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Solutions for $\mathbb{E}$xercise sheet 7: Donsker's invariance principle (v2)

Solution 1 — *Recurrence and Donsker.*
We cancelled this exercise but might come back to it later.

Solution 2 — *Skorokhod's embedding.*
To be updated. Let X be a centered random variable with variance 1.

(1) This is done by repetetively applying Skorokhod's embedding and the strong Markov property.

(2) The strong LLN tells us that almost surely, for every $t \in \mathbb{Q}$, $\phi_n(t) \to t$ as $n \to \infty$. Then Dini's theorem gives that ϕ_n converges to the identity uniformly on every compact.

(3) Denote $B^n : t \mapsto n^{-1/2} B_{nt}$. Remark that all B^n are Brownian motions. Now if t is a multiple of $1/n$, $\widetilde{S}^n_t = n^{-1/2} S_{nt} = n^{-1/2} B_{nn^{-1}T_{\lfloor nt \rfloor}} = B^n(\phi_n(t))$. Then for $0 \leq t \leq T$, there is $u \in [0,T] \cap n^{-1}\mathbb{Z}$, $|u - t| < 1/n$. As a result,

$$\begin{aligned} |\widetilde{S}^n_t - B^n_t| &\leq |\widetilde{S}^n_u - B^n_u| + |\widetilde{S}^n_u - \widetilde{S}^n_t| + |B^n_u - B^n_t| \\ &\leq |B^n \circ \phi_n(u) - B^n_u| + n^{-1/2}|X_{nu}| + |B^n_u - B^n_t| \end{aligned}$$

Taking the sup,

$$\|\widetilde{S}^n - B^n\|_{[0,T]} \leq m_{2T}(B^n, \|\phi_n - id\|_{[0,T]}) + \infty \, \mathbb{1}_{\|\phi_n - id\| > 1} + n^{-1/2} \sup_{1 \leq i \leq nT} X_i + m_T(B^n, n^{-1}),$$

and each term indeed goes to 0 in probability. For the third one, thanks to the second moment hypothesis, we have by Chebychev's inequality $\mathbb{P}(X > \varepsilon n^{1/2}) = o(n)$.

(4) We have $B^n \to B$ in distribution (actually the distribution is constant!) and $d(\widetilde{S}^n, B^n) \to 0$ in probability. Then a classic generalization of Slutsky's lemma tells us that $\widetilde{S}^n \to B$ in distribution.

Solution 3 — *Donsker's theorem for bridges.*
In this exercise, let $b(n,p,k)$ denothe the probability that a binomial of parameters (n,p) equals k, and f denote the standard Gaussian density. We will make use of the following *local limit theorem*, which is a refinement of the central limit theorem.

Theorem. (De Moivre–Laplace) As $n \to \infty$,

$$\sup_{k \in \mathbb{Z}} \left| \sqrt{p(1-p)n}\, b(n,p,k) - f\left(\frac{k - np}{\sqrt{p(1-p)n}}\right) \right| = o(n^{-1/2}).$$

Recall (a few sessions back) that the Brownian bridge β (from 0 to 0) has the following property: for every integrable function H and $\epsilon > 0$,

$$\mathbb{E}\left[H(\beta_{|[0,1-\epsilon]})\right] = \mathbb{E}\left[H(B_{|[0,1-\epsilon]})\frac{\epsilon^{-1/2} f(B_{1-\epsilon})}{f(0)}\right].$$

Define the simple random walk S and its interpolated and rescaled version $\widetilde{S}^n$. Our goal is to show that the distribution of $\widetilde{S}^{2n}$ **given that it is a bridge** (i.e. $\widetilde{S}_{2n} = 0$), converges to that of β as $n \to \infty$.

