Master 1 — Spring 2019
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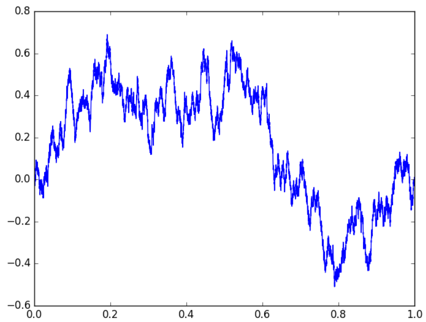
Homework assignment : On the Brownian bridge. (v3)

Let B be a standard Brownian motion. The standard Brownian bridge is defined as follows: $\beta_t = B_t - tB_1$ for $t \in [0,1]$.



Exercise 1 — *Absolute continuity.*
You have shown in the second exercise session that

$$\mathrm{Law}(B_{|[0,1]} | B_1 \in dx) = \mathrm{Law}(x\mathrm{Id} + \beta),$$

In other words, for every bounded measurable H,

$$\mathbb{E}[H(B_{|[0,1]}, B_1)] = \int_{\mathbb{R}} \mathbb{E}[H(x\mathrm{Id} + \beta, x)]\, \mathbb{P}_{B_1}(dx).$$

(1) For $\varepsilon > 0$, let $\nu_\epsilon = \mathrm{Law}(B_{|[0,1]} | -\varepsilon \leq B_1 \leq \varepsilon)$ be the (deterministic!) probability measure such that for every bounded measurable H,

$$\int_{\mathcal{C}([0,1])} H(\varphi)\nu_\varepsilon(d\varphi) = \frac{\mathbb{E}[H(B_{|[0,1]})\, \mathbb{1}_{|B_1| \leq \varepsilon}]}{\mathbb{P}(|B_1| \leq \varepsilon)}.$$

Show that it converges (in the weak topology of measures), as $\varepsilon \to 0$, to $\mathrm{Law}(\beta)$.

(2) For $0 < a < 1$, what does the Markov property say about the joint distribution of $(B_{|[0,a]}, B_1)$? Deduce that, for H positive bounded continuous $\mathcal{C}([0,a]) \to \mathbb{R}$, the following quantity:

$$\mathbb{E}[H(B_{|[0,a]}) | |B_1| < \varepsilon] = \frac{\mathbb{E}[H(B_{|[0,a]})\, \mathbb{1}_{|B_1| \leq \varepsilon}]}{\mathbb{P}(|B_1| \leq \varepsilon)} = \int_{\mathcal{C}([0,1])} H(\varphi_{|[0,a]})\nu_\varepsilon(d\varphi).$$

converges, as $\varepsilon \to 0$, to

$$\int_{\mathcal{C}([0,a])} H(\phi) \frac{1}{\sqrt{1-a}} \exp\left(-\frac{\phi(a)^2}{2(1-a)}\right) \mathbb{P}_{B_{|[0,a]}}(d\phi).$$

(3) Deduce that the distribution of $\beta_{|[0,a]}$ is absolutely continuous with regard to that of $B_{|[0,a]}$ when $a < 1$. Is it the case when $a = 1$?

Exercise 2 — *Location of the minimum.*
We want to compute the distribution of $T = \inf\{t \geq 0, \beta_t = \min_{[0,1]} \beta\}$.

(1) Show that the global minimum of β is almost surely reached exactly once. You may use the fact that for every $a < b < c < d \in \mathbb{Q} \cap (0,1)$, the global minimum of B on $[a,b]$ and $[c,d]$ are almost surely different (4th exercise session)
(2) Show that the Brownian bridge is cyclically exchangeable, i.e. that for every $x \in [0,1)$, the process $t \mapsto \beta_{(x+t) \bmod 1} - \beta_x$ is still distributed like β. (You may start by reasoning on the Brownian motion.)
(3) Deduce the law of T.

Exercise 3 — *Maximum of $|\beta|$.*
We now wish to compute the distribution of the random variable

$$K = \sup_{[0,1]} |\beta|.$$

To that end, let us study $S = \sup_{[0,1]} |B|$ under the conditioning $|B_1| \leq \varepsilon$.

(1) Our first goal is to compute $\mathbb{P}(S > a, |B_1| < \varepsilon)$, where we assume that $0 < \varepsilon < a$.
  (a) Show that
$$\mathbb{P}(S > a, |B_1| < \varepsilon) = 2\,\mathbb{P}(T_a < T_{-a}, |B_1 - 2a| < \varepsilon)$$
  (b) Show that
$$\mathbb{P}(T_a < T_{-a}, |B_1 - 2a| < \varepsilon) = \mathbb{P}(|B_1 - 2a| < \varepsilon) - \mathbb{P}(T_a < T_{-a}, |B_1 - 4a| < \varepsilon)$$
  (c) Keep working and deduce an explicit series which equals $\mathbb{P}(S > a, |B_1| < \varepsilon)$.
(2) Deduce the cumulative distribution function of K.

The distribution of K is called the Kolmogorov distribution *and shows up naturally in statistics in the* Kolmogorov-Smirnov test*, for reasons that we will not try to explain in this homework.*