The Lévy-Itô Decomposition (Applebaum, §2.4)

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Basics

Lévy Process

(L1) \( X(0) = 0 \) a.s.
(L2) \( X \) has independent stationary increments.
(L3) \( X \) is strong continuous, i.e. \( X_t \xrightarrow{P} 0 \).

They are Feller, have càdlàg modifications,
their characteristic function is

\[ \mathbb{E}[e^{iX_t}] = e^{i\eta(t)} \]

Hence the distribution of \( X \) characterizes the Lévy Process.
and \( X \) is ID.

(Conversely, for any ID distribution, there is a Lévy process corresponding.)

\( \eta \) is called the Lévy symbol of \( X \).

Then (Lévy-Khintchine) if \( X \) is ID,

There exists \( \beta \in \mathbb{R}, \sigma^2 \geq 0 \) such that

\[ \mathbb{E}[e^{i\mu X}] = e^{\eta(\mu)} \]

with \( \eta(\mu) = i\beta\mu + \frac{\sigma^2 \mu^2}{2} + \int (e^{i\mu x} - 1 - i\mu x) \nu(dx) \).

\( \beta \) : we have not proved LK yet! The proof comes today.
A refresher of Mathew's talk: Poisson Random Measures

Definition: Let \( \mu \) be a \( \sigma \)-finite measure on some space \( X \) with \( \sigma \)-algebra \( \mathcal{X} \). A Poisson random measure on \( X \) with intensity \( \mu \) is a random measure on \( X \) such that:

1. \( M \) is a counting measure almost surely.
2. If \( \mu(A) < \infty \) then \( M(A) \sim \text{Poisson}(\mu(A)) \).
3. \( A \cap B = \emptyset \Rightarrow M(A) \perp M(B) \).

Illustration:

\[ \begin{align*}
\mu(A) &= 5.6 \\
M(A) &\sim \text{Poisson}(5.6) \\
M(A) &\perp M(B) \\
M(B) &\sim \text{Poisson}(\mu(B)) \\
\text{hence } M(B) &\neq 0
\end{align*} \]

Notation: If \( \alpha \) is a measure on \( X \), and \( f: X \rightarrow \mathbb{R}^d \) positive and integrable, let \( \alpha(f) \) denote \( \int f(x) \, d\alpha(x) \) (so that \( \alpha(A) = \alpha(1_A) \)).

Applebaum does not use it but it's a useful shorthand.

Theorem (Campbell's Formula for sums over points of a Poisson Random Measure):

Assume we have a Poisson random measure \( M \) with intensity \( \mu \) on space \( X \), then for \( f: X \rightarrow \mathbb{R}^d \) measurable if \( \mu(\mathbb{R}^d) < \infty \) then \( M(f) < \infty \) a.s. (i.e. \( f \) is \( \mu \)-integrable a.s.),

and
\[
\mathbb{E}[e^{i\int f(x) \, dM(x)}] = \int (e^{ixf(x)} - 1) \, \mu(dx),
\]

in particular,
\[
\mathbb{E}[M(f)] = \int f(x) \, \mu(dx) \quad \text{if } \mu(\mathbb{R}^d) < \infty
\]
\[
\text{Var}(M(f)) = \int f(x)^2 \, \mu(dx) \quad \text{if } \mu(\mathbb{R}^d) < \infty.
\]
Let \( \mu \) be a finite measure on \( X \).

The Poisson point process on \( X \) with intensity \( \mu \)

is the Poisson random measure on \( \mathbb{R}^+ \times X \) with intensity Lebesgue \( \otimes \mu \).

We then use the notation \( N(t, \cdot) = M([0, t) \times \cdot) \).

Theorem (2.3.15): \( \mu \) measure on \( \mathbb{R} \setminus \{0\} \), \( \forall \theta \in \mathbb{R} \), \( \mu(\mathbb{R} \setminus [\theta, \infty)) < \infty \)

such that the jump-process \( M = \sum_{t: \Delta X(t) > 0} \delta_{(t, \Delta X(t))} \)

is a PPP of intensity \( \mu \) on \( \mathbb{R} \times \mathbb{R}^{+} \)

for equivalently a Poisson random measure on \( \mathbb{R}^+ \times \mathbb{R}^{+} \) with intensity \( \text{Leb} \otimes \mu \).

