$\mathbb{E}$xercise sheet 6: Some more martingales & Donsker's invariance principle

Exercise 1 — *A weaker condition for the first Wald's lemma.*
We wish to show that when T is a stopping time with $\mathbb{E}[T^{1/2}] < \infty$, Wald's lemma still applies and $\mathbb{E}[B_T] = 0$

(1) Define $\tau := \min\{k : 4^k \geq T\}$. Set $M(t) := \max_{[0,t]} B$ and $X_k := M(4^k) - 2^{k+2}$. Show that (X_k) is a supermartingale for the filtration $(\mathcal{F}_{4^k})_k$, and that τ is a stopping time.
(2) Show that $\mathbb{E}[M(4^\tau)] < \infty$ and conclude.
(3) Show that when T is the hitting time of 1, then $\mathbb{E}[T^\alpha] < \infty$ for all $\alpha < 1/2$, yielding that our result is in some sense optimal.

Exercise 2 — *An application of Donsker's invariance principle.*
Let B be a standard Brownian motion on $[0, 1]$, and $D = \sup\{t \in [0, 1], B_t = 0\}$ and $\widetilde{B}$ be B reflected at D (i.e. $\widetilde{B}_t = B_t \mathbb{1}_{t<D} - B_t \mathbb{1}_{t \geq D}$). Show that $\widetilde{B}$ is distributed like B. Same question for the process reflected at $E = \inf\{t \in [0, 1], B_t = B_1\}$.

Exercise 3 — *First arcsine law.*
The goal of this exercise is to find the distribution of $P = \mathrm{Leb}\{t \in [0, 1], B_t \geq 0\}$.

(1) Let $n \geq 1$ and $(X_1^n, \dots X_n^n)$ be independent Rademacher steps. Set $S_0^n = 0$ and $S_k^n = S_{k-1}^n + X_k$ inductively for $1 \leq k \leq n$. Let $Y^n = (Y_1^n, \dots, Y_n^n)$ be constructed by taking the X_k^n for which $S_k > 0$ in decreasing order, then the Y_k^n for which $S_k^n \leq 0$ in increasing order. Show that $X^n \overset{d}{=} Y^n$ (Hint: draw a picture).
(2) Let R^n be the walk associated with Y^n. Show that

$$A_n := \#\{k \in [\![1, n]\!] : S_k^n > 0\} = \inf\{k \in [\![0, n]\!] : R_k^n = \max_{j \in [\![0,n]\!]} R_j^n\} =: B_n.$$

(3) Show that B_n/n converges in distribution to $\inf\{t \in [0, 1], B_t = \max_{[0,1]} B\}$. You can use the fact that almost surely the maximum of B on some closed interval is reached at a unique point (Mörters-Peres Thm. 2.11)
(4) Deduce that P is arcsine distributed.

Exercise 4 — *Convergence in distribution of random continuous functions.*
The criterion you were given in class for convergence in $\mathcal{C}(\mathbb{R}_+)$ is a consequence of the following celebrated theorem that gives a compactness criterion for narrow convergence of probability measures:

Theorem *(Prokhorov). Let $(\mu_n)_n$ be a sequence of probability measures on a Polish (complete metric separable) space E. Suppose that it is tight, i.e. for every $\epsilon > 0$ there exists a compact K_ϵ such that for every n, $\mu_n(E \setminus K_\epsilon) < \epsilon$. Then μ_n admits a narrowly convergent subsequence.*

Recall that we say that $\mu_n \to \mu$ *narrowly* if $\mu_n f \to \mu f$ for every $f \in \mathcal{C}_b(E)$, and *vaguely* if $\mu_n f \to \mu f$ for every $f \in \mathcal{C}_c(E)$ (continuous with compact support). We will first prove Prokhorov's theorem then deduce the criterion.

(1) (a) $E = \mathbb{R}^d$, show that every sequence of probability measures admits a vaguely convergent subsequence (use standard functional analysis theorems).
   (b) Deduce Prokhorov's theorem in the case $E = \mathbb{R}^d$ (using the fact that $\mu_n \to \mu$ narrowly $\iff$ ($\mu_n \to \mu$ vaguely and $\mu(E) = 1$)).
   (c) Use a diagonal argument to show that it is still the case when $E = \mathbb{R}^\mathbb{N}$ (you need to use Kolmogorov's extension theorem, which states that given a collection of probability measures $(\pi_I)_{I \subset \mathbb{N} \text{ finite}}$ with the compatibility condition $(\mathrm{proj}_J)_* \pi_I = \pi_J$ for every $J \subset I$, then there exists a probability measure π on $\mathbb{R}^\mathbb{N}$ with $(\mathrm{proj}_I)_* \pi = \pi_I$ for every I).
   (d) Show it for a general E, by first showing that E is then homeomorphic to a subset of $[0,1]^\mathbb{N}$.

(2) Let $X^{(n)}$ be a sequence of random variables in $\mathcal{C}(\mathbb{R}_+)$ such that
   (a) $\sup_n \mathbb{P}(|X^{(n)}(0)| > M) \xrightarrow[M\to\infty]{} 0$
   (b) for every $\eta > 0$, $T > 0$, we have $\sup_n \mathbb{P}(m_{[0,T]}(X^{(n)}, \delta) > \eta) \xrightarrow[\delta \to 0]{} 0$

   Show that the sequence of the distributions of the $X^{(n)}$ for $n \in \mathbb{N}$ is tight.

(3) Show that if we have conditions (a) and (b) above, and furthermore the f.d.m.s of $X^{(n)}$ converge to the f.d.m.s of X, then $X^{(n)} \to X$ in distribution.

Remark 1: Prokhorov's theorem, along with question (2) and (3), all admit a converse.
Remark 2: If you don't like functional analysis, the vague sequential compactness property can be proved in $\mathbb{R}$ (and even in $\mathbb{R}^d$) by showing that the cumulative distribution functions admit a convergent subsequence in the sense of convergence at limit continuity points (this is Helly's selection theorem)