Solutions for $\mathbb{E}$xercise sheet 8: Miscellaneous

Solution 1 — *Zeros of B.*
Let $Z = \{t \geq 0 : B_t = 0\}$.

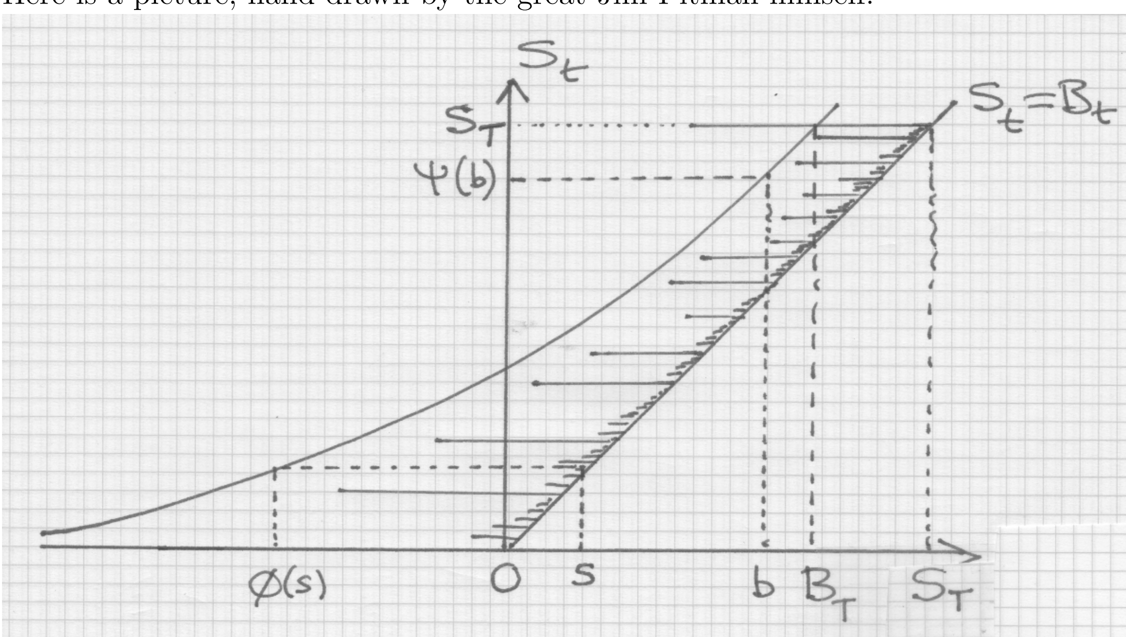
(1) We denote O_t the first zero after t. It is a stopping time. By the strong Markov property, $(B_{O_t+u} - B_{O_t})_u = (B_{O_t+t})_u$ is distributed like B. Hence it oscillates almost surely near 0. Hence B oscillates almost surely at the right of

(2) Almost surely 0 is an accumulation point of Z (lecture). By countable union, and strong Markov, every first 0 after any rational is an accumulation point of Z (at its right). If Z had an isolated point, it would be a first 0 after a rational. Hence it couldn't be isolated in Z.

(3) The last zero U before 1 is isolated at its right, hence almost surely it is not isolated at its left. $1 - U$ is not a stopping time for the reversed Brownian motion $\widetilde{B} = (B_1 - B_{1-t})_t$, because of the equality $\widetilde{B}_1 = \widetilde{B}_{1-U}$,which negates strong Markov's property.

Solution 2 — *Azéma-Yor embedding.*

(1) Here is a picture, hand-drawn by the great Jim Pitman himself:



(2) Show that events $\{B_T \geq a\}$ and $\{T_{\psi(a)} \leq T\}$ are the same, when T_y denotes the hitting time of y. Assume $B_T \geq a$. By definition of T, $M_{T+u} \geq \psi(B_T + u)$ for

arbitrary small u, and hence $M_T \geq \psi(a+) \geq \psi(a)$. Of course this means that $T_{\psi(a)} \leq T$.
Assume $T_{\psi(a)} \leq T$. Let U be the first hitting time of a by B after $T_{\psi(a)}$. Then of course $T_{\psi(a)} \leq T \leq U$. This implies that $B_T \geq a$.

