Solutions of the homework assignement: on the zero set of B

Exercise 1 — *Triviality.*
We have $\mathrm{Leb}(Z) = \int_0^\infty \mathbb{1}_{B_t=0}\,dt$. But since $(\omega, t) \mapsto \mathbb{1}_{B_t(\omega)=0}$ is measurable positive, then Fubini's theorem tells us that $\mathrm{Leb}(Z)$ is a measurable random variable whose expectation is $\mathbb{E}[\mathrm{Leb}(Z)] = \int_0^\infty \mathbb{P}(B_t = 0)dt = \int 0 = 0$.

Exercise 2 — *For your personal enjoyment.*
This is a classic application of Baire's category theorem: If E is a countable complete metric space, then $\emptyset = \cap_{x\in E} E \setminus \{x\}$. But for every x, $E \setminus \{x\}$ is open and dense in E (otherwise x would be isolated). Hence $\emptyset$ is dense in E and E is empty.

Exercise 3 — *For your personal enjoyment.*
The lim is a sup because as $\delta \to 0$ we take an inf on smaller and smaller sets. Moreover, with $\epsilon > 0$, we have

$$\inf_{\substack{(U_i)_i \in \mathcal{P}(E)^{\mathbb{N}} \\ \forall i, \mathrm{diam}(U_i) \le \delta \\ \bigcup_i U_i \supset A}} \left(\sum_{i\in\mathbb{N}} \mathrm{diam}(U_i)^{(\alpha+\epsilon)} \right) \le \delta^\epsilon \inf_{\substack{(U_i)_i \in \mathcal{P}(E)^{\mathbb{N}} \\ \forall i, \mathrm{diam}(U_i) \le \delta \\ \bigcup_i U_i \supset A}} \left(\sum_{i\in\mathbb{N}} \mathrm{diam}(U_i)^{\alpha} \right)$$

which gives lemma 1. Now if E is a metric space and λE is obtained by scaling the distances by $\lambda > 0$, it is clear that $\mathcal{H}_\alpha(\lambda A) = \lambda^\alpha(E)$.
Apparently there is no such thing as finite additivity for Hausdorff measure. So computing the Hausdorff dimension of self-similar sets is harder than I thought (of course upper bounds are always easy...) Sorry I was misleading you...

Exercise 4 — *Last* 0 *before time* 1 *(Second arcsine Law).*
Denote by $\widetilde{B}$ another BM, independent of B.

$$\begin{aligned}
\mathbb{P}(G_1 \le t) &= \mathbb{P}(B_t > 0, \min_{s\in[0,1-t]} B_s^{(t)} > -B_t) + \mathbb{P}(B_t < 0, \max_{s\in[0,1-t]} B_s^{(t)} < -B_t) \\
&= \mathbb{P}(B_t > 0, \max_{s\in[0,1-t]} B_s^{(t)} < B_t) + \mathbb{P}(B_t < 0, \max_{s\in[0,1-t]} B_s^{(t)} < -B_t) \\
&= \mathbb{P}(\max_{s\in[0,1-t]} B_s^{(t)} < |B_t|) \\
&= \mathbb{P}(|\widetilde{B}_{1-t}| < |B_t|) = \mathbb{P}(\sqrt{1-t}|\widetilde{B}_1| < \sqrt{t}|B_1|).
\end{aligned}$$

Let $\theta = \arg(\widetilde{B}_1 + iB_1)$. Then θ is uniform in $[-\pi, \pi]$ and our probability rewrites as

$$\mathbb{P}(|\tan(\theta)| < |\tan(\arcsin(\sqrt{t}))|) = \tfrac{2}{\pi} \arcsin \sqrt{t}$$

Then the equality $\frac{\pi}{2} - \arcsin(\sqrt{1-t}) = \arcsin(\sqrt{t})$ implies a rather surprising symmetry property of G_1: $\mathbb{P}(G_1 > 1-t) = \mathbb{P}(G_1 < t)$. Now by Brownian scaling, the probability that there is a 0 in $[x, x+\epsilon]$ is the same as the probability that there is a 0 in $[x/(x+\epsilon), 1]$, which is $\mathbb{P}(G_1 > x/(x+\epsilon)) = \mathbb{P}(G_1 > 1-\epsilon/(x+\epsilon)) = \frac{2}{\pi}\arcsin(\sqrt{\epsilon/(x+\epsilon)}) \leq 2\sqrt{\epsilon/(x+\epsilon)}$.

