MATH 121-b TD3

Exercice 1:

1) $\int_0^1 x^3 - 5x^2 + 3\,dx = \left[\frac{x^4}{4} - 5\frac{x^3}{3} + 3x\right]_0^1 = \left(\frac{1}{4} - \frac{5}{3} + 3\right) - 0 = \frac{3-20+36}{12} = \frac{19}{12}$

2) $\int_0^1 (2x+3)(x+2)^3\,dx$

* Méthode 1 : on développe

$$\int_0^1 (2x+3)(x+2)^3\,dx = \int_0^1 (2x+3)(x^3+6x^2+12x+8)\,dx$$
$$= \int_0^1 2x^4 + 12x^3 + 24x^2 + 16x + 3x^3 + 18x^2 + 36x + 24\,dx$$
$$= \int_0^1 2x^4 + 15x^3 + 42x^2 + 52x + 24\,dx$$
$$= \left[2\frac{x^5}{5} + \frac{15x^4}{4} + \frac{42x^3}{3} + \frac{52x^2}{2} + 24x\right]_0^1$$
$$= \left(\frac{2}{5} + \frac{15}{4} + 64\right) - 0$$
$$= \frac{8+75+1280}{20} = \frac{1363}{20}$$

* Méthode 2 : ipp

On choisit $\begin{cases} u(x) = 2x+3 & \Rightarrow u'(x) = 2 \\ v'(x) = (x+2)^3 & \Rightarrow v(x) = \frac{(x+2)^4}{4} \end{cases}$

$\# \int_a^b uv' = [uv]_a^b - \int_a^b u'v$ car $[uv]_a^b = \int_a^b u'v + v'u \#$

On a

$$\int_0^1 (2x+3)(x+2)^3\,dx = \left[(2x+3)\frac{(x+2)^4}{4}\right]_0^1 - \int_0^1 2\frac{(x+2)^4}{4}$$
$$= \frac{1}{4}(5\times 3^4 - 3\times 2^4) - \frac{1}{2}\left[\frac{(x+2)^5}{5}\right]_0^1$$
$$= \frac{357}{4} - \frac{1}{10}(3^5 - 2^5)$$
$$= \frac{357}{4} - \frac{211}{10}$$
$$= \frac{1785 - 422}{20} = \frac{1363}{20}$$

3) $\int_1^3 \left(x + \frac{1}{x}\right)^2 dx = \int_1^3 x^2 + 2 + \frac{1}{x^2}\,dx = \left[\frac{x^3}{3} + 2x - \frac{1}{x}\right]_1^3 = \left(9 + 6 - \frac{1}{3}\right) - \left(\frac{1}{3} + 2 - 1\right) = \frac{42}{3} - \frac{2}{3} = \frac{40}{3}$

4) $\int_1^2 \frac{1}{x\sqrt{x}}\,dx = \int_1^2 \frac{1}{x^{3/2}}\,dx = \int_1^2 x^{-3/2}\,dx = \left[\frac{x^{-3/2+1}}{-3/2+1}\right]_1^2 = \left[\frac{x^{-1/2}}{-1/2}\right]_1^2 = \left[\frac{-2}{\sqrt{x}}\right]_1^2 = -\frac{2}{\sqrt{2}} + 2 = \frac{-2\sqrt{2}}{2} + 2 = 2 - \sqrt{2}$

$\# \int x^n = \left[\frac{x^{n+1}}{n+1}\right] \#$

5) $\int_2^0 \sqrt{|1-x|}\,dx = -\int_0^2 \sqrt{|1-x|}\,dx$

Enlevons maintenant la valeur absolue. On a $1-x \geq 0 \Leftrightarrow x \leq 1$. D'où $|1-x| = \begin{cases} 1-x & \text{si } x \in [0,1] \\ x-1 & \text{si } x \in [1,2] \end{cases}$.
Alors d'après la relation de Chasles, on a

$$-\int_0^2 \sqrt{|1-x|}\,dx = -\int_0^1 \sqrt{|1-x|}\,dx - \int_1^2 \sqrt{|1-x|}\,dx = -\int_0^1 \sqrt{1-x}\,dx - \int_1^2 \sqrt{x-1}\,dx = -\int_0^1 (1-x)^{1/2}\,dx - \int_1^2 (x-1)^{1/2}\,dx$$
$$= \left[\frac{(1-x)^{1/2+1}}{1/2+1}\right]_0^1 - \left[\frac{(x-1)^{1/2+1}}{1/2+1}\right]_1^2$$

