Master 1 — Spring 2018
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# $\mathbb{E}$xercise sheet 11: Miscellanea (v3)

**Exercise 1** — *Capacity and Hausdorff dimension.*
Let $f$ be a positive function on $\mathbb{R}^d$ called *potential*. The energy of a measure $\mu$ is $I_f(\mu) = \iint f(x-y)\mu(dx)\mu(dy)$. The capacity of some set $A$ is

$$\mathrm{Cap}_f(A) = [\inf\{I_f(\mu) : \mu \text{ probability measure on } A\}]^{-1}$$

At some point you will see that a closed set is polar in dimension $d \geq 2$ if and only if it has zero capacity for the radial potential $f(\epsilon) = |\log(\epsilon)|$ if $d = 2$ and $f(\epsilon) = \epsilon^{2-d}$ if $d \geq 3$. We wish to show a connexion between the notion of capacity and Hausdorff dimension.

(1) Show that if $\mu$ is a measure on $A \subset \mathbb{R}^d$,

$$\inf_{\substack{(U_i)_i \in \mathcal{P}(A)^{\mathbb{N}} \\ \forall i, \mathrm{diam}(U_i) \leq \delta \\ \bigcup_i U_i = A}} \left( \sum_{i \in \mathbb{N}} \mathrm{diam}(U_i)^\alpha \right) \geq \frac{\mu(A)^2}{\iint_{|x-y|<\delta} \mu(dx)\mu(dy)|x-y|^{-\alpha}}$$

and deduce that a set of nonzero capacity for $f(\epsilon) = \epsilon^{-\alpha}$ has Hausdorff dimension $\geq \alpha$.

(2) Show also that the image of a segment by a $\alpha$-Hölder function is of Hausdorff dimension bounded by $\frac{1}{\alpha}$.

(3) What is the Hausdorff dimension of $B([0,1])$ in $\mathbb{R}^d$ ?

**Exercise 2** — *Some more boundary value problems.*
In this exercise we will admit that for $x, y \in \mathbb{R}^d$, $t > 0$, we have $\partial_t p_t(x,t) = \frac{1}{2}\Delta_y p_t(x,y)$. (Fokker-Planck equation)

(1) Show that if $f$ is $\mathcal{C}^2$ with compact support, then under $\mathbb{P}_x$, $(f(B_t) - \frac{1}{2}\int_0^t \Delta f(B_s)ds)_t$ is a martingale. (Dynkin's formula)

(2) Let $D$ be a bounded domain and $f : \overline{D} \to \mathbb{R}$ continuous and $\mathcal{C}^2$ on the interior with bounded second derivatives. Let $T$ be the hitting time of the complement of $D$. Show that $(f(B_{t\wedge T}) - \frac{1}{2}\int_0^{t\wedge T} \Delta f(B_s)ds)_t$ is a martingale (*Hint*: use a regularization procedure to apply question 1).

(3) Show that in the sense of distributions, we have $\Delta G(x, \cdot) = -2\delta_x$, where $G$ is the Green function of the Brownian motion in the whole of $\mathbb{R}^3$ or in a bounded domain of $\mathbb{R}^2$.

(4) Show that in a bounded domain $D \subset \mathbb{R}^d$ with $f$ continuous, a solution of the *Poisson problem*

$$\Delta u = f \text{ on } D$$
$$u = 0 \text{ on } \partial D$$

must verify $u(x) = -\frac{1}{2}\,\mathbb{E}_x[\int_0^T f(B_s)ds]$.

(5) Conversely, if $f$ is Hölder and $D$ is bounded and verifies the Poincaré cone condition, show that this formula (which can be rewritten $u(x) = -\frac{1}{2}\int f(y)G(x,y)dy$) gives a solution of the Poisson problem.

⚠ *It is doable to verify that u is continuous at the boundary and solves the Poisson problem in the weak sense (see solution). To extend this to the strong sense seems harder. It is done in S. Port, Brownian Motion and Classical Potential Theory, from page 114 onwards (available at the library). Maybe there is a simpler way but I haven't found it yet!*

**Exercise 3** — *Transition probabilities and Green's function on the disc.*

⚠ *This is taken from Mörters-Peres, lemma 3.36, lemma 3.37 and exercise 3.12. I actually don't know how to do question 3 (I don't understand their proof of lemma 3.37), so for now I can't help you with this rather boring exercise...*

Let $p^*(t,x,y)$ be the transition probabilities for the Brownian killed when exiting the Disc $B(0,1)$, verifying

$$\mathbb{E}_x[f(B_t)\,\mathbb{1}_{t\le T}] = \int f(y)p^*(t,x,y)dy$$

and $G(x,y) = \int p^*(t,x,y)dt$ the Green function.

(1) Show that $p^*(t,x,y) = p_t(x,y) - \mathbb{E}_x[p_{t-T}(B(T),y)\,\mathbb{1}_{T<t}]$
(2) Show that $\int_0^\infty p_s(x,y) - p_s(0,1)ds = -\frac{1}{\pi}\log|x-y|$.
(3) Deduce that $G(x,y) = -\frac{1}{\pi}\log|x-y| - \mathbb{E}_x[-\frac{1}{\pi}\log|B(T)-y|]$.
(4) Compute this with Poisson's formula.

## Appendix A. Hausdorff dimension

Let $(E,d)$ be a metric space. For $\alpha \ge 0$ and $A \subset E$, we define the *$\alpha$-dimensional Hausdorff measure* of $A$ follows:

$$\mathcal{H}_\alpha(A) := \lim_{\delta\to 0}\left(\inf_{\substack{(U_i)_i\in\mathcal{P}(E)^{\mathbb{N}} \\ \forall i, \mathrm{diam}(U_i)\le\delta \\ \bigcup_i U_i\supset A}}\left(\sum_{i\in\mathbb{N}}\mathrm{diam}(U_i)^\alpha\right)\right).$$

It is well defined because the lim is actually a sup, and verifies the following property:
**Lemma** Let $\alpha \in [0,\infty)$.If $\mathcal{H}_\alpha(A) < \infty$ then for $\beta > \alpha$ $\mathcal{H}_\beta(A) = 0$. If $\mathcal{H}_\alpha(A) > 0$ then for $\beta < \alpha$ $\mathcal{H}_\beta(A) = \infty$.

This tells us that there is a transition point $\alpha \in [0, \infty]$ where the Hausdorff measure jumps from $\infty$ to 0, and we want to call that point the Hausdorff dimension of $A$.

$$\dim_{\mathcal{H}}(A) := \sup\{\alpha, \mathcal{H}_\alpha(A) = \infty\} = \inf\{\alpha, \mathcal{H}_\alpha(A) = 0\}.$$

This $\alpha$ is the only dimension for which $A$ admits a possibly non-trivial Hausdorff measure (but it may still be 0 or $\infty$ in some cases).
For instance, in $\mathbb{R}^d$, the $d$-dimensional Hausdorff measure is equal to the Lebesgue measure (you probably constructed the Lebesgue measure this way), and open sets have necessarily Hausdorff dimension $d$. Of course sets with 0 Lebesgue measure might have a strictly smaller Hausdorff dimension.