Solutions for $\mathbb{E}$xercise sheet 5: Markov processes (v2)

Solution 1 — *Counter-example.*

(1) Consider the filtration $\mathcal{F}_t = \sigma(X_s, 0 \le s \le t)$. Then we can write $X_{t+s} = X_t + (B_{t+s} - B_t)\,\mathbb{1}[A \neq 0]$. But almost surely, $\mathbb{1}[A \neq 0] = \mathbb{1}[X_t \neq 0]$, which means that we can rewrite $X_{t+s} = X_t + (B_{t+s} - B_t)\,\mathbb{1}[X_t \neq 0]$. Since $X_t \in \mathcal{F}_t$ and $B_{t+s} - B_t \perp\!\!\!\perp \mathcal{F}_t$, we get the Markov property with

$$p_t(x, dy) = \delta_0(dy) \text{ if } x = 0 \text{ and } \mathbb{P}(x + B_t \in dy) \text{ otherwise.}$$

(2) If it did, then it would mean that the process sticks to 0 after its first hitting time of 0, which is indeed not the case.

Solution 2 — *The stationary Ornstein-Uhlenbeck process.*
Firstly, $\mathrm{Cov}(X_t, X_s) = e^{-|t-s|}$. So at each time t, X_t is a standard Gaussian. For the Markov property, start from the standard filtration $\mathcal{F}$ of B. The standard filtration of X is then $t \mapsto \mathcal{F}_{e^{2t}}$. Then we easily compute

$$\mathbb{P}(X_{t+s} \in dy \mid \mathcal{F}_{e^2t}) = \mathbb{P}(e^{-s}x + B_{1-e^{-2s}} \in dy) = \frac{1}{\sqrt{2\pi(1-e^{-2s})}} \exp\left(\frac{(y - e^{-s}x)^2}{2(1-e^{-2s})}\right) dy.$$

Solution 3 — *Brownian bridges.*
Here we denote $p(t, x, y)$ the Brownian transition kernel density $p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/(2t)}$ for $t > 0$.

(1) Set $X_t = x + B_t$ and $\beta x, y_t = x + B_t - tB_1 + t(y - x)$. Remark that $\beta^{x,y}_t = \beta^{x,0}_t + yt$. But $X_t = \beta^{x,0}_t + tX_1$. Since $\beta^{x,0}$ is independent from X_1, it comes that $\mathbb{E}[H(X_{|[0,1]})|X_1] = \mathbb{E}[H(\beta^{x,0}_t + ty)]_{y=X_1} = \mathbb{E}[H(\beta^{x,y})]_{y=X_1}$. Hence the claim that $\mathbb{P}(X_{|[0,1]} \in \cdot | X_1 \in dy) = \mathbb{P}(\beta^{x,y} \in \cdot)$.
(2) The Markov property at a tells us that

$$\mathbb{E}[(H(X_{|[0,a]}, X_1)] = \int_{\mathcal{C}([0,a])} \mathbb{P}(X_{|[0,a]} \in d\phi) \int_{\mathbb{R}} dy\, p(1-a, \phi(a), y) H(\phi, y).$$

But then, multiplying and dividing by $p_1(x,y) = \frac{\mathbb{P}(X_1\in dy)}{dy}$, and using Fubini

$$\begin{aligned}\mathbb{E}[(H(X_{|[0,a]}, X_1)] &= \int_{\mathbb{R}} dy\, p(1,x,y) \int_{\mathcal{C}([0,a])} \mathbb{P}(X_{|[0,a]} \in d\phi) \frac{p(1-a,\phi(a),y)}{p(1,x,y)} H(\phi,y). \\ &= \int_{\mathbb{R}} \mathbb{P}(X_1 \in dy) \int_{\mathcal{C}([0,a])} \mathbb{P}(X_{|[0,a]} \in d\phi) \frac{p(1-a,\phi(a),y)}{p(1,x,y)} H(\phi,y). \\ &= \int_{\mathbb{R}} \mathbb{P}(X_1 \in dy) \int_{\mathcal{C}([0,a])} U(y,d\phi) H(\phi,y)\end{aligned}$$

But we recognize a desintegration of measure formula as in the definition of conditional probability[1], with the conditional probability kernel $U(y,d\phi) = \mathbb{P}(X_{|[0,a]} \in d\phi)\frac{p(1-a,\phi(a),y)}{p(1,x,y)}$. Hence we have

$$\mathbb{P}(X_{|[0,a]} \in d\phi | X_1 \in dy) = \mathbb{P}(X_{[0,a]} \in d\phi) \frac{p(1,\phi(a),y)}{p(1,x,y)}.$$