Remark: If \( f: X \to \mathbb{R}^d \), \( \mu(1_{\{X \}}) < \infty \), Campbell's formula tells us we can define \( N(t, \varphi) = \int f(x) N(t, dx) = \int f(x) M(dt, dx) \).

One can easily show this is a Lévy process in the variable \( t \), and Campbell's formula tells us its Lévy symbol is \( \mu \mapsto \int (e^{iux} - 1) \mu(dx) \).

Subremark 1: If \( \mu \) is finite and \( f = 1_{A} \), it's the compound Poisson process with rate \( \mu(X) \)

and distribution \( \frac{\mu(\cdot)}{\mu(X)} \). This is a generalization with \( \infty \) of summable jumps.

Subremark 2: The application of the remark above to the PPP coming from the jumps of \( X \)

and \( f = 0 \) around zero is called Thm 2.3.7 by Applebaum. I prefer to emphasize this is a general fact about PPPs.
Recall the total variation of a function \( f \) over \([0,t]\) is denoted \( V_f(t)\).

**Theorem 2.4.1**

Assume \( M_1, M_2 \) are càdlàg centred martingales.
Assume \( M_1(t), M_2(t) \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \) and furthermore \( V_{M_2}(t) \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \).

Then \( \mathbb{E}[M_1(t)M_2(t)] = \mathbb{E} \left[ \sum_{s \leq t} \Delta M_1(s) \Delta M_2(s) \right] \).

**(1)** First of all, is \( \sum_{s \leq t} \Delta M_1(s) \Delta M_2(s) \) summable?

Answer: yes, this is nonzero only for countably many \( s \) and

\[
\sum_{s \leq t} \Delta M_1(s) \Delta M_2(s) \leq 2 \sup_{t \in [0,t]} M_1 \cdot V_{M_2}(t).
\]

Moreover, \( \mathbb{E} \left[ \sum_{s \leq t} \Delta M_1(s) \Delta M_2(s) \right] \leq 4 \sup_{t \in [0,t]} \mathbb{E} \left[ M_1(t)^2 \right] + 4 \mathbb{E} \left[ V_{M_2}(t)^2 \right]. \)

So partial sums are dominated by a \( L^1 \)-variable and the convergence of the series is in \( L^1 \) by dominated convergence.

**(2)** Take an arbitrary partition \( \Pi = \{0 = t_0 < t_1 < \ldots < t_n = t\} \) of \([0,t]\).

\[ \mathbb{E}[M_1(t)M_2(t)] = \mathbb{E} \left[ \sum_{i=1}^{n} (H_i(t_{i+1}) - H_i(t)) \cdot (M_2(t_{i+1}) - M_2(t)) \right]. \]

**(3)** Fix \( \varepsilon > 0 \)

for a càdlàg function, one can enumerate jumps in the decreasing order of magnitude. Assume \( j_1, \ldots, j_k \) are the jumps \( \geq \varepsilon \) of \( M_1 \),

There is a partition \( \Pi_E \) fine enough such that

* if \([t_i, t_{i+1})\) contains a jump \( j \), \( \varepsilon \geq \varepsilon \) then it's the only such \( t \) and

\[ |M_1(t_{i+1}) - M_1(t_i) - \Delta M_1(t_j)| < \frac{\varepsilon}{2\varepsilon} \left( \sup_{t \in [0,t]} M_1(t) + \sup_{t \in [0,t]} |M_2(t)| \right)^{-1} \]

* if \([t_i, t_{i+1})\) does not, then \( |M_1(t_{i+1}) - M_1(t_i)| < 2\varepsilon \)

Hence theorem as \( \mathbb{E} \left[ \sum_{s \leq t} \Delta M_1(s) \Delta M_2(s) \right] \leq \varepsilon \sum_{s \leq t} \Delta M_1(s) \Delta M_2(s) \)