(3) Our restrictive hypothesis implies that $|B_{t\wedge(T\vee T_{\psi(a)})}| < C$. We can then apply the optional stopping theorem to $T_{\psi(a)}$ and $T \vee T_{\psi(a)}$. This yields

$$\mathbb{E}[B_{T\vee T_{\psi(a)}} \mid \mathcal{F}_{T_{\psi(a)}}] = B_{T_{\psi(a)}} = \psi(a).$$

(4)

$$\begin{aligned}\mathbb{E}[B_T \mid B_T \geq a] &= \mathbb{E}[B_T\, \mathbb{1}_{T_{\psi(a)}\leq T}] / \mathbb{P}(T_{\psi(a)} \leq T)\\ &= \mathbb{E}[B_{T\vee T_{\psi(a)}}\, \mathbb{1}_{T_{\psi(a)}\leq T}] / \mathbb{P}(T_{\psi(a)} \leq T)\\ &= \mathbb{E}[\mathbb{1}_{T_{\psi(a)}\leq T}\, \mathbb{E}[B_{T\vee T_{\psi(a)}} \mid \mathcal{F}_{T_{\psi(a)}}]] / \mathbb{P}(T_{\psi(a)} \leq T)\\ &= \mathbb{E}[\mathbb{1}_{T_{\psi(a)}\leq T}\, \psi(a)] / \mathbb{P}(T_{\psi(a)} \leq T)\\ &= \psi(a)\, \mathbb{P}(T_{\psi(a)} \leq T) / \mathbb{P}(T_{\psi(a)} \leq T) = \psi(a)\end{aligned}$$

We have shown that B_T has the same barycenter function as X, hence B_T is distributed like X. Moreover $\mathbb{E}[B_T^2] = \mathbb{E}[X]$ because $|B_{t\wedge T}| \leq C$ (Wald's second lemma).

(5) Assume $X \sim \sum_{i=1}^n p_i \delta_{x_i}$, with $x_1 < \ldots < x_n$. Setting $y_i = \psi(x_i)$, we have $0 = y_1 < \ldots < y_n = x_n$. Then writing

$$I = \inf\{i \geq 1, B \text{ hits } x_i \text{ between } T_{y_i} \text{ and } T_{y_{i+1}}\},$$

we have $B_T = x_I$.

Let us show that $\mathbb{P}(I = i) = p_i$. We have by strong Markov at time T_{y_i} and gambler's ruin, that

$$\mathbb{P}(I > i \mid I \geq i) = \mathbb{P}_{y_i}(T_{y_{i+1}} \leq T_{x_i}) = \frac{y_i - x_i}{y_{i+1} - x_i}.$$

Let us compute this last quantity. For convenience, we write $s = p_i + \cdots + p_n$ and $S = p_i x_i + \cdots p_n x_n$.

$$\cdots = \frac{\frac{S}{s} - x_i}{\frac{S - p_i x_i}{s - p_i} - x_i} = \frac{\frac{S - s x_i}{s}}{\frac{S - p_i x_i - s x_i + p_i x_i}{s - p_i}} = \frac{\frac{S - s x_i}{s}}{\frac{S - s x_i}{s - p_i}} = \frac{s - p_i}{s} = \frac{\sum_{j=i+1}^n p_j}{\sum_{j=i}^n p_j}.$$

Clearly by induction it implies that I has the desired distribution.

(6) We have that B_T is uniform when $T = \inf\{t \geq 0, B_t = 2M_t - 1\} = \inf\{t \geq 0, 2M_t - B_t = 1\}$.
We have that $B_T + 1 \sim \mathcal{E}xp(1)$ when $T = \inf\{t \geq 0, M_t - B_t = 1\}$.