Solutions for $\mathbb{E}$xercise sheet 10: Brownian motion, harmonic functions and measures

Solution 1 — *Conformal invariance in dimension 2.*
We recall that a map $U \subset \mathbb{R}^n \to \mathbb{R}^n$ is conformal if it is differentiable and its differential is the multiple of an isometry at every point. For $n = 2$, a map is conformal if and only if it is holomorphic.

(1) We could proceed by computations, but we will use the classic fact that an harmonic function on a simply connected domain is the real part of some holomorphic function. Let $x \in U$ and $B(x, \epsilon)$ be a small ball contained in U small enough so that ϕ maps $B(x, r)$ inside some other small ball $B(y, \delta)$ inside V. On $B(y, \delta)$, we can rewrite $h = \mathrm{Re} f$ with f holomorphic. Hence on $B(x, \epsilon)$, we have $\widetilde{h} = h \circ \phi = \mathrm{Re} f \circ \phi$, and h is harmonic at x.

(2) Let $D, \widetilde{D}$ be two open sets verifying the Poincaré cone condition, with $\phi : \overline{D} \to \overline{\widetilde{D}}$ an homeomorphism which restricts to a conformal homeomorphism between D and $\widetilde{D}$. For $x \in D$, show that $\phi_* \mu_{\partial D}(x, \cdot) = \mu_{\partial \widetilde{D}}(\phi(x), \cdot)$. (Hint: verify this for bounded continuous functions).

As hinted it is sufficient to verify that for every $f : \partial \widetilde{D} \to \mathbb{R}$ bounded continuous, $\int f(y) \phi_* \mu_{\partial D}(x, dy) = \int f(y) \mu_{\partial \widetilde{D}}(\phi(x), dy)$. But

$$\text{(A)} \qquad \int f(y) \phi_* \mu_{\partial D}(x, dy) = \int f(\phi(y)) \mu_{\partial D}(x, dy) = \mathbb{E}_x[f(\phi(B_{T_{\partial D}}))] = u(x)$$

where u is the unique harmonic function on D with boundary value $f \circ \phi$. At the same time,

$$\text{(B)} \qquad \int f(y) \mu_{\partial \widetilde{D}}(\phi(x), dy) = \mathbb{E}_{\phi(x)}[f(B_{T_{\partial \widetilde{D}}})] = \widetilde{u}(\phi(x))$$

where $\widetilde{u}$ is the unique harmonic function on $\widetilde{D}$ with boundary value f. But now by question 1 we know that $\widetilde{u} \circ \phi$ is harmonic on D, continuous on $\overline{D}$ and has boundary values $f \circ \phi$. Thus by the maximum principle $\widetilde{u} \circ \phi = u$, hence (A) equals (B), and we are done.

(3) Using the fact that an unbounded domain that verifies the Poincaré cone condition, and a continuous and bounded boundary condition, the Brownian expectation still defines a continuous solution of the Dirichlet problem, the above proof transfers without modification to the present situation.
When $x = i$, $\phi(x) = 0$, and by rotation invariance of B we know that $\mu_{\partial \mathbb{D}}(0, \cdot) = \nu_{0,1}$, the uniform measure on the circle. Furthermore we can check that for $x \in \mathbb{R} = \partial \mathbb{H}$,

$\phi(x) = e^{-2i\arctan x}$. Hence for f bounded continuous $\overline{\mathbb{D}} \to \mathbb{R}$,

$$\int_{\partial\mathbb{D}} f(y)\mu_{\partial\mathbb{D}}(0, dy) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\pi t})dt$$

$$\int_{\mathbb{R}} f(y)\phi_*\mu_{\partial\mathbb{H}}(i, dy) = \int_{\mathbb{R}} f(\phi(u))\mu_{\partial\mathbb{H}}(i, du) = \int_{\mathbb{R}} f(e^{-2i\arctan u})\mu_{\partial\mathbb{H}}(i, du)$$

By the previous question, these two expressions are equal. Hence

$$\int_{\mathbb{R}} f(e^{-2i\arctan u})\mu_{\partial\mathbb{H}}(i, du) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\pi t})dt = \int f(e^{-2i\arctan u})\frac{1}{\pi(1+u^2)}du$$

where the last equality is obtained through a change of variables. Hence the measures $\mu_{\partial\mathbb{H}}(i, du)$ and $\frac{1}{\pi(1+u^2)}du$ are equal when tested against all functions of the form $u \mapsto f(e^{-2i\arctan u})$. This space of functions containts in particular all continuous functions with compact support on $\mathbb{R}$, which is enough to characterize equality. Hence $\mu_{\partial\mathbb{H}}(i, du) = \frac{1}{\pi(1+u^2)}du$, the Cauchy distribution.

Remark: the Cauchy distribution for the hitting point on a line was already obtained in a previous exercise by direct computations.

Solution 2 — *Singularity removal.*

Assume without loss of generality that U is a ball centered at x. Let $\widetilde{h}(y) = \mathbb{E}_y[h(B_T)]$, where $T = T_{U^\complement}$. This is well defined because almost surely $B_T \in \partial U$, and of course $\widetilde{h}$ is harmonic on the whole of U. To show that $h(y) = \widetilde{h}(y)$ forall $y \neq x$, proceed as follows. Define $T_\epsilon = T_{U^\complement \cup B(x,\epsilon)}$. Then by harmonicity of h, $h(y) = \mathbb{E}_y[h(B_{T_\epsilon})]$. Furthermore, since almost surely x is not hit by B, we have $B_{T_\epsilon} \to B_T$ as $\epsilon \to 0$. Applying the dominated convergence theorem yields $h(y) = \mathbb{E}_y[h(B_{T_\epsilon})] \underset{\epsilon\downarrow 0}{\longrightarrow} \mathbb{E}_y[h(B_T)] = \widetilde{h}(y)$ and we are done.

Whith the relaxed condition that $u(x+\epsilon) = o(f(\epsilon))$ where f is a fundamental solution, we define the same $T, \widetilde{h}, T_\epsilon$. Now

$$h(y) = \mathbb{E}_y[h(B_{T\epsilon})] = \mathbb{E}_y[\mathbb{1}_{T_\epsilon < T}\, h(B_{T_\epsilon})] + \mathbb{E}[\mathbb{1}_{T_\epsilon = T}\, h(B_T)]$$

The first term is bounded by $\frac{C}{f(\epsilon)}o(f(\epsilon)) \to 0$ and the second term goes to $\mathbb{E}_y[h(B_T)] = \widetilde{h}(y)$. Hence we still have $h(y) = \widetilde{h}(y)$.

Solution 3 — *Inversions in all dimensions.*

If I find a more interesting way than just computing the Laplacian I will update this solution!