$\mathbb{E}$xercise sheet 7: Donsker's invariance principle

Exercise 1 — *Recurrence and Donsker.*
You know that almost surely the random walk on $\mathbb{Z}^2$ visits 0 infinitely often. Is it the case for the bidimensional Brownian motion? What does Donsker's invariance principle tell us here?

Exercise 2 — *Skorokhod's embedding.*
Let X be a centered random variable with variance 1.

(1) Argue for the existence of a sequence of random times T_k such that $(T_k - T_{k-1})_k$ is an i.i.d. sequence of mean 1 and $(S_k)_k = (B_{T_k})_k$ is a random walk whose increments are distributed like X. Define $(\widetilde{S}^n_t)_t = (\frac{S_{nt}}{\sqrt{n}})_t$ its (properly interpolated) rescaled version.
(2) Let $\phi_n(t) = n^{-1}T_{\lfloor nt \rfloor}$. Show that almost surely this random function converges uniformly on every compact to the identity $\mathbb{R}_+ \to \mathbb{R}_+$.
(3) Show that $\widetilde{S}^n = (t \mapsto n^{-1/2}B_{nt}) \circ \phi_n$ at points that are multiples of $1/n$. Deduce that $\|\widetilde{S}^n - (t \mapsto n^{-1/2}B_{nt})\|_{[0,A]}$ goes to 0 in probability for every A.
(4) Deduce Donsker's theorem.

Exercise 3 — *Donsker's theorem for bridges.*
In this exercise, let $b(n,p,k)$ denothe the probability that a binomial of parameters (n,p) equals k, and f denote the standard Gaussian density. We will make use of the following *local limit theorem*, which is a refinement of the central limit theorem.
Theorem. (De Moivre–Laplace) As $n \to \infty$,

$$\sup_{k \in \mathbb{Z}} \left| \sqrt{p(1-p)n}\, b(n,p,k) - f\left(\frac{k-np}{\sqrt{p(1-p)n}}\right) \right| = o(n^{-1/2}).$$

Recall (a few sessions back) that the Brownian bridge β (from 0 to 0) has the following property: for every integrable function H and $\epsilon > 0$,

$$\mathbb{E}\left[H(\beta_{|[0,1-\epsilon]})\right] = \mathbb{E}\left[H(B_{|[0,1-\epsilon]})\frac{\epsilon^{-1/2} f(B_{1-\epsilon})}{f(0)}\right].$$

Define the simple random walk S and its interpolated and rescaled version $\widetilde{S}^n$. Our goal is to show that the distribution of $\widetilde{S}^{2n}$ **given that it is a bridge** (i.e. $\widetilde{S}_{2n} = 0$), converges to that of β as $n \to \infty$.

(1) Let H be a bounded continuous or positive measurable function. Fix $n \geq 1$ and $k_n \leq n-1$. Show that

$$\mathbb{E}[H(\widetilde{S}^{2n}{}_{|[0,k_n/n]}) \mid \widetilde{S}_{2n} = 0] = \mathbb{E}\left[H(\widetilde{S}^{2n}{}_{|[0,k_n/n]})\frac{b(\frac{1}{2}, 2n-2k_n, n-k_n+\frac{\sqrt{2n}}{2}\widetilde{S}_{k/n})}{b(\frac{1}{2}, 2n, n)}\right]$$

(2) Suppose that $k_n/n \to 1-\epsilon$. Denote by A_n the fraction appearing in the right-hand side. Show that A_n is bounded by $b(\frac{1}{2}, 2n-2k_n, n-k_n)/b(\frac{1}{2}, 2n, n)$, which (deterministic) bound converges to a constant. Deduce that the tightness criterion (the bound on $\mathbb{E}[|\widetilde{S}^{2n}_t - \widetilde{S}^{2n}_s|^4]$) that applies to $\widetilde{S}^{2n}$ still applies to the conditioned version.

(3) Show that there exists a deterministic $o(n^{-1/2})$ such that almost surely we have $\left|A_n - \frac{\epsilon^{-1/2} f(\widetilde{S}^{2n}_{k/n})}{f(0)}\right| = o(n^{-1/2})$. Deduce that all finite-dimensional marginals of $\widetilde{S}^{2n}$ given $\widetilde{S}^{2n} = 0$ converge to that of the Brownian bridge.

(4) Conclude.

Exercise 4 — *The binary splitting martingale.*
Let X be centered with finite variance and $(X_n)_n$ be the associated binary splitting martingale, defined as follows: Let $\mathcal{G}_0$ the trivial σ-field, and for $n \geq 0$, set $X_n = \mathbb{E}[X \mid \mathcal{G}_n]$, $\xi_n = \operatorname{sgn}(X - X_n)$ and $\mathcal{G}_{n+1} = \sigma(\xi_0, \ldots, \xi_n)$. You know that $(X_n)_n$ is a martingale for the filtration $(\mathcal{G}_n)_n$, that it is bounded in L^2 hence converges a.s. and L^1 to some random variable X_∞. We still need to show that $X_\infty = X$ a.s.

(1) Express $X_{n+1} - X_n$ so that its positive and negative part are explicit. Use this to compute $|X_{n+1} - X_n|$.

(2) Deduce that $|X_n - X|$ goes to 0 in L^1 and conclude.