E. Jacob & M. Maazoun

$\mathbb{E}$xercise sheet 9: Harmonic functions and Brownian motion

Exercise 1 — *Harmonic functions and martingales.*
Let $D \subset \mathbb{R}^d$ be a domain and T its exit time. Let $h : \overline{D} \to \mathbb{R}$ be continuous, bounded and harmonic inside D. Fix $x \in D$.

(1) Assume that D is bounded and verifies the Poincaré cone condition. Show that under $\mathbb{P}_x$, the process $(h(B_{t\wedge T}))_t$ is a closed martingale.
(2) In the general case, show that under $\mathbb{P}_x$, the process $(h(B_{t\wedge T}))_t$ is a martingale. (Hint: approximate D by domains that verify the cone condition.)
(3) Deduce that for an arbitrary domain, *when a bounded solution to the Dirichlet problem exists*, the Brownian expectation finds it.
(4) Look at $\log(\|B_t\|)$ if $d = 2$ and $\|B_t\|^{2-d}$ if $d \geq 3$. Are those martingales ?
(5) A converse: show that if h is defined on some domain U with the property that for every $\overline{B}(x,\epsilon) \subset U$, $t \mapsto h(B_{t\wedge T_{\partial B(x,\epsilon)}})$ is a martingale under $\mathbb{P}_x$, then h is harmonic.

Exercise 2 — *A lemma for the Poincaré cone condition.*
Let C be an open cone based in 0. We wish to show that the function $\varphi(x) = \mathbb{P}_x(T_{\partial B(0,1)} < T_{\partial C})$ is bounded away from 1 on $\overline{B}(0,1/2) \setminus C$.

(1) Why can't we use the maximum principle for φ on $\overline{B}(0,1/2) \setminus C$?
(2) Bound φ by some (similarly defined) function to which the maximum principle can be applied, and conclude.

Exercise 3 — *Counterexample.*
Let $D = B(0,1) \setminus \{0\} \subset \mathbb{R}^2$ and consider the Laplace equation $\Delta u = 0$ with Dirichlet boundary conditions $u(0) = 0$ and $u(x) = 1$ for $x \in \partial B(0,1)$. Show that the Brownian expectation does not define a solution. Show that there can't exist a solution (you may use exercise 1 or the fact that if $u(x) = g(|x|)$, then $\Delta u(x) = g''(|x|) + \frac{1}{x}g'(|x|)$.)

Exercise 4 — *Singularity removal.*
Let $d \geq 2$ and $x \in U \subset \mathbb{R}^d$ open. Suppose $h : U \setminus \{x\} \to \mathbb{R}$ is harmonic and bounded around x. Show that u can be extended to a harmonic function on the whole of U.
Better yet, show the same outcome with the relaxed condition that $u(x + \cdot)$ is negligible near x compared to the fundamental solution ($\log(\|\cdot\|)$ if $d = 2$ and $\|\cdot\|^{2-d}$ otherwise).