$\mathbb{E}$xercise sheet 3: stopping times and Markov property

Exercise 1 — *Stopping times.*
Let (T_n) be a sequence of stopping times for a filtration $\mathcal{F}$. We call $\mathcal{F}^+$ the right-continuous version of $\mathcal{F}$.

(1) Assume that for every ω, the sequence $(T_n(\omega))$ is increasing with limit $T(\omega)$. Show that T is a $\mathcal{F}$-stopping time.
(2) Assume that for every ω, the sequence $(T_n(\omega))$ is decreasing with limit $T(\omega)$. Show that T is a $\mathcal{F}^+$-stopping time.
(3) Assume that for every ω, the sequence $(T_n(\omega))$ is decreasing and is eventually equal to $T(\omega)$. Show that T is a $\mathcal{F}$-stopping time.

Exercise 2 — *Measurability of the stopped process.*
Let B be a $\mathcal{F}$-Brownian motion and T a $\mathcal{F}$-stopping time. Show that $(T, B_{\min(t,T)})_{t\geq 0}$ is $\mathcal{F}_T$-measurable.

Exercise 3 — *Counter-example.*
Show that the first hitting time by B of the maximum of B on $[0,1]$ is not a stopping time.

Exercise 4 — *Another counter-example.*
Let $X_t = AB_t$ where B is a Brownian motion started from 1 and A an independent balanced Bernoulli.

(1) Show that X is a Markov process and give its transition kernel.
(2) Show that it does not verify the strong Markov property.

Exercise 5 — *Brownian motion on the circle.*
Define a Brownian motion on the circle $\mathbb{S}^1$ by setting $X_t = e^{iB_t}$ for $t \geq 0$. What is the distribution of the last point hit by X in $\mathbb{S}^1$?

Exercise 6 — *The set of zeros of B is perfect.*
Let B be a Brownian motion, and $Z = \{t \geq 0 : \ B_t = 0\}$. Show that almost surely, Z is a closed set without isolated points.