\( \mathbb{E} \left[ \sum_{s \leq t} \Delta M_1(s) \Delta M_2(s) \right] \leq \varepsilon \sum_{s \leq t} \Delta M_1(s) \Delta M_2(s) \)

and \( \sum_{s \leq t} \Delta M_1(s) \Delta M_2(s) \) converges in \( L^1 \).
Application:

Thm (2.4.6) If $A_1, A_2$ are disjoint and bounded

$$\int_{A_1} x \, N(t, dx) \quad \text{and} \quad \int_{A_2} x \, N(t, dx)$$

are independent stochastic processes.

More general theorem, same proof

If $X_1$ and $X_2$ are Lévy processes and $X_2$ has square-integrable truncated variation

(i.e. $\sup_{n=1}^\infty \sum_{i=0}^{n-1} |X_2(t_{i+1}) - X_2(t_i)|^2$ is in $L^2$ for all $t > 0$)

then they are disjoint, they are independent.

Proof. For $b=1, 2$, and $\theta \in \mathbb{R}$, define

$$M^b_\theta(t) = e^{i\theta X_b(t)} - 1 \mathbb{I}_{X_b(t)}(\theta)$$

This is a martingale.

Then one can apply (2.4.1) to $M^1_{\theta}$ and $M^2_{\theta}$, hence

$$E[M^1_\theta(t) M^2_\theta(s)] = 0 \forall t, s$$

Then

$$E[M^1_\theta(t) M^2_\theta(s)] = E\left[\frac{M^1_\theta(s) M^2_\theta(s)}{M^1_\theta(t)} + \frac{M^1_\theta(t) - M^1_\theta(s)}{M^1_\theta(s)}\right] = 0$$

(As $M^1, M^2$ martingales, same filtration)

As a result

$$E\left[e^{i\theta X_1(t)} e^{i\theta X_2(s)}\right] = E\left[e^{i\theta X_1(t)}\right] E\left[e^{i\theta X_2(s)}\right]$$

Hence $X_1(t) \perp X_2(s) \forall s, t$. Independence of $X_1, X_2$ as processes

($\Rightarrow$ independence of finite-dim marginals) follows by Lévy's (P by induction)
Thm (Lévy-Itô) \[(2.4.16) + (2.4.12) + (2.4.13)\]

Let \(X\) be Lévy, recall the jump Poisson process \(N(t, dx)\)

- There exists \(x \in \mathbb{R}\)
  - a centered BM \(B\) w/ covariance \(\sigma^2 > 0\) adapted to the filtration of \(X\)

Such that

\[
X(t) = bt + B(t) + Yd(t) + \int_{|x| > 1} x N(t, dx)
\]

\(Yd(t)\) is a centered martingale and a Lévy process, whose jumps are the points of \(N\) of magnitude < 1. It is adapted to \(N\) and its Lévy symbol is \(\mu \mapsto \int_{|x| < 1} (e^{iux} - 1 - iux) \mu(dx)\)

- All terms of the sum are \(L\) processes

\[
\int_{|x| < 1} x^2 \mu(dx) < \infty
\]

Corollary LK P8 take characteristic finding □
Theorem 2.4.7
A Lévy process has bounded jumps, (i.e. $\exists C > 0$ s.t. almost surely $|\Delta X_s| < C$ for $s > 0$).
Then it has bounded moments of all orders (i.e. $E[|X_s|^m] < \infty$ for all $m > 0$).

Proof
Let $C$ be the bound on jumps.
Let $T_0 = 0$ and inductively $T_{i+1} = \inf \{ t > T_i : |X_t - X_{T_i}| > C \}$
Then for $t \leq T_i$, $|X_t| \leq 2C$. And for $t \leq T_n$ $|X_t| \leq 2nC$.

By Strong Markov, $T_n$ is a sum of iid copies of $T_i$. $T_i > 0$ a.s. by stochasticity.

$$P(|X_t| > 2nC) \leq P(T_n > t) \leq e^{tE[e^{-T_i}]} \leq 1$$

$X_t$ has exponential tails, hence all moments.