$\# \int u'u^n = \left[\frac{u^{n+1}}{n+1}\right] \#$

$$= \left[\frac{(1-x)^{3/2}}{3/2}\right]_0^1 - \left(\frac{(x-1)^{3/2}}{3/2}\right]_1^2$$

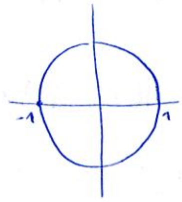
$$= \frac{2}{3}(0-1) - \frac{2}{3}(1-0) = -\frac{4}{3}$$

6) $\displaystyle\int_0^3 3^t\,dt = \int_0^3 e^{t\ln 3}\,dt = \left[\frac{e^{t\ln 3}}{\ln 3}\right]_0^3 = \frac{e^{3\ln 3}}{\ln 3} - \frac{e^0}{\ln 3} = \frac{27}{\ln 3} - \frac{1}{\ln 3} = \frac{26}{\ln 3}$

$\int u'e^u = [e^u]$

7) $\displaystyle\int_{-\pi}^{\pi} t\sin(3t)\,dt$

On fait une ipp avec $\begin{cases} u(t) = t \Rightarrow u'(t) = 1 \\ v'(t) = \sin(3t) \Rightarrow v(t) = -\dfrac{\cos(3t)}{3}\end{cases}$. On a alors



$$\int_{-\pi}^{\pi} t\sin(3t)\,dt = \left[-t\,\frac{\cos(3t)}{3}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -\frac{\cos(3t)}{3}\,dt$$

$$= -\frac{1}{3}\Big(\pi\underbrace{\cos(3\pi)}_{=-1} + \pi\underbrace{\cos(-3\pi)}_{=-1}\Big) + \frac{1}{3}\left[\frac{\sin(3t)}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{2\pi}{3} + \frac{1}{9}\Big(\underbrace{\sin(3\pi)}_{0} - \underbrace{\cancel{\sin(-3\pi)}}_{0}\Big)$$

$$= \frac{2\pi}{3}$$

8) $\displaystyle\int_1^2 x^2\ln x\,dx$

On fait une ipp avec $\begin{cases} u(x) = \ln x \Rightarrow u'(x) = \dfrac{1}{x} \\ v'(x) = x^2 \Rightarrow v(x) = \dfrac{x^3}{3}\end{cases}$. On a alors

$$\int_1^2 x^2\ln x\,dx = \left[\frac{x^3}{3}\ln x\right]_1^2 - \int_1^2 \frac{1}{\cancel{x}}\,\frac{x^{\cancel{3}2}}{3}\,dx = \frac{1}{3}\Big(8\ln 2 - \underbrace{\cancel{\ln 1}}_{0}\Big) - \frac{1}{3}\int_1^2 x^2\,dx = \frac{8\ln 2}{3} - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{8\ln 2}{3} - \frac{1}{9}(8-1) = \frac{8\ln 2}{3} - \frac{7}{9}$$

9) $\displaystyle\int_0^{\pi/2} t^2\cos t\,dt$

On fait une ipp avec $\begin{cases} u(t) = t^2 \Rightarrow u'(t) = 2t \\ v'(t) = \cos t \Rightarrow v(t) = \sin t\end{cases}$. On a alors

$$\int_0^{\pi/2} t^2\cos t\,dt = \left[t^2\sin t\right]_0^{\pi/2} - \int_0^{\pi/2} 2t\sin t\,dt = \Big(\frac{\pi^2}{4}\underbrace{\sin\frac{\pi}{2}}_{1} - 0\Big) - 2\int_0^{\pi/2} t\sin t\,dt$$

On fait une nouvelle ipp, avec $\begin{cases} u(t) = t \Rightarrow u'(t) = 1 \\ v'(t) = \sin t \Rightarrow v(t) = -\cos t\end{cases}$. On obtient

$$\frac{\pi^2}{4} - 2\int_0^{\pi/2} t\sin t\,dt = \frac{\pi^2}{4} - 2\left(\left[-t\cos t\right]_0^{\pi/2} - \int_0^{\pi/2} -\cos t\,dt\right)$$

$$= \frac{\pi^2}{4} + 2\left[t\cos t\right]_0^{\pi/2} - 2\left[\sin t\right]_0^{\pi/2}$$

$$= \frac{\pi^2}{4} + 2\Big(\frac{\pi}{2}\underbrace{\cancel{\cos\frac{\pi}{2}}}_{0} - 0\Big) - 2\Big(\underbrace{\sin\frac{\pi}{2}}_{=1} - \underbrace{\cancel{\sin 0}}_{0}\Big)$$

$$= \frac{\pi^2}{4} - 2$$

Exercice 2 :