On the other hand, the previous question gives us (restricting a process is a measurable transformation), that

$$\mathbb{P}(X_{|[0,a]} \in d\phi | X_1 \in dy) = \mathbb{P}(\beta^{x,y}_{|[0,a]} \in d\phi).$$

(3) The rhs of the last two equations are measure-valued functions of y that are $\mathbb{P}(X_1 \in dy)$-a.e. equal. Since the distribution of X_1 and Leb are mutually absolutely continuous, then they are equal dy-a.e. Now we claim that they are both continuous (measure valued, with the narrow convergence topology) functions of y. For the first one, it follows from Scheffé's lemma. For the second one, it is because as $y_n \to y$, $\beta^{x,y_n} \to \beta^{x,y}$ in $\mathcal{C}([0,1])$ for every ω and of course ω-wise convergence implies convergence in distribution. Hence for every y,

$$\mathbb{P}(\beta^{x,y}_{|[0,a]} \in d\phi) = \mathbb{P}(X_{|[0,a]} \in d\phi) \frac{p(1,\phi(a),y)}{p(1,x,y)}.$$

In particular, if $\beta = \beta^{0,0}$ denotes a standard Brownian bridge,

$$\frac{\mathbb{P}(\beta_{|[0,a]} \in d\phi)}{\mathbb{P}(B_{|[0,a]} \in d\phi)} = \frac{\exp\left(\frac{\phi(a)^2}{2(1-a)}\right)}{\sqrt{1-a}}.$$

[1]To go back to the main definition of conditional probability distributions and precisely show that $\mathbb{E}[F(X_{|[0,1]})|X_1] = \int U(X_1,d\phi)F(\phi)d\phi$, you can test with H in the product form $H(\phi,y) = F(\phi)G(y)$.

Solution 4 — *Cauchy process.*

(1) We have $C_a = B^{(2)}_{T_a} = \Psi(B^{(2)}, T_a)$ where $\Psi(\phi, t) = \phi_t$ is a measurable (actually continuous) functional $\mathcal{C}(\mathbb{R}_+) \times \mathbb{R}_+ \to \mathbb{R}$. Then for some positive measurable H,

$$\begin{aligned} E[H(C_a)] = \mathbb{E}[H(\Psi(B^{(2)}, T_a))] &= \int_{R_+} \mathbb{P}(T_a \in dt) \int_{\mathcal{C}(\mathbb{R}_+)} \mathbb{P}(B^{(2)} \in d\phi) H(\Psi(\phi, t)) \\ &= \int_{R_+} \mathbb{P}(T_a \in dt)\, \mathbb{E}[H(B^{(2)}_t)] = \int_{R_+} \mathbb{P}(T_a \in dt) \int_{\mathbb{R}} \mathbb{P}(B_t \in du) H(u) \\ &= \int_{\mathbb{R}} H(u) \left(\int_{R_+} \frac{\mathbb{P}(T_a \in dt)}{dt} \frac{\mathbb{P}(B_t \in du)}{du} dt \right) du. \end{aligned}$$

(Fubini has been used several times). Hence the thing inside the parentesis is the density of C_a at u. Let's compute it

$$\begin{aligned} \frac{\mathbb{P}(C_a \in du)}{du} &= \int_{R_+} \frac{\mathbb{P}(T_a \in dt)}{dt} \frac{\mathbb{P}(B_t \in du)}{du} dt \\ &= \int_{R_+} \frac{a}{\sqrt{2\pi} t^{3/2}} e^{-a^2/2t} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dt = \frac{a}{\pi(x^2 + a^2)} \end{aligned}$$

(2) We have that $(C_{a+\cdot} - C_a)$ is constructed from $B_{T_a+\cdot} - B_{T_a}$ the same way C is constructed from B. Hence the first claim follows from the strong Markov property of B. Now $C_{t+s} = C_t + C_{t+s} - C_t$, and $C_{t+s} - C_t$ is independent of $\sigma(C_u, u \le t) \subset F_{T_a}$, hence $\mathbb{P}(C_{t+s} \in dy | C_t \in dx) = \mathbb{P}(x + C_s \in dy) = \frac{s}{\pi((y-x)^2 + s^2)} dy$ (The usual stuff with Lévy processes)

(3) C is càglàd because T is. Let $G = T_{B^{(1)*}(1)}$ and $D = T_{B^{(1)*}(1)+}$. Then almost surely $G < 1 < D$. If C were continous, we'd get $B^{(2)}(G) = B^{(2)}(D)$ almost surely, negating the independence of $B^{(2)}$ and $(G, D) \in \sigma(B^{(1)})$.