$\mathbb{E}$xercise sheet 5: Markov processes (v2)

Exercise 1 — *Counter-example.*
Let $X_t = AB_t$ where B is a Brownian motion started from 1 and A an independent balanced Bernoulli.

(1) Show that X is a Markov process and give its transition kernel.
(2) Show that it does not verify the strong Markov property.

Exercise 2 — *The stationary Ornstein-Uhlenbeck process.*
For $t \in \mathbb{R}$, set $X_t = e^{-t} B_{e^{2t}}$, where B is a Brownian motion. Show that X is a continuous Gaussian process, compute its covariance function. For any given t, what is the distribution of X_t ? Show that it is a Markov process and compute its transition kernel.

Exercise 3 — *Brownian bridges.*
For $x, y \in \mathbb{R}$, we define the Brownian bridge of length one between x and y as follows: let B be a standard Brownian motion and set $\beta_t^{x,y} = x + B_t - tB_1 + t(y - x)$ for $t \in [0, 1]$.

(1) Show that if X is a Brownian motion started from x, then the conditional distribution of the process $X_{|[0,1]}$ given $X_1 \in dy$ is $\beta^{x,y}$.
(2) For $0 < a < 1$, what expression does the Markov property applied at a give for the joint distribution of $(X_{|[0,a]}, X_1)$? Deduce an expression for the conditional distribution of $X_{|[0,a]}$ given $X_1 \in dy$. Deduce a second expression from the previous question.
(3) Show that the distribution of $\beta_{|[0,a]}^{x,y}$ is absolutely continuous with regard to that of $X_{|[0,a]}$ where $a < 1$.
Hint: use the fact that the conditional distributions are a.e. uniquely defined, along with some continuity argument.

Exercise 4 — *Cauchy process.*
Let $(B^{(1)}, B^{(2)})$ be a two-dimensional Brownian motion, and $T_a = \inf\{t \geq 0, B_t^{(1)} \geq a\}$ for $a \geq 0$. Set $C_a = B_{T_a}^{(2)}$.

(1) Compute the distribution of C_a.
(2) Show that $(C_{a+\cdot} - C_a)$ is independent of $\mathcal{F}_{T_a}$ and distributed like C. Deduce that C is a Markov process and give its transition kernel.
(3) Is C continuous ?