$\displaystyle\int_a^b \varphi'(t) f(\varphi(t))\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx$ $\quad(x = \varphi(t))$

1) $\displaystyle\int_0^2 e^{\sqrt{t}}\,dt$

On fait le changement de variable $\begin{cases} x = \sqrt{t} \\ t = x^2 \Rightarrow dt = 2x\,dx\end{cases}$. On a alors

$$\int_0^2 e^{\sqrt{t}}\,dt = \int_{\sqrt{0}}^{\sqrt{2}} e^x \cdot 2x\,dx = 2\int_0^{\sqrt{2}} x e^x\,dx$$

On fait une ipp avec $\begin{cases} u(x) = x \Rightarrow u'(x) = 1 \\ v'(x) = e^x \Rightarrow v(x) = e^x\end{cases}$. On obtient

$$2\int_0^{\sqrt{2}} x e^x\,dx = 2\left[x e^x\right]_0^{\sqrt{2}} - 2\int_0^{\sqrt{2}} e^x\,dx = 2\left(\sqrt{2}e^{\sqrt{2}} - 0\right) - 2\left[e^x\right]_0^{\sqrt{2}} = 2\sqrt{2}e^{\sqrt{2}} - 2\left(e^{\sqrt{2}} - e^0\right) = 2(\sqrt{2}-1)e^{\sqrt{2}} + 2$$

2) $\int_0^1 \frac{2e^x}{e^x+e^{-x}}\,dx = \int_0^1 \frac{2e^x}{e^x+\frac{1}{e^x}}\,dx$

On fait le changement de variable $\begin{cases} u = e^x \\ x = \ln u \Rightarrow dx = \frac{1}{u}\,du \end{cases}$ On a alors

$$\int_0^1 \frac{2e^x}{e^x+\frac{1}{e^x}}\,dx = \int_{e^0}^{e^1} \frac{2u}{u+\frac{1}{u}} \cdot \frac{1}{u}\,du$$

$$= \int_1^e \frac{2u}{u^2+1}\,du \quad \rightsquigarrow \text{ on reconnaît } \int \frac{v'}{v} = [\ln|v|]$$

$$= \left[\ln|\underbrace{u^2+1}_{>0}|\right]_1^e$$

$$= \left[\ln(u^2+1)\right]_1^e = \ln(e^2+1) - \ln 2$$

3) $\int_0^{\pi/3} \frac{\sin^3 x}{\sqrt{\cos x}}\,dx$

On fait le changement de variable $t = \sqrt{\cos x}$. On a besoin d'exprimer dx, ie de calculer $(\sqrt{\cos x})'$. On a

$$(\sqrt{\cos x})' = (\sqrt{u})' \quad \text{où } u(x) = \cos x \Rightarrow u'(x) = -\sin x$$

$$= \frac{u'}{2\sqrt{u}}$$

$$= \frac{-\sin x}{2\sqrt{\cos x}}$$

Donc $dt = \frac{-\sin x}{2\sqrt{\cos x}}\,dx \iff dx = \frac{-2\sqrt{\cos x}}{\sin x}\,dt$. On doit maintenant exprimer $\frac{-2\sqrt{\cos x}}{\sin x}$ en fonction de t. Pour cela, on remarque que

$$\sin^2 x + \cos^2 x = 1 \iff \sin^2 x = 1 - \cos^2 x \iff \sin^2 x = 1 - \sqrt{\cos x}^4 = 1 - t^4$$

$$\iff \sin x = \sqrt{1-t^4} = (1-t^4)^{1/2}$$

D'où $dx = \frac{-2t}{(1-t^4)^{1/2}}$. On obtient donc

$$\int_0^{\pi/3} \frac{\sin^3 x}{\sqrt{\cos x}}\,dx = \int_{\sqrt{\cos 0}}^{\sqrt{\cos\frac{\pi}{3}}} \frac{(1-t^4)^{3/2}}{\cancel{t}} \cdot \frac{-2\cancel{t}}{(1-t^4)^{1/2}}\,dt$$

