$\mathbb{E}$xercise sheet 1 : Review of Gaussian vectors and conditional expectation, and a first approach of Brownian Motion. (v2)

Exercise 1 — *Gaussian vectors.*
Let X be a random vector in $\mathbb{R}^n$. We say that it is a Gaussian vector (i.e. has a multidimensional Gaussian distribution) if for every $t \in \mathbb{R}^n$, the r.v. $\langle t, X\rangle \in \mathbb{R}$ has a (possibly degenerate) Gaussian distribution.

(1) Recall the parameters, the characteristic function, and (when it exists) the p.d.f. of a Gaussian distribution on $\mathbb{R}$.
(2) Show that $t \mapsto \mathbb{E}[\langle t, X\rangle]$ is a linear form, and $(s,t) \mapsto \mathrm{Cov}[\langle s, X\rangle, \langle t, X\rangle]$ is a *positive semi-definite*[1] bilinear form. Let them be represented by $\langle \cdot, m\rangle$ and $\langle \cdot, \Sigma \cdot\rangle$. What would be the coordinates of respectively this vector and this matrix? How would you call them?
(3) Deduce the (multidimensional) characteristic function of X, and that the distribution of X is characterized by the parameters m and Σ. Show that conversely any vector with a characteristic function of this form is Gaussian.
(4) Show that a linear transform AX of a Gaussian vector X is Gaussian, and compute its parameters.
(5) Let V_1 and V_2 be two subspaces of $\mathbb{R}^n$. Give a necessary and sufficient condition for the independence of the σ-algebras $\sigma(\langle t, X\rangle, t \in V_1)$ and $\sigma(\langle t, X\rangle, t \in V_2)$.
(6) Build two standard Gaussian variables X and Y that are uncorrelated yet not independent (they obviously do not form a Gaussian vector !)
(7) Show that the vector $(X_1, \dots X_n)$ with $X_1, \dots, X_n$ independent standard Gaussian variables, is Gaussian. Use it to build a Gaussian vector with arbitrary parameters. Deduce its p.d.f. when it has one.

Exercise 2 — *Central Limit Theorem and random walks.*
Consider a random walk $S_n = \sum_{i=1}^n X_i$ for $n \geq 0$, where X_i are i.i.d. centered increments with variance $\sigma^2 < \infty$. For $n \geq 0$ and $t \in \mathbb{R}_+$, set $\widetilde{S}_n(t) = \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor}$, the rescaled version of S. Now we set $0 \leq t_0 \leq t_1 \leq \dots \leq t_k$ and wish to show convergence in distribution of the random vector $(\widetilde{S}_n(t_0), \dots, \widetilde{S}_n(t_k))$.

(1) Show that for every n, the increments $(\widetilde{S}_n(t_i) - \widetilde{S}_n(t_{i-1}))_{1 \leq i \leq k}$ are independent.
(2) What is the limit of distribution of each increment? What is the joint limit in distribution of the vectors of the increments?

[1] *positive* in French

(3) Deduce that the vector $(\widetilde{S}_n(t_0), \dots, \widetilde{S}_n(t_k))$ converges in distribution towards a given centered Gaussian random vector, that we will denote $(B_{t_0}, \dots, B_{t_k})$. What is its covariance matrix? Its p.d.f.?

(4) Show that $(B_{1/2}, B_1)$ is distributed like $(\frac{X}{\sqrt{2}}, \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}})$, where X and Y are two independent standard Gaussian random variables. Rewrite this distribution as the distribution of (something, X).

Exercise 3 — *Limit in distribution of Gaussian vectors.*
Let $(X_n)_{n\geq 0}$ be a sequence of Gaussian variables $(X_n)_{n\geq 0}$. Give a necessary and sufficient condition for convergence in distribution, show that the limit is always Gaussian, and determine its parameters.

Exercise 4 — *Conditional Fubini's theorem.*
Let $\mathcal{G}$ be a σ-algebra, $X \in \mathcal{G}$ and $Y \perp\!\!\!\perp \mathcal{G}$ be two random variables, and $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(X,Y) \in L^1$. Compute $\mathbb{E}[f(X,Y) \mid \mathcal{G}]$.

Exercise 5 — *"Conditional probability".*
Let X and Y be independent standard Gaussians.

(1) Let $(B_{1/2}, B_1)$ be defined as in exercise 2, question 3. Let f be a function such that $f(B_{1/2}, B_1) \in L^1$. Compute $\mathbb{E}[f(B_{1/2}, B_1) \mid B_1]$ as a deterministic function applied to B_1.

(2) Let f such that $f(X) \in L^1$. Compute $\mathbb{E}[f(X) \mid \cos(X)]$ as a deterministic function applied to $\cos(X)$.