Master 1 — Spring 2018
E. Jacob & M. Maazoun

Solutions for $\mathbb{E}$xercise sheet 9: Harmonic functions and Brownian motion (v2)

Solution 1 — *Harmonic functions and martingales.*
Let D be a bounded domain satisfying the Poincaré cone condition and T its exit time. Let $h : \overline{D} \to \mathbb{R}$ be continuous and harmonic inside D. Show that for $x \in D$, under $\mathbb{P}_x$, the process $t \mapsto h(B_{t\wedge T})$ is a closed martingale.
Conversely show that if h is defined on some domain U with the property that for every $\overline{B}(x,\epsilon) \subset U$, $t \mapsto h(B_{t\wedge T_{\partial B(x,\epsilon)}})$ is a martingale under $\mathbb{P}_x$, then h is harmonic.
Let us show that for every x, under $\mathbb{P}_x$, $t \mapsto h(B_{t\wedge T})$ is a martingale that is closed by $h(B_T)$. Thus we shall show that for every x, $\mathbb{E}_x[h(B_T) \mid \mathcal{F}_t] = h(B_{t\wedge T})$. Indeed we can compute

$$\begin{aligned}
\mathbb{E}_x[h(B_T) \mid \mathcal{F}_t] &= \mathbb{E}_x[h(B_{t\wedge T})\mathbb{1}_{T<t} + h(B_T)\mathbb{1}_{t\le T} \mid \mathcal{F}_t] \\
&= h(B_{t\wedge T})\mathbb{1}_{T<t} + \mathbb{E}_x[h(B_T)\mathbb{1}_{t\le T} \mid \mathcal{F}_t] \\
&= h(B_{t\wedge T})\mathbb{1}_{T<t} + \mathbb{E}_x[h(B^t_{T^t})\mathbb{1}_{t\le T} \mid \mathcal{F}_t] \\
&= h(B_{t\wedge T})\mathbb{1}_{T<t} + \mathbb{E}_x[h(B^t_{T^t})\mathbb{1}_{t\le T} \mid \mathcal{F}_t] \\
&= h(B_{t\wedge T})\mathbb{1}_{T<t} + \mathbb{E}_{B_t}[h(B_T)]\mathbb{1}_{t\le T} \\
&= h(B_{T\wedge t})\mathbb{1}_{T<t} + h(B_t)\mathbb{1}_{T\ge t} = h(B_{t\wedge T})
\end{aligned}$$

Where B^t denoted the Brownian motion restricted from time t onward, and T^t the hitting time of ∂D for this process.

Solution 2 — *A lemma for the Poincaré cone condition.*
Let C be an open cone based in 0. We wish to show that the function $\phi(x) = \mathbb{P}_x(T_{\partial B(0,1)} < T_{\partial C})$ is bounded away from 1 on $\overline{B}(0,1/2) \setminus C$.

(1) We have that ϕ is harmonic on the interior of $B(0,1) \setminus C$. That is because it can be rewritten as $\mathbb{E}[u(B_{T_{\partial(B(0,1)\setminus C)}})]$ for $u = \mathbb{1}_{B(0,1)^\complement \setminus C}$.
We can't use the maximum principle for ϕ on $\overline{B}(0,1/2) \setminus C$ because we don't know if ϕ is continuous on the boundary of this set.

(2) Let $\widetilde{C}$ be another open cone such that $\overline{\widetilde{C}} \subset C$ (for instance take $\widetilde{C}$ to be C translated away from 0). Define $\psi(x) = \mathbb{P}_x(T_{\partial B(0,1)} < T_{\partial\widetilde{C}})$. It is clear from an inclusion of events, that $\psi(x) \ge \phi(x)$. It is also clear that ψ is harmonic on $U = B(0,1) \setminus \overline{\widetilde{C}}$, for the same reasons as ϕ. Now $P = \overline{(B(0,1/2) \setminus C)}$ is completely included in U, so ψ is continuous on the compact P. If $\sup_P \psi = 1$, then by compactness we found a point of $P \subset U$ where ψ reaches one. As $\psi \le 1$, the maximum principle tells us

that $\psi \equiv 1$, which seems absurd.
To see why this is absurd, take a finite union F (for *flower*) of rotations of $\widetilde{C}$ that disconnects 0 from infinity. If $\phi(0) = 1$, then almost surely B does not touch $\widetilde{C}$ before exiting $B(0,1)$ and by rotation invariance and countable union, it also almost surely does not touch F before exiting $B(0,1)$. Hence it does not touch F, hence it stays bounded almost surely. This is clearly absurd.

Solution 3 — *Counterexample.*
Set $T = T_{\partial D}$ and $h(x) = \mathbb{E}_x[u(B_T)]$. This does not define a solution to the Laplace equation, because since the Brownian motion started outside of 0 almost surely does not hit 0, we have $h(0) = 0$ and $h(x) = 1$ for all $x \in \overline{D} \setminus \{0\}$. Hence h is not continuous.
Suppose a solution h exists. Then a rotation of h is still a solution, and hence equals h thanks to the maximum principle. Thus h is rotation invariant hence radial ($h(x) = g(|x|), x \in \overline{D}$, for some $g : \mathbb{R}_+ \to \mathbb{R}$ that must be twice differentiable.) We deduce that $0 = g''(x) + \frac{1}{x}g'(x)$ for all $0 < x < 1$, an ODE whose solutions are of the form $x \mapsto A + B\log(x)$, none of which fits our purpose. Hence a solution cannot exist.

Solution 4 — *The binary splitting martingale.* (1) We write

$$\begin{aligned} X_{n+1} - X_n &= \mathbb{E}[X - X_n \mid \mathcal{G}_n] \\ &= \mathbb{E}[(X - X_n)\,\mathbb{1}_{X>X_n} \mid \mathcal{G}_n]\,\mathbb{1}_{X>X_n} + \mathbb{E}[(X - X_n)\,\mathbb{1}_{X<X_n} \mid \mathcal{G}_n]\,\mathbb{1}_{X<X_n}. \end{aligned}$$

where we used the fact that the sign of $(X - X_n)$ is $\mathcal{G}_n$-measurable. The first term is almost surely positive, the second one is almost surely negative, and almost surely only one of them is nonzero. Hence they almost surely they form a decomposition of $X_{n+1} - X_n$ into a positive and negative part. Then

$$\begin{aligned} |X_{n+1} - X_n| &= \mathbb{E}[(X - X_n)\,\mathbb{1}_{X>X_n} \mid \mathcal{G}_n]\,\mathbb{1}_{X>X_n} - \mathbb{E}[(X - X_n)\,\mathbb{1}_{X<X_n} \mid \mathcal{G}_n]\,\mathbb{1}_{X<X_n} \\ &= \mathbb{E}[|X - X_n| \mid \mathcal{G}_n]. \end{aligned}$$

(2) We deduce $\mathbb{E}[|X_n - X|] = \mathbb{E}[|X_{n+1} - X_n|]$, and this last expression goes to 0 as $(X_n)_n$ is L^1-convergent. Thus $|X_n - X|$ goes to 0 in L^1 and by uniqueness (up to a.s. equality) of the L^1 limit we get that $X_\infty = X$ a.s. Hence X_n converges a.s. and L^1 to X.