Master 1 — Spring 2019
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Solutions for $\mathbb{E}$xercise sheet 11: Miscellanea

Solution 1 — *Capacity and Hausdorff dimension.*
We first notice that we can change the definition of the Hausdorff measure, as to require that the sets U_i are disjoint.

(1) Let $(U_i)_i \in \mathcal{P}(A)^{\mathbb{N}}$ be such that for all i, $\operatorname{diam}(U_i) \leq \delta$ and the $(U_i)_i$ forms a partition of A.

$$\begin{aligned}\iint_{|x-y|<\delta} \mu(dx)\mu(dy)|x-y|^{-\alpha} &\geq \iint_{|x-y|<\delta} \mu(dx)\mu(dy)(\sum_i \mathbb{1}_{x,y\in U_i})|x-y|^{-\alpha} \\ &\geq \sum_i \iint_{U_i^2} \mu(dx)\mu(dy)|x-y|^{-\alpha} \\ &\geq \sum_i \mu(U_i)^2 \operatorname{diam}(U_i)^{-\alpha}\end{aligned}$$

Hence

$$\begin{aligned}&\left(\iint_{|x-y|<\delta} \mu(dx)\mu(dy)|x-y|^{-\alpha}\right)\left(\sum_{i\in\mathbb{N}} \operatorname{diam}(U_i)^{\alpha}\right) \\ &\geq \left(\sum_i \mu(U_i)^2 \operatorname{diam}(U_i)^{-\alpha}\right)\left(\sum_{i\in\mathbb{N}} \operatorname{diam}(U_i)^{\alpha}\right) \\ &\geq \left(\sum_i \mu(U_i)\operatorname{diam}(U_i)^{\alpha/2}\operatorname{diam}(U_i)^{-\alpha/2}\right)^2 = \left(\sum_i \mu(U_i)\right)^2 = \mu(A)^2\end{aligned}$$

by Cauchy-Schwarz, yielding the desired inequality. Taking the infimum on all $(U_i)_i$ then the limit $\delta \to 0$ yields

$$\mathcal{H}^{\alpha}(A) \geq \frac{\mu(A)^2}{\iint_A \mu(dx)\mu(dy)|x-y|^{-\alpha}}$$

Hence for a set of nonzero finite α-capacity, by definition there exists $\mu > 0$ such that $\iint_A \mu(dx)\mu(dy)|x-y|^{-\alpha} < \infty$, so the right-hand-side is bounded below away from 0. Hence $H^{\alpha}(A) > 0$ and the Hausdorff dimension is larger than α.

(2) Assume wlog that the segment is $[0,1]$. Let C be the α-Hölder constant. For $n \geq 1$ take $U_k = f([k/n,(k+1)/n])$ for $0 \leq k \leq n-1$. Then it is a cover of $f([0,1])$ and $\operatorname{diam}(U_k) \leq C(1/n)^{\alpha}$. Hence $\sum_k \operatorname{diam}(U_k)^{1/\alpha} \leq \sum_k C^{1/\alpha} 1/n \leq C^{1/\alpha}$. So we found arbitrarily fine covers with bounded α-sum. Hence $\mathcal{H}^{\alpha}(A) < \infty$ and $\dim_{\mathcal{H}}(A) \leq \epsilon$.

(3) If $d = 1$ $B([0,1])$ almost surely contains a ball so has Hausdorff dimension 1.
If $d \geq 2$, we use question 2 and the fact that B is almost surely $(1/2-\epsilon)$-Hölder

on $[0,1]$ to show that $\dim_{\mathcal{H}}(B([0,1])) \leq 2$. For the lower bound we consider the (random) occupation measure $\mu = B_\star \operatorname{Leb}_{[0,1]}$. If we take $\alpha < 2$ and compute

$$\begin{aligned}
\mathbb{E}\left[\iint_{B([0,1])^2} \mu(dx)\mu(dy)(x-y)^{-\alpha}\right] &= \mathbb{E}\left[\iint_{[0,1]^2} dxdy(B(x)-B(y))^{-\alpha}\right] \\
&= \iint_{[0,1]^2} dxdy\, \mathbb{E}[(B(x)-B(y))^{-\alpha}] \\
&= \iint_{[0,1]^2} dxdy(x-y)^{-\alpha/2}\, \mathbb{E}[(B(1))^{-\alpha}]
\end{aligned}$$

This is a product of two integrals, the first one boils down to $\int_0^1 r^{-\alpha/2}dr < \infty$, the second one to $\int_0^\infty r^{d-1}r^{-\alpha}e^{-r^2/2}dr < \infty$, since $\alpha < 2$. Hence the random variable $\iint_{B([0,1])^2} \mu(dx)\mu(dy)(x-y)^{-\alpha}$ has finite expectation and is almost surely finite. Hence almost surely $\dim_{\mathcal{H}}(B([0,1])) > \alpha$. Hence $\dim_{\mathcal{H}}(B([0,1])) = 2$ almost surely.