$$= -2\int_1^{\sqrt{2}/2} (1-t^4)\,dt$$

$$= -2\left(t - \frac{t^5}{5}\right)_1^{\sqrt{2}/2}$$

$$= -2\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}^5}{2^5 \times 5}\right) + 2\left(1 - \frac{1}{5}\right)$$

$$= -2\left(\frac{\sqrt{2}}{2} - \frac{4\sqrt{2}}{\cancel{2^2} 2^3 \times 5}\right) + 2 \times \frac{4}{5}$$

$$= \frac{8}{5} - 2\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{40}\right)$$

$$= \frac{8}{5} - 2\,\frac{19\sqrt{2}}{40}$$

$$= \frac{8}{5} - \frac{19\sqrt{2}}{20}$$

Exercice 3 :

Calculer $F(x) = \int_2^x \frac{t+1}{t(t-1)(t+2)}\,dt$, $x > 1$.

Cherchons $a, b, c \in \mathbb{R}$ tels que $\frac{t+1}{t(t-1)(t+2)} = \frac{a}{t} + \frac{b}{t-1} + \frac{c}{t+2}$. On développe et on identifie :

$$\frac{a}{t} + \frac{b}{t-1} + \frac{c}{t+2} = \frac{a(t-1)(t+2) + bt(t+2) + ct(t-1)}{t(t-1)(t+2)}$$

$$= \frac{at^2 + at - 2a + bt^2 + 2bt + ct^2 - ct}{t(t-1)(t+2)}$$

$$= \frac{(a+b+c)t^2 + (a+2b-c)t - 2a}{t(t-1)(t+2)} = \frac{t+1}{t(t-1)(t+2)}$$

Par identification, on en déduit le système suivant

$$\begin{cases} a+b+c=0 \\ a+2b-c=1 \\ -2a = 1 \end{cases} \Leftrightarrow \begin{cases} b+c = \frac{1}{2} & (L1) \\ 2b+c = \frac{3}{2} & (L2) \\ a = -\frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} b+c = \frac{1}{2} \\ 3b = 2 & (L1)+(L2) \\ a = -\frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} c = \frac{1}{2} - \frac{2}{3} = -\frac{1}{6} \\ b = \frac{2}{3} \\ a = -\frac{1}{2} \end{cases}$$

Du coup,

$$\frac{t+1}{t(t-1)(t+2)} = -\frac{1}{2}\frac{1}{t} + \frac{2}{3}\frac{1}{t-1} - \frac{1}{6}\frac{1}{t+2}$$

Et ainsi,

$$\int_2^x \frac{t+1}{t(t-1)(t+2)}\,dt = \int_2^x -\frac{1}{2}\frac{1}{t} + \frac{2}{3}\frac{1}{t-1} - \frac{1}{6}\frac{1}{t+2}\,dt$$

$$= -\frac{1}{2}\int_2^x \frac{1}{t}\,dt + \frac{2}{3}\int_2^x \frac{1}{t-1}\,dt - \frac{1}{6}\int_2^x \frac{1}{t+2}\,dt$$

$$= -\frac{1}{2}\left[\ln|t|\right]_2^x + \frac{2}{3}\left[\ln|t-1|\right]_2^x - \frac{1}{6}\left[\ln|t+2|\right]_2^x$$

$$= -\frac{1}{2}(\ln x - \ln 2) + \frac{2}{3}(\ln(x-1) - \underbrace{\ln 1}_{0}) - \frac{1}{6}(\ln(x+2) - \ln 4)$$ car $x > 1 \Rightarrow \begin{cases} x>0 \\ x-1>0 \\ x+2>0 \end{cases}$

$$= -\frac{1}{2}\ln x + \frac{2}{3}\ln(x-1) - \frac{1}{6}\ln(x+2) + \frac{1}{2}\ln 2 + \underbrace{\frac{1}{6}\ln 4}_{\frac{1}{6}\ln(2^2) = \frac{1}{3}\ln 2}$$

Et donc $\boxed{F(x) = -\frac{1}{2}\ln x + \frac{2}{3}\ln(x-1) - \frac{1}{6}\ln(x+2) + \frac{5}{6}\ln 2}$

<u>Exercice 4</u> :

Calculer $I := \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin^3 t}\,dt$.

