Master 1 — Spring 2018
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$\mathbb{E}$xercise sheet 10: Brownian motion, harmonic functions and measures (v3)

Exercise 1 — *Conformal invariance in dimension 2.*
We recall that a map $U \subset \mathbb{R}^n \to \mathbb{R}^n$ is conformal if it is differentiable and its differential is the multiple of an isometry at every point. For $n = 2$, a map is conformal if and only if it is holomorphic.

(1) Let U, V open in $\mathbb{C}$ and $\phi : U \to V$ a conformal homeomorphism. Show that a map $h : U \to \mathbb{R}$ is harmonic if and only if $\widetilde{h} = h \circ \phi$ is.

(2) Let $D, \widetilde{D}$ be two open sets verifying the Poincaré cone condition, with $\phi : \overline{D} \to \overline{\widetilde{D}}$ an homeomorphism which restricts to a conformal homeomorphism between D and $\widetilde{D}$. For $x \in D$, show that $\phi_* \mu_{\partial D}(x, \cdot) = \mu_{\partial \widetilde{D}}(\phi(x), \cdot)$. (Hint: verify this for bounded continuous functions).

(3) Let $\phi : \mathbb{H} \to \mathbb{D}$, $\phi(z) = -\frac{z-i}{z+i}$. By a slight modification of the previous argument, show that $\phi_* \mu_{\partial \mathbb{H}}(x, \cdot) = \mu_{\partial \mathbb{D}}(\phi(x), \cdot)$ (You can use the fact that for an unbounded domain that verifies the Poincaré cone condition, and a continuous and bounded boundary condition, the Brownian expectation still defines a continuous solution of the Dirichlet problem). When $x = i$, compute explicitly $\mu_{\partial \mathbb{H}}(x, \cdot)$.

Exercise 2 — *Singularity removal.*
Let $d \geq 2$ and $x \in U \subset \mathbb{R}^d$ open. Suppose $h : U \setminus \{x\} \to \mathbb{R}$ is harmonic and bounded around x. Show that u can be extended to a harmonic function on the whole of U.
Better yet, show the same outcome with the relaxed condition that $u(x + \cdot)$ is negligible near x compared to the fundamental solution ($\log(\|\cdot\|)$ if $d = 2$ and $\|\cdot\|^{2-d}$ otherwise).

Exercise 3 — *Inversions in all dimensions.*
Show that $u : \mathbb{R}^d \setminus \overline{B}(0,1) \to \mathbb{R}$ is harmonic if and only if $u^* : B(0,1) \setminus \{0\} \to \mathbb{R}, u^*(x) = u(x/|x|^2)|x|^{2-d}$ is.
⚠ This is a rather tedious computation. This transform is classically called *Kelvin transform* if you want to look it up.