𝔼xercise sheet 6: Martingales

Exercise 1 — *All hypotheses matter.*
Find two stopping times S, T with $S \leq T < \infty$ a.s., with $\mathbb{E}[S] < \infty$, such that $\mathbb{E}[B_S^2] > \mathbb{E}[B_T^2]$

Exercise 2 — *Brownian gambler's ruin.*
Let $a, b > 0$ and T be the hitting time of $\{-a, b\}$.

(1) Compute the probability of $\{T = T_{-a}\}$
(2) Compute $\mathbb{E}\, T$.

Exercise 3 — *Exponential martingale and computations.*
We recall that for every $\lambda \in \mathbb{R}$, the process $e^{\lambda B_t - t\lambda^2/2}$ is a martingale, called the exponential martingale.

(1) Use the exponential martingale to compute the Laplace transform of the hitting time T_a of a given level $a > 0$
(2) Let B be a complex Brownian motion. Show that $e^{\lambda i B_t}$ is a complex martingale and deduce the Characteristic function of the Cauchy process (see previous exercise sheet)
(3) Show that if X and $t \mapsto e^{\lambda X_t - t\lambda^2/2}$ for every $\lambda \in \mathbb{R}$ are martingales, then X is a Brownian motion.

Exercise 4 — *Hitting time of a line.*
Let $a, b \geq 0$ and $T = \inf\{t \geq 0, B_t = at + b\}$. Compute $\mathbb{P}(T < \infty)$.

Exercise 5 — *Martingales derived from B.*
Show that $B_t^2 - t$ and $B_t^3 - 3tB_t$ are martingales. Guess the other ones.