Let $a > 0$
Define now $Y_a = X - \int_{|x| > a} x N(t, dx)$

Theorem 2.4.8 $Y_a$ is a Lévy process

But for all $t > 0$, $(Y_a(t+s) - Y_a(t))_{s \geq 0} = F((X(t+s) - X(t))_{s \geq 0})$ a.s., where $a$ is the functional that takes a càdlàg function and removes all jumps $\geq a$. Moreover $Y_a(t)$ eò $(X_s, s \leq t)$. This allows to show that $Y_a$ has stationary independent increments.
Theorem 2.4.15  A centered Lévy process (hence martingale) with no jumps (hence continuous) is a centered Brownian motion with variance $\sigma^2 > 0$. By 2.4.7, it has all moments. Moments are obtained by derivation of $\eta \rightarrow E[\exp(i \theta \eta)] = e^{\theta^2 \eta(0)}$. Since $\eta(0) = 0$ and $\eta''(0) = 0$ because centered, hence 3-polynomial s.t. $E[X_m] = a t + a_2 t^2 + \ldots + a_m t^m$. In particular $E[X_t^2] = a t$ (Hence sense be Lévy) and as $t \to 0$ $E[X_t^m] = a t + o(t^2)$ for $m > 2$.

Now consider $S(t) = \mathbb{E}[\exp(i \theta X_t)]$. Take a partition $0 = t_0 \leq \ldots \leq t_n = t$. Denote $S_j = X_{t_{j+1}} - X_{t_j}$. $S(t) - S(0) = \mathbb{E}[\sum_{i=0}^{n-1} e^{i \theta X_{t_{i+1}}} - e^{i \theta X_{t_i}}]$. By Taylor applied to $x \rightarrow e^{i \theta x}$ at the order 1, $E[X_{t_{i+1}}, X_{t_i}]$ $\leq \frac{\theta^2}{2} |\sum_{i=0}^{n-1} (i \delta i)^2 | \leq \frac{\theta^2}{2} (\sum_{i=0}^{n-1} \delta i^2)$

Hence $\sum_{i=0}^{n-1} \mathbb{E}[e^{i \theta X_{t_{i+1}}} - e^{i \theta X_{t_i}} - i \theta e^{i \theta X_{t_i}} S_i + \frac{\theta^2}{2} e^{i \theta X_{t_i}} S_i^2] \leq 101^2 + \frac{\theta^2}{2} (\sum_{i=0}^{n-1} \delta i^2) \leq 101^2 \mathbb{E}[\max_{i=0}^{n-1} \theta \delta i^2 \wedge 2] + \frac{\theta^2}{2} (\sum_{i=0}^{n-1} \delta i^2)$

Now make the mesh of the partition go to zero ($n \to \infty$). Bounded in $L^2$.

$S(t) - S(0) + \alpha \frac{\theta^2}{2} \int_0^t S(s) ds$

Go to zero a.s. and bounded by dominated convergence. Hence $S(t)$ goes to zero in $L^2$. Because $\mathbb{E}[\sum_{i=0}^{n-1} S_i^2] = \sum_{i=0}^{n-1} \mathbb{E}[S_i^2] + 2 \sum_{i<j} \mathbb{E}[S_i S_j] \leq \sum_{i=0}^{n-1} (t_{i+1} - t_i) \mathbb{E}[S_i^2] + 2 \sum_{i<j} \mathbb{E}[S_i S_j] \leq 101 + 4 t^2$ for $n$ large.

As a result $S(t) = S(0) + \alpha \frac{\theta^2}{2} \int_0^t S(s) ds$, hence $S(t) = e^{-\alpha \frac{\theta^2}{2} t}$ implying increments of $X$ are Gaussian. $X$ is BM.
2 Proof of Lévy-Itô, easy case

Recall \( Y_a(t) = X(t) - \int x N(t, dx) \). Since \( Y_a \) is Lévy (thm 2.4.8) and has bounded jumps, it has bounded moments (thm 2.4.7).