Exercise 5 — *Upper bound.*

(1)

$$\begin{aligned}\mathbb{E}[\sum_{I \in C_n} \operatorname{diam}(I)^\alpha] &= \sum_{k=0}^{2^n-1} 2^{-\alpha n}\, \mathbb{P}(Z \text{ intersects } [k2^{-n}, (k+1)2^{-n}]) \\ &\leq 2 \sum_{k=0}^{2^n-1} 2^{-\alpha n} \sqrt{1/(k+1)} \\ &\leq 2^{(\frac{1}{2}-\alpha)n+1} \sum_{k=1}^{2^n} \frac{1}{\sqrt{2^{-n}k}}.\end{aligned}$$

The prefactor goes to 0 when $\alpha > 1/2$, and the sum goes to $\int_0^1 t^{-1/2} dt = 1$.

(2) We want to show that when $\alpha > 1/2$, then $\liminf_{n\to\infty} \sum_{I \in C_n} \operatorname{diam}(I)^\alpha = 0$ almost surely. The previous question and Fatou's lemma give this immediately.

(3) This shows that when $\alpha > 1/2$, we can almost surely find a sequence of coverings of largest diameter going to 0, such that the sum of diameters to the α goes to 0. This implies that $\mathcal{H}_\alpha(Z) = 0$ almost surely for every $\alpha > 1/2$ and hence $\dim_{\mathcal{H}}(Z) \leq 1/2$ almost surely.

Exercise 6 — *Lower bound.*

(1) Let U_i be a covering. Then if $\sup_i \operatorname{diam}(U_i) < \delta$, then

$$\sum_{i \in \mathbb{N}} \operatorname{diam}(U_i)^\alpha = \sum_{i \in \mathbb{N}} \operatorname{diam}(\overline{U_i})^\alpha \geq \tfrac{1}{C} \sum_{i \in \mathbb{N}} \mu(U_i) \geq \tfrac{1}{C}\mu(E).$$

Taking the infimum on all coverings of max diameter $< \epsilon < \delta$ and letting $\epsilon \to 0$ gives theorem 1.

(2) Let B be a Brownian motion. Then Lévy's M-B theorem says that $B^* - B$ is distributed as $|B|$. But the zero set of B is the same as the zero set of $|B|$, which is then distributed as the zero set of $B^* - B$, which is $R = \{t \geq 0, B_t = B_t^*\}$.

(3) B^* is a weakly increasing continuous function, so we can build a random measure μ on $\mathbb{R}_+$ by setting $\mu((a,b)) = B_b^* - B_a^*$. Then let us show that open intervals that avoid R have zero measure. By contraposition, if $\mu((x,y)) > 0$, then $\max_{[x,y]} B > B^*(x)$. Take t to be the first time in $[x,y]$ where B hits $u = (\max_{[x,y]} B + B^*(x))/2$. Then $y > t > x$ and t is the first time in $\mathbb{R}_+$ where B hits u. Hence $t \in R$ and R intersects (x,y). We have shown that almost surely μ is supported on R.

(4) Almost surely $\mu([0,1])$ is nonzero and μ is supported on R so $\mu([0,1] \cap R) > 0$. Let $\alpha < 1/2$. Then we know that almost surely B is α-Hölder on $[0,1]$. Let $C < \infty$

a.s. be the α-Hölder constant and consider U closed in $[0,1]$. Then $U \subset [x,y]$ with $y-x = \operatorname{diam} U$. We have $\mu(U) \leq B^*_y - B^*_x \leq B_\xi - B_x$ where ξ is the first hitting time of the maximum of B on $[x,y]$. This last quantity is bounded by $C(\xi - x) \leq C(y-x) = C \operatorname{diam}(U)$. Then we can apply theorem 1 and show that $\dim_{\mathcal{H}} R \geq \dim_{\mathcal{H}}(R \cap [0,1]) \geq \alpha$ almost surely. This transfers to Z as Z and R have the same distribution.

(5) Combining the two bounds gives the final answer.