On sait que $\forall t \in \mathbb{R}$, $\cos^2 t + \sin^2 t = 1 \Leftrightarrow \sin^2 t = 1 - \cos^2 t$ et donc $\sin^4 t = (1-\cos^2 t)^2$.

On fait apparaître $\sin^4 t$ en multipliant le tout par $\frac{\sin t}{\sin t}$:

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin^3 t} \times \frac{\sin t}{\sin t}\,dt = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin^4 t} \times \sin t\,dt = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{(1-\cos^2 t)^2}\sin t\,.\,dt$$

On fait le changement de variable $\begin{cases} u = \cos t \\ du = -\sin t\,.\,dt \end{cases}$. On a alors :

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{(1-\cos^2 t)^2} \underbrace{\sin t\,dt}_{=-du} = \int_{\cos(\frac{\pi}{3})}^{\cos(\frac{\pi}{2})} -\frac{1}{(1-u^2)^2}\,du = -\int_{\frac{1}{2}}^{0} \frac{1}{(1-u^2)^2}\,du = \int_0^{\frac{1}{2}} \frac{1}{(1-u^2)^2}\,du$$

Ecrivons $\frac{1}{(1-u^2)^2} = \frac{1}{[(1-u)(1+u)]^2} = \frac{1}{(1-u)^2(1+u)^2}$ sous la forme $\frac{a}{1-u} + \frac{b}{(1-u)^2} + \frac{c}{1+u} + \frac{d}{1+u^2}$ avec $a, b, c, d \in \mathbb{R}$. On a :

$$\frac{a}{1-u} + \frac{b}{(1-u)^2} + \frac{c}{1+u} + \frac{d}{(1+u)^2} = \frac{a(1-u)(1+u)^2 + b(1+u)^2 + c(1+u)(1-u)^2 + d(1-u)^2}{(1-u)^2(1+u)^2}$$

Développons le numérateur N :

$$N = a(1-u)(1+2u+u^2) + b(1+2u+u^2) + c(1+u)(1-2u+u^2) + d(1-2u+u^2)$$

$$= a + 2au + au^2 - au - 2au^2 - au^3 + b + 2bu + bu^2 + c - 2cu + cu^2 + cu - 2cu^2 + cu^3 + d - 2du + du^2$$

$$= (c-a)u^3 + (b-a+d-c)u^2 + (a+2b-c-2d)u + (a+b+c+d)$$

Par identification, $N = 1$, donc on en déduit le système suivant :

$$\begin{cases} c-a=0 \\ b-a+d-c=0 \\ a+2b-c-2d=0 \\ a+b+c+d=1 \end{cases} \iff \begin{cases} c=a \\ b-2a+d=0 \\ 2b-2d=0 \\ 2a+b+d=1 \end{cases} \iff \begin{cases} c=a \\ 2b-2a=0 \\ d=b \\ 2a+2b=1 \end{cases} \iff \begin{cases} c=a \\ b=a \\ d=b \\ 4a=1 \end{cases} \iff a=b=c=d=\frac{1}{4}$$

Du coup

$$\frac{1}{(1-u^2)^2} = \frac{1}{4(1-u)} + \frac{1}{4(1-u)^2} + \frac{1}{4(1+u)} + \frac{1}{4(1+u)^2}$$

Et donc finalement,

$$I = \frac{1}{4}\int_0^{\frac{1}{2}} \frac{1}{1-u} + \frac{1}{(1-u)^2} + \frac{1}{1+u} + \frac{1}{(1+u)^2}\, du$$

$$= \frac{1}{4}\left[-\ln|1-u| + \frac{1}{1-u} + \ln|1+u| - \frac{1}{1+u}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{4}\left[-\ln\left(\frac{1}{2}\right) + \frac{1}{\frac{1}{2}} + \ln\left(\frac{3}{2}\right) - \frac{1}{\frac{3}{2}}\right] - \frac{1}{4}\left[-\underbrace{\ln(1)}_{0} + 1 + \underbrace{\ln(1)}_{0} - 1\right]$$

$$= \frac{1}{4}\left(\cancel{\ln 2} + 2 + (\ln 3 - \cancel{\ln 2}) - \frac{2}{3}\right)$$

$$I = \frac{1}{3} + \frac{\ln 3}{4}$$