One can define \( \hat{Y}_a(t) = Y_a(t) - \mathbb{E}[Y_a(1)] \)

\[ = t \mathbb{E}[Y_a(1)] \]

Let us sketch the proof in the special case \( \int |x| N(1, dx) < \infty \) then \( Y_0 \) is well-defined and

\[ X(t) = t \mathbb{E}[Y_1(1)] + \hat{Y}_0(t) + \frac{\hat{Y}_1(t) - \hat{Y}_0(t)}{t} + \int_{|x| > 1} x N(t, dx) \]

Continuous centered Lévy = BM.
by (2.4.15)

\[ = \int_{|x| < 1} x N(t, dx) - \mathbb{E}\left[\int_{|x| < 1} x N(t, dx)\right] \]

hence has Lévy symbol \( \nu \mapsto \int (e^{ix\nu} - 1 - ix\nu) \mu(d\nu) \)

By Campbell's thm.

For the independence property first of all \( \int_{|x| > 1} x N(t, dx) \perp \hat{Y}_1 \) By 2.4.5

Secondly, \( \hat{Y}_0 \) and \( \hat{Y}_1 - \hat{Y}_0 \in \mathcal{F}(\hat{Y}_1) \) and \( \hat{Y}_1 - \hat{Y}_0 \perp \hat{Y}_1 \) By 2.4.5 again. \( \square \)
Proof general case.

\[ X(t) = t \mathbb{E}[Y_1(1)] + \mathcal{Y}_a(t) + \left( \frac{1}{\lambda_a} \right) \left( \mathcal{Y}_1(t) - \mathcal{Y}_a(t) \right) + \int_{|x| > 1} x \, N(t, dx) \]

- **Lévy process with jumps \( \lambda_a \)**
- **Compensated compound Poisson**
- **So Lévy symbol is**
  \[ \mu(x) = \int_{|x| > 1} (e^{ix} - 1 - iux) \, \nu(dx) \]

The three terms are independent by a double application of Thm 2.4.5 as in the special case above.

Now we want to take \( \alpha \to 0 \).

Because bounded jumps, \( \hat{\mathcal{Y}}_a(t), \hat{Z}_a(t) \in L^2 \). Because orthogonal (2.4.1), \( \text{Var} (\hat{\mathcal{Y}}_a(t)) + \text{Var} (\hat{Z}_a(t)) = \text{Var} (\hat{\mathcal{Y}}_1(t)) \)

Hence \( \text{Var}(\hat{Z}_a(t)) \) is bounded as \( \alpha \downarrow 0 \).

Moreover if \( \alpha' < \alpha \), \( \hat{Z}_{a'}(t) \equiv \frac{\hat{Z}_a(t) + \mathcal{Z}_a(t) - \mathcal{Z}_a(t)}{\alpha} \) hence \( \text{Var}(\hat{Z}_{a'}(t)) = \text{Var}(\hat{Z}_a(t)) - \text{Var}(\hat{Z}_{a'}(t)) \)

As a result \( \text{Var}(\hat{Z}_a(t)) \) is increasing and bounded as \( \alpha \downarrow 0 \) hence it converges, hence it is Cauchy and by (9) \( \hat{Z}_a(t) \) itself is Cauchy in \( L^2 \).

Denote \( B(t) \) the \( L^2 \)-limit of \( \mathcal{Y}_a(t) \) and \( Y_1(t) \) the \( L^2 \)-limit of \( Z_a(t) \). Independence and being Levy pass to the limit in \( L^2 \). Being centered passes to the limit in \( L^2 \).
Now, because of Doob's inequality for martingales \( \mathbb{E} \left[ \sup_{s \in [0, T]} |M_s| \right] \leq \mathbb{E} \left[ \sup_{s \in [0, T]} |M_s^2| \right]^{1/2} \leq \mathbb{E} [M_T^2]^{1/2} \), we can obtain some regularity of convergence.