(1) For $j_0, j_1, \ldots, j_{2k_n}$ fixed integers,

$$\begin{aligned}
&\mathbb{P}(S_0 = j_0, \ldots, S_{2k_n} = j_{2k_n} \mid S_{2n} = 0) \\
&= \frac{\mathbb{P}(S_0 = j_0, \ldots, S_{2k_n} = j_{2k_n})\, \mathbb{P}(S_{2n} = 0 \mid S_0 = j_0, \ldots, S_{2k_n} = j_{2k_n})}{\mathbb{P}(S_{2n} = 0)} \\
&= \mathbb{P}(S_0 = j_0, \ldots, S_{2k_n} = j_{2k_n}) \frac{\mathbb{P}(S_{2n} = 0 \mid S_{2k_n} = j_{2k_n})}{\mathbb{P}(S_{2n} = 0)} \\
&= \mathbb{P}(S_0 = j_0, \ldots, S_{2k_n} = j_{2k_n}) \frac{b(\frac{1}{2}, 2n - 2k_n, n - k_n + \frac{j_{2k_n}}{2} \widetilde{S}_{k/n})}{b(\frac{1}{2}, 2n, n)}
\end{aligned}$$

Now since $H(\widetilde{S}^{2n}{}_{|[0,k_n/n]})$ is a deterministic function of $S_0, \ldots S_{2k_n}$, we get the desired result.

(2) It is clear that the central binomial coefficient bounds all the others. Hence the bound of A_n by $b(\frac{1}{2}, 2n - 2k_n, n - k_n)/b(\frac{1}{2}, 2n, n)$. By de Moivre-Laplace, this sequence converges to $\frac{f(0)}{\sqrt{\epsilon} f(0)}$ so is bounded uniformly in n (by C_ϵ, say). Then for s, t with $|s - t| < 1/2$, we can without loss of generality assume that $s < t < 3/4$ (otherwise reverse everything). Then $\mathbb{E}[|\widetilde{S}^{2n}_t - \widetilde{S}^{2n}_s|^4 \mid \widetilde{S}_{2n} = 0] \le C_{3/4}\, \mathbb{E}[|\widetilde{S}^{2n}_t - \widetilde{S}^{2n}_s|^4]$. You know from the proof of Donsker's theorem that this $\mathbb{E}[|\widetilde{S}^{2n}_t - \widetilde{S}^{2n}_s|^4$ is bounded by $c|s - t|^{1+\gamma}$ with $c, \gamma > 0$. We get

$$\forall s, t, |t - s| < 1/2, \quad \mathbb{E}[|\widetilde{S}^{2n}_t - \widetilde{S}^{2n}_s|^4 \mid \widetilde{S}_{2n} = 0] \le C_{3/4} c |t - s|^{1+\gamma}$$

If $|t - s| \ge 1/2$, then $\mathbb{E}[|\widetilde{S}^{2n}_t - \widetilde{S}^{2n}_s|^4 \mid \widetilde{S}_{2n} = 0]$ is uniformly bounded, (by the L^4 triangle inequality we can reuse the case $|t - s| < 1/2$, and bound by some constant D independent of n, $t - s$). Hence we get

$$\forall s, t, \quad \mathbb{E}[|\widetilde{S}^{2n}_t - \widetilde{S}^{2n}_s|^4 \mid \widetilde{S}_{2n} = 0] \le D 2^{1+\gamma} C_{3/4} c |t - s|^{1+\gamma}$$

proving a tightness bound for the random walk under the conditioned measure.

(3) From now on we suppose $k_n/n = 1-\epsilon+o(1/n)$, for instance by taking $k_n = n^{-1}\lfloor n(1-\epsilon)\rfloor$.

$$\begin{aligned}
A_n &= \frac{b(\frac{1}{2}, 2n-2k_n, n-k_n+\frac{\sqrt{2n}}{2}\widetilde{S}_{k/n})}{b(\frac{1}{2},2n,n)} \\
&= \frac{1}{\sqrt{1-\frac{k_n}{n}}}\frac{\sqrt{2n-2k}\,b(\frac{1}{2}, 2n-2k_n, n-k_n+\frac{\sqrt{2n}}{2}\widetilde{S}_{k/n})}{\sqrt{2n}\,b(\frac{1}{2},2n,n)} \\
&= (\epsilon^{-1/2}+o(n^{-1}))\frac{f(\widetilde{S}^{2n}_{k/n})+o(n^{-1/2})}{f(0)+o(n^{-1/2})} = \frac{\epsilon^{-1/2}f(\widetilde{S}^{2n}_{k/n})}{f(0)}+o(n^{-1/2})
\end{aligned}$$

