

Solutions for Exercise sheet 6: Martingales

Solution 1 — *All hypotheses matter.*

Take $S = 3$ and T to be the first zero after 3. Of course the problem is that $\mathbb{E}[T] = \infty$.

Solution 2 — *Brownian gambler's ruin.*

Let $a, b > 0$ and T be the hitting time of $\{-a, b\}$.

- (1) Let $p = \mathbb{P}\{T = T_{-a}\}$. We can apply the optional stopping theorem to B at T since $|B_{t \wedge T}| < a \vee b < \infty$. Hence $0 = \mathbb{E}[B_T] = -ap + (1-p)b$. Solving for p yields $p = b/(a+b)$.
- (2) We want to apply Wald's second lemma, and need to show first that T is integrable. By scaling it is sufficient to show it for $a = b = 1$. We have that

$$\begin{aligned} \mathbb{P}(T \geq n) &= \prod_{k=0}^{n-1} \mathbb{P}(T \geq k+1 \mid T \geq k) \\ &= \prod_{k=0}^{n-1} \mathbb{E} \left(\frac{\mathbb{1}[T \geq k+1]}{\mathbb{P}(T \geq k)} \mid \mathcal{F}_k \right) \\ &= \prod_{k=0}^{n-1} \mathbb{E} \left(\frac{\mathbb{1}[T \geq k+1]}{\mathbb{P}(T \geq k)} \mathbb{E} \left(\mathbb{1} \left[\max_{[0,1]} B^{(k)} < 1 - B_k \text{ and } \min_{[0,1]} B^{(k)} > -1 + B_k \right] \mid \mathcal{F}_k \right) \right) \\ &= \prod_{k=0}^{n-1} \mathbb{E} \left(\frac{\mathbb{1}[T \geq k+1]}{\mathbb{P}(T \geq k)} \mathbb{P} \left(\max_{[0,1]} B < 1 - x \text{ and } \min_{[0,1]} B > -1 + x \right)_{x=B_k} \right) \end{aligned}$$

The probability inside depends on x and we need to bound it uniformly for $x \in [-1, 1]$. But it is clearly bounded by $\alpha \mathbb{P}(\max_{[0,1]} |B| < 2)$, which is < 1 . Hence $\mathbb{P}(T \geq n) \leq \alpha^n$ and T is integrable.

Now by Wald, we get $\mathbb{E}[T] = \mathbb{E}[B_T^2] = ab$.

Solution 3 — *Exponential martingale and computations.*

We recall that for every $\lambda \in \mathbb{R}$, the process $e^{\lambda B_t - t\lambda^2/2}$ is a martingale, called the exponential martingale.

- (1) Let $a, \lambda > 0$ and $X_t = e^{\lambda B_t - t\lambda^2/2}$. Then $|X_{t \wedge T}| \leq e^{\lambda a}$. So the optional stopping theorem applies and $1 = \mathbb{E}[X_T] = e^{\lambda a - T\lambda^2/2}$. Setting $\lambda^2/2 = \mu$, we get $\mathbb{E}[e^{-\mu T}] = e^{-\sqrt{2\mu}a}$.
- (2) Let B be a complex Brownian motion and $a \geq 0$. We set $T_a = \inf\{t, \Im B_t \leq -a\}$ and $C_a = \Re B_{T_a}$. By a simple computation, we have that for $\lambda > 0$, $X_t = e^{\lambda i B_t}$ is a complex martingale. Moreover $|X_{t \wedge T_a}| = e^{\Re(-i\lambda B_{t \wedge T_a})} = e^{-\lambda \Im B_{t \wedge T_a}} \leq e^{\lambda a}$. Hence the

optional stopping theorem applies and $1 = \mathbb{E}[X_{T_a}] = \mathbb{E}[e^{i\lambda(-ia+C_a)}] = e^{\lambda a} \mathbb{E}[e^{i\lambda C_a}]$. Hence $\mathbb{E}[e^{i\lambda C_a}] = e^{-\lambda a}$ for $\lambda \geq 0$. For negative λ we use the fact that $C_a \stackrel{d}{=} -C_a$, and finally get $\mathbb{E}[e^{i\lambda C_a}] = e^{-|\lambda|a}$.

- (3) These hypotheses imply that $\mathbb{E}[e^{\lambda(X_{t+s}-X_t)} \mid \mathcal{F}_t] = e^{-s\lambda^2/2}$. So almost surely for every $\lambda \in \mathbb{Q}$,

$$\int e^{\lambda x} \mathbb{P}((X_{t+s} - X_t) \in dx \mid \mathcal{F}_t) = \int e^{\lambda x} \mathbb{P}(B_s \in dx)$$

Hence by injectivity of the moment-generating function, almost surely $\mathbb{P}((X_{t+s} - X_t) \in dx \mid \mathcal{F}_t) = \mathbb{P}(B_s \in dx)$, which is exactly to say that $(X_{t+s} - X_t)$ is a Gaussian of variance s , independent of \mathcal{F}_t . This characterizes the Brownian motion.

Solution 4 — *Hitting time of a line.*

Let $X_t = e^{2aB_t - 2a^2t}$ be the exponential martingale with $\lambda = 2a$. For $n \geq 0$, $X_{t \wedge T}$ is trivially bounded by $e^{2ab} < \infty$, and the optional stopping theorem applied to $n \wedge T$ gives

$$1 = \mathbb{E}[X_{T \wedge n}] = \mathbb{E}[X_T \mathbf{1}_{T < n}] + \mathbb{E}[X_n \mathbf{1}_{n \leq T}]$$

As $n \rightarrow \infty$, the first term goes to $\mathbb{E}[X_T \mathbf{1}_{T < \infty}]$ by monotone convergence. The integrand in the second term is bounded by $e^{2ab} < \infty$, and goes to 0 almost surely, since $B_t = o(t)$ almost surely. So by dominated convergence the expectation goes to 0. We get $1 = \mathbb{E}[X_T \mathbf{1}_{T < \infty}] = \mathbb{E}[e^{2ab} \mathbf{1}_{t < \infty}]$ so $\mathbb{P}(T < \infty) = e^{-2ab}$.

Solution 5 — *Martingales derived from B .*

Those martingales are the derivative w.r.t λ of the exponential martingale.