Solutions for $\mathbb{E}$xercise sheet 4: stopping times and Markov property (v2)

Solution 5 — *Hitting time.*
Using cdf's, we get $\mathbb{P}(T_a \leq t) = \mathbb{P}(B_t^* \geq a) = \mathbb{P}(|B_t| \geq a) = \mathbb{P}(\sqrt{t}|B_1| \geq a) = \mathbb{P}(\frac{a^2}{B_1^2} \leq t)$. So $T_a \overset{d}{=} \frac{a^2}{B_1^2}$, and Lebesgue's change of variable theorem gives

$$\mathbb{P}(T_a \in dt) = \frac{a}{\sqrt{2\pi}t^{3/2}} e^{-a^2/2t}$$

Solution 6 — *A bit more on differentiability.*
We know that almost surely, B is nowhere differentiable. Set $D^*B(t) = \limsup_{h\downarrow 0} \frac{1}{t}(B_{t+h} - B_t)$ and $D_*B(t) = \liminf_{h\downarrow 0} \frac{1}{t}(B_{t+h} - B_t)$.

(1) We showed earlier that almost surely, $\limsup B_t = +\infty$ and $\liminf B_t = -\infty$ almost surely (actually we showed that the rate stricly more than $\sqrt{t}$) Hence the claim by time inversion.
(2) $\mathbb{E}[\text{Leb}\{t \geq 0, D^*B(t) \neq +\infty \text{ or } D_*B(t) \neq -\infty\}] = \int_{\mathbb{R}} dt\, \mathbb{P}(D^*B(t) \neq +\infty \text{ or } D_*B(t) \neq -\infty) = \int_R 0 = 0$, where we used Fubini and Markov.
(3) We know that 0 is almost surely not a local extremum at its right because there is an accumulation of instants where B is strictly positive and negative near 0. For a fixed point t, we treat the right side by Markov and the left side by time reversal. Now for fixed $p, q \in \mathbb{Q}_+$ almost surely p, q are not one-sided local extrema. Hence the maximum of B on $[p, q]$ is reached somewhere in the interior, and that is a point inside (p, q) where $D^*B \leq 0$. We get the claim by countable union.
(4) We consider $\tau(x) = \inf\{t \geq 0, B_t = x\}$. This is by definition strictly increasing function, and if it were continuous on some open interval, then B would be monotonous on some open interval, which it is almost surely not. Now if we consider $V_n = \{x \geq 0, \exists h \in (0, 1/n), \tau(x-h) < \tau(x) - nh\}$, it is open because τ is càglàd strictly increasing. It is dense because otherwise we found an open interval of x where $\forall h \in (0, 1/n)$, $\tau(x) - nh \leq \tau(x-h) \leq \tau(x)$, implying continuity on some open interval. Then by the Baire category theorem, $\bigcap_{n\geq 1} V_n$ is uncountable and dense. Let x be in this set, and $t = \tau(x)$. Then there exists a sequence $t_n \uparrow t$, $B^*(t_n) > t - 1/n$, $t_n < t - nB^*(t_n)$. Hence the lower left derivative of B at t is 0. The upper left derivative is 0 too by definition. We get the claim by time reversal.

Solution 7 — *The set of zeros of B is perfect.*
Almost surely 0 is an accumulation point of Z (lecture). By countable union, and strong Markov, every first 0 after any rational is an accumulation point of Z (at its right). If Z

had an isolated point, it would be a first 0 after a rational. Hence it couldn't be isolated in Z.