Consider some f.d.m. Without loss of genrality we can always assume that it contains 1. Hence set $0 \le t_1 < \ldots < t_r = 1$ and $G : \mathbb{R}^r \to \mathbb{R}$ continuous with compact support. Take ϵ such that $1-\epsilon > t_{r-1}$ Now consider only n large enough so that $k_n/n > t_{r-1}$. where H is some continuous functional. Hence we can use question 2. Thus

$$\begin{aligned}
&\mathbb{E}[G(\widetilde{S}^{2n}_{t_1},\ldots,\widetilde{S}^{2n}_{t_r}) \mid \widetilde{S}_{2n}=0] \\
&= \mathbb{E}[G(\widetilde{S}^{2n}_{t_1},\ldots,\widetilde{S}^{2n}_{t_{r-1}},0) \mid \widetilde{S}_{2n}=0] \quad (\text{a.s. under}\,\mathbb{P}(\cdot \mid \widetilde{S}_{2n}=0),\ S^{2n}=0) \\
&= \mathbb{E}\left[G(\widetilde{S}^{2n}_{t_1},\ldots,\widetilde{S}^{2n}_{t_{r-1}},0)A_n\right] \quad (\text{question 2, the integrand is a function of } \widetilde{S}^{2n}{}_{|[0,k_n/n]}) \\
&= \|G\|_\infty o(n^{-1/2}) + \mathbb{E}\left[G(\widetilde{S}^{2n}_{t_1},\ldots,\widetilde{S}^{2n}_{t_{r-1}},0)\frac{\epsilon^{-1/2}f(\widetilde{S}^{2n}_{k/n})}{f(0)}\right] \quad (\text{first part of the question}) \\
&= o(1)+o(1)+\mathbb{E}\left[G(B_{t_1},\ldots,B_{t_{r-1}},0)\frac{\epsilon^{-1/2}f(B_{1-\epsilon})}{f(0)}\right] \quad (\text{unconditioned Donsker}) \\
&= o(1)+\mathbb{E}\left[G(\beta_{t_1},\ldots,\beta_{t_{r-1}},0)\right] \quad (\text{absolute continuity property of the bridge}) \\
&= o(1)+\mathbb{E}\left[G(\beta_{t_1},\ldots,\beta_{t_{r-1}},\beta_{t_r})\right] \quad (\beta_1 = 0 \text{ almost surely})
\end{aligned}$$

(4) By the usual criterion for convergence in distribution of functions, we are done.

Solution 4 — *The binary splitting martingale.* (1) We write

$$\begin{aligned}
X_{n+1}-X_n &= \mathbb{E}[X-X_n \mid \mathcal{G}_n] \\
&= \mathbb{E}[(X-X_n)\mathbb{1}_{X>X_n} \mid \mathcal{G}_n]\mathbb{1}_{X>X_n} + \mathbb{E}[(X-X_n)\mathbb{1}_{X<X_n} \mid \mathcal{G}_n]\mathbb{1}_{X<X_n}.
\end{aligned}$$

where we used the fact that the sign of $(X-X_n)$ is $\mathcal{G}_n$-measurable. The first term is almost surely positive, the second one is almost surely negative, and almost surely only one of them is nonzero. Hence they almost surely they form a decomposition of $X_{n+1}-X_n$ into a positive and negative part. Then

$$\begin{aligned}
|X_{n+1}-X_n| &= \mathbb{E}[(X-X_n)\mathbb{1}_{X>X_n} \mid \mathcal{G}_n]\mathbb{1}_{X>X_n} - \mathbb{E}[(X-X_n)\mathbb{1}_{X<X_n} \mid \mathcal{G}_n]\mathbb{1}_{X<X_n} \\
&= \mathbb{E}[|X-X_n| \mid \mathcal{G}_n].
\end{aligned}$$

(2) We deduce $\mathbb{E}[|X_n - X|] = \mathbb{E}[|X_{n+1} - X_n|]$, and this last expression goes to 0 as $(X_n)_n$ is L^1-convergent. Thus $|X_n - X|$ goes to 0 in L^1 and by uniqueness (up to a.s. equality) of the L^1 limit we get that $X_\infty = X$ a.s. Hence X_n converges a.s. and L^1 to X.