Solution 2 — *Some more boundary value problems.*
In this exercise we admit that for $x, y \in \mathbb{R}^d$, $t > 0$, we have $\partial_t p_t(x,y) = \frac{1}{2}\Delta_y p_t(x,y)$. (Fokker-Planck equation)

(1) This process has clearly independent increments, so we need only show that it is centered.

$$\begin{aligned}
\mathbb{E}_x[X_t] &= \mathbb{E}_x\left[f(B_t) - \frac{1}{2}\int_0^t \Delta f(B_s)ds\right] \\
&= \int_y f(y)p_t(x,y)dy - \frac{1}{2}\int_0^t \left(\int_y p_s(x,y)\Delta f(y)dy\right) ds \\
\frac{\partial}{\partial t}\mathbb{E}_x[X_t] &= \int_y f(y)\frac{\partial}{\partial t}p_t(x,y)dy - \frac{1}{2}\int_y p_t(x,y)\Delta f(y)dy \\
&= \int_y f(y)\frac{\partial}{\partial t}p_t(x,y)dy - \frac{1}{2}\int_y \Delta p_t(x,y)f(y)dy \\
&= \int_y \left(\frac{\partial}{\partial t}p_t(x,y) - \frac{1}{2}\Delta p_t(x,y)\right) f(y)dy = 0
\end{aligned}$$

Where we used Fubini, Lebesgue's differentiation theorem, and integration by part (the fact that f has compact support makes the boundary term vanish). Hence $\mathbb{E}_x[X_t] = \mathbb{E}_x[X_0]$ and we are done.

(2) Once again we need only show that the increments are centered. We want to reuse question 1. Let $\epsilon > 0$ and ϕ_ϵ a $\mathcal{C}^\infty$ approximation of unity with support contained in $B(0,\epsilon)$. Let also $D_\epsilon = \mathbb{R}^d \setminus B(D^\complement, \epsilon)$. Set $f_\epsilon = (\mathbb{1}_{D_{\epsilon/2}} * \phi_{\epsilon/4})f$. Then f_ϵ verifies the hypotheses of question 1. Hence, setting T_ϵ to be the hitting time of $D_\epsilon^\complement$, and using the optional stopping theorem for $f_\epsilon(B_t) - \int_0^t \Delta f_\epsilon(B_s)ds$ at stopping time

$t \wedge T_\epsilon$, we get

$$\begin{aligned} f(x) &= \mathbb{E}_x\left[f_\epsilon(B_{t\wedge T_\epsilon}) - \frac{1}{2}\int_0^{t\wedge T_\epsilon} \Delta f_\epsilon(B_s)ds\right] \\ &= \mathbb{E}_x\left[f(B_{t\wedge T_\epsilon}) - \frac{1}{2}\int_0^{t\wedge T_\epsilon} \Delta f(B_s)ds\right] \\ &\xrightarrow{\epsilon\to 0} \mathbb{E}_x\left[f(B_{t\wedge T}) - \frac{1}{2}\int_0^{t\wedge T} \Delta f(B_s)ds\right] \end{aligned}$$

where we used the fact that f and f_ϵ coincide on D_ϵ at the second line, and the continuity of paths with the dominated convergence theorem at the last line (this uses the boundedness of f and its derivatives, along with integrability of the first exit time of bounded domains). This finishes the question.

(3) Show that in the sense of distributions, we have $\Delta G(x,\cdot) = \delta_x$, where G is the Green function of the Brownian motion in the whole of $\mathbb{R}^3$ or in a bounded domain of $\mathbb{R}^2$.

Let D be the domain in which we are working, possibly $\mathbb{R}^d$ for $d \geq 3$. We need to show that for ϕ $\mathcal{C}^\infty$ and compactly supported (in particular ϕ vanishes at the boundary of D), we have

$$\int \Delta\phi(y)G(x,y)dy = -2\int \delta_x(y)\phi(y)dy = -2\phi(x).$$

But by definition of G, for all θ, $\int \theta(y)G(x,y)dy = \mathbb{E}_x[\int_0^T \theta(B_s)ds]$. Hence if we go back to the result of question 2, we have

$$\begin{aligned} \phi(x) &= \mathbb{E}_x\left[\phi(B_{t\wedge T}) - \frac{1}{2}\int_0^{t\wedge T} \Delta\phi(B_s)ds\right] \\ &\xrightarrow{t\to\infty} \mathbb{E}_x\left[\phi(B_T) - \frac{1}{2}\int_0^{T} \Delta\phi(B_s)ds\right] \\ &= 0 - \frac{1}{2}\mathbb{E}_x\left[\int_0^T \Delta\phi(B_s)ds\right] = -\frac{1}{2}\int_y \Delta\phi(y)G(x,y)dy. \end{aligned}$$

which is what we wanted. We used a dominated convergence theorem at line 2:

- when D is bounded the almost sure convergence is immediate, and when D is unbounded, in dimension ≥ 3, it comes from the transience of Brownian motion and compactness of $\mathrm{supp}(\phi)$.
- the domination is by $\|\phi\|_\infty + \|\Delta\phi\|_\infty \int_0^T \mathbb{1}_{B_s\in\mathrm{supp}(\phi)}\, ds$, whose expectation is bounded by $C\int_{\mathrm{supp}(\phi)} G(x,y)dy < \infty$.

We also used the fact that ϕ vanishes at the boundary of D at line 3.

(4) This is only a matter of applying question 2 to u and once again the dominated convergence theorem as $t \to \infty$.

(5) The fact that u is continuous at the boundary follows from the same proof as for the Laplace problem, using the Poincaré cone condition.

It solves the equation in the weak sense because it is the integral against the fundamental solution ("classic" PDE stuff).

Remark : This last claim also holds in the strong sense, under suitable regularity assumptions on f. It is done in S. Port, Brownian Motion and Classical Potential Theory, from page 114 onwards (available at the library). It should also be in the books of Evans or Gilbarg-Trüdinger.