Extract a \( \omega_0 \) such that \( \mathbb{E} \left[ \sup_{t \in [0, T]} |\hat{Y}_{\omega_0}^t - \hat{a}_{\omega_0}^t| \right] \leq 2 \mathbb{E} \left[ (\hat{Y}_{\omega_0}^T - \hat{a}_{\omega_0}^T)^2 \right]^{1/2} \leq \frac{1}{2} \).

(possible bc \( \hat{Y}_{\omega_0} \) is (u.a) when \( \omega_0 \)).

Almost surely \( \sum_{t \in [0, T]} |\hat{Y}_{\omega_0}^t - \hat{a}_{\omega_0}^t| \) is a convergent series. On that event of probability one, the sequence \( (\hat{Y}_{\omega_0}) \) of càdlàg functions on \([0, T]\) is (u.a) for the uniform norm hence there exists a function \( \hat{B} \) that is its uniform limit.

For fixed \( T \) we have found a subsequence such that almost surely \( \hat{Y}_{\omega_0} \) CV uniformly on \([0, T]\).

Now by diagonal extraction there is a subsequence such that a.s. \( \hat{Y}_{\omega_0} \) CV uniformly on every compact to a function \( \hat{B} \) (and similarly for \( \hat{Z}_a \)).

The uniform limit of càdlàg functions is càdlàg and if \( f_n \to f \) uniformly then \( \Delta f_n \to \Delta f \) uniformly.

Hence \( \hat{B} \) is continuous a.s. because \( \Delta \hat{Y}_a \leq a \) a.s.

* The jumps of \( \hat{Y}_a \) are exactly given by the restriction of \( N \) to \( \{ |x| < 1 \} \)

* The jumps of \( \hat{Z}_a \) are exactly given by the restriction of \( N \) to \( \{ a \leq |x| < 1 \} \)

We already knew the Lévy processes \( \hat{B} \) and \( \hat{Y}_a \) had càdlàg modifications, but here we explicited those modifications \( \hat{B} \) and \( \hat{Y}_a \), yielding further properties.

As a byproduct, \( \hat{B} \) being a continuous Lévy, is a BH (2.4.15).
Finally we need to show that \( \int x^2 \cdot 1 \, \mu(dx) < \infty \). We know \( \int \frac{1}{x^2} \, \mu(dx) < \infty \).

But \( \text{Var}\left( Z_a \right) = \int x^2 \, \mu(dx) \) converges to \( \text{Var}\left( Y_d(1) \right) \) for \( a \leq |x| < 1 \).

Hence \( \int x^2 \, \mu(dx) < \infty \) and hence \( \int x^2 \cdot 1 \, \mu(dx) < \infty \).

This shows that \( \mu \) integrated \( \frac{1}{|x|} (e^{ix} - 1 - ix) \) and one can take a limit in (\( a \)) showing that the Lévy symbol of \( Y_d \) is \( \int (e^{ix} - 1 - ix) \, \mu(dx) \) for \( |x| < 1 \).

We're done! \( \square \)
4) A few remarks

1) In the easy case \( \int |x|^{\lambda} \mu(da) < \infty \), the compensation of jumps of magnitude \( \leq 1 \) is not really necessary. Lévy-Itô becomes

\[
X(t) = b't + B(t) + \int xN(t,da)
\]

(the drift has changed: \( b' = b + \int x\mu(da) \))

This is the preferred form to write the Lévy-Itô decomposition and the Lévy-Khintchine representation \( \eta(u) = iku' + \frac{\sigma^2u^2}{2} + \int (e^{imu} - 1)\mu(da) \)

for such cases.

2) Thm 2.4.25 Lévy processes have finite variation if \( \lambda \geq 2 \)

* easy case described above

* \( \sigma = 0 \) (no Brownian part)

3) In the case \( \int |x|^{\lambda} \mu(da) = \infty \), then the small jumps are not summable even though they manage to form a càdlàg function. This is really "martingale magic".

4) The cutoff at 1 for which jumps are compensated is artificial. One could replace by a cutoff at \( M > 0 \) or even a smooth cutoff function, up to updating the drift term.