$\mathbb{E}$xercise sheet 8: Miscellaneous

Exercise 1 — *Zeros of B.*
Let $Z = \{t \geq 0 : B_t = 0\}$.

(1) For $t > 0$, show that almost surely the first zero after t is not isolated in Z.
(2) Deduce that almost surely, no point of Z is isolated in Z.
(3) What happens at the left of the last zero before 1? Is it a stopping time for the reversed Brownian motion ?

Exercise 2 — *Azéma-Yor embedding.*
The goal of this exercise is to give a different and more explicit solution to the Skorokhod embedding problem. Let X a centered, finite-variance, random variable, and set

$$\psi : \mathbb{R} \to \mathbb{R}_+, \psi(x) = \begin{cases} \mathbb{E}[X \mid X \geq x] & \text{if } \mathbb{P}(X \geq x) > 0 \\ x & \text{otherwise.} \end{cases}$$

We call ψ the *barycenter function* of X. It is a left-continuous increasing function with $\psi(-\infty) = 0$ and $\psi(x) \geq x, x \in \mathbb{R}$. **We admit that ψ characterizes the law of X.**
Let B be a standard Brownian motion started from 0, M its maximum process $M_t = \sup_{0 \leq u \leq t} B_u$, and

$$T = \inf\{t \geq 0, M_t \geq \psi(B_t)\}.$$

We will show that $\mathbb{E}[T] = \mathbb{E}[X^2]$ and $B_T \stackrel{d}{=} X$.

(1) Draw a picture in the plane (choosing an arbitrary example for ψ) with the graph of ψ, the graph of the identity, the trajectory of $t \mapsto (B_t, M_t)$, and the point (B_T, M_T). Go back to this picture (or draw a new one) when you are lost.
(2) Show that events $\{B_T \geq a\}$ and $\{T_{\psi(a)} \leq T\}$ are the same, when T_y denotes the hitting time of y.
(3) To simplify the proof, we restrict to the case where $|X| < C$ almost surely. Let a be such that $\mathbb{P}(X \geq a) > 0$. Evaluate $\mathbb{E}[B_{T \vee T_{\psi(a)}} \mid \mathcal{F}_{T_{\psi(a)}}]$.
(4) Deduce $\mathbb{E}[B_T \mid B_T \geq a]$, and conclude.
(5) When X has a discrete finite support, give a simple description of T. Optionally, show that B_T has the prescribed distribution using elementary arguments.
(6) What do we get when X is uniform? When X is an exponential random variable minus one?