$4ae^{2t} - 4ae^{2t} + ae^{2t} = e^{2t}$ $\quad a = 1$

$\underline{y(t) = (A+Bt)e^t + e^{2t}}$

5) $y''(t) - 2y'(t) + 5y(t) = t$

$r^2 - 2r + 5 = 0$ $\quad \Delta = 4 - 4\times5 = -16$ $\quad r = \frac{2}{2} = 1$ $\quad \omega = \frac{\sqrt{-\Delta}}{2} = 2$

$y_0(t) = e^t\,(A\cos(2t) + B\sin(2t))$

$y_1(t) = at + b$

$-2a + 5at + 5b = t$ $\quad \begin{cases} 5a = 1 \\ -2a + 5b = 0 \end{cases}$ $\quad a = 1/5$, $\quad b = \frac{2a}{5} = \frac{2}{25}$

$y_1(t) = \frac{1}{5}t + \frac{2}{25}$ $\quad \underline{y(t) = e^t(A\cos(2t) + B\sin(2t)) + \frac{1}{5}t + \frac{2}{25}}$

6) $y''(t) + y'(t) = t^2$

$r^2 + r = 0$ $\quad \Delta = 1$ les racines sont $r_0 = 0$ $r_1 = -1$

$y_0(t) = A + Be^{-t}$

0 est racine donc on cherche y_1 de la forme $y_1(t) = t(at^2+bt+c)$

$= at^3 + bt^2 + ct$

$y'_1(t) = 3at^2 + 2bt + c$, $y''_1(t) = 6at + 2b$

$6at + 2b + 3at^2 + 2bt + c = t^2$ $\quad \Rightarrow \begin{cases} 3a = 1 \\ 6a + 2b = 0 \\ 2b + c = 0 \end{cases}$ $\quad a = 1/3$, $\quad b = -3a = -1$, $\quad c = -2b = 2$

$y_1(t) = \frac{1}{3}t^3 - t^2 + 2t$ $\quad \underline{y(t) = A + Be^{-t} + \frac{1}{3}t^3 - t^2 + 2t}$