Scaling limits of pattern-avoiding permutations

Mickaël Maazoun — UMPA, ENS de Lyon
Joint work with F. Bassino, M. Bouvel, V. Féray, L. Gerin and A. Pierrot
(LIPN-P13, Zürich[2], CMAP-Polytechnique, LMO-Orsay)

Journées MAS
30 août 2018

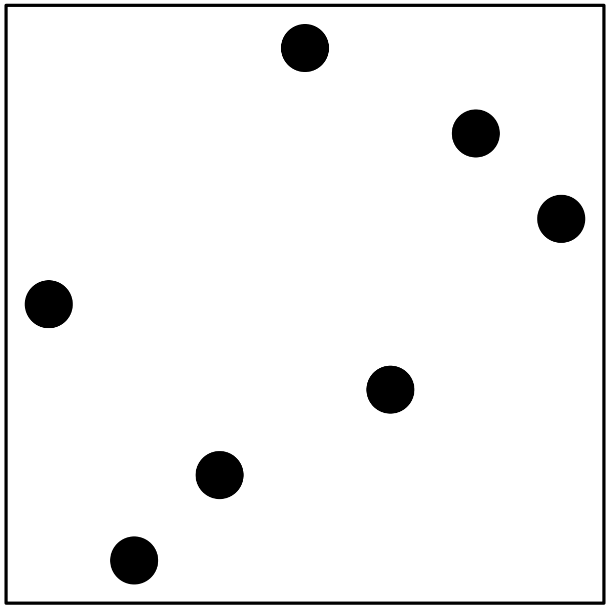
1 – The scaling limit of separable permutations

After Bassino, Bouvel, Féray, Gerin, Pierrot 2016

Permutations

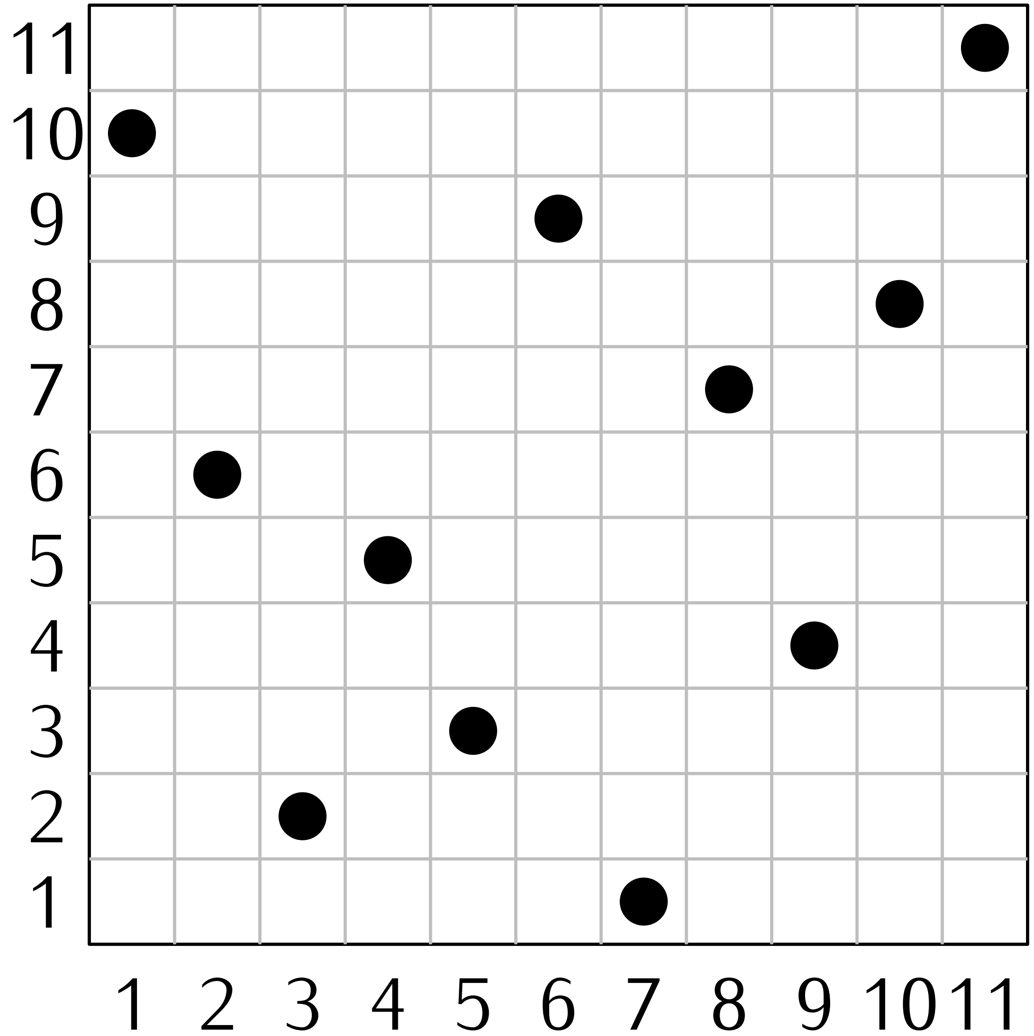
A permutation $\sigma \in \mathfrak{S}_n$ is a word $(\sigma(1), \ldots, \sigma(n))$ which contains every element of $\{1, \ldots, n\}$.

Diagram of $(4128376) \in \mathfrak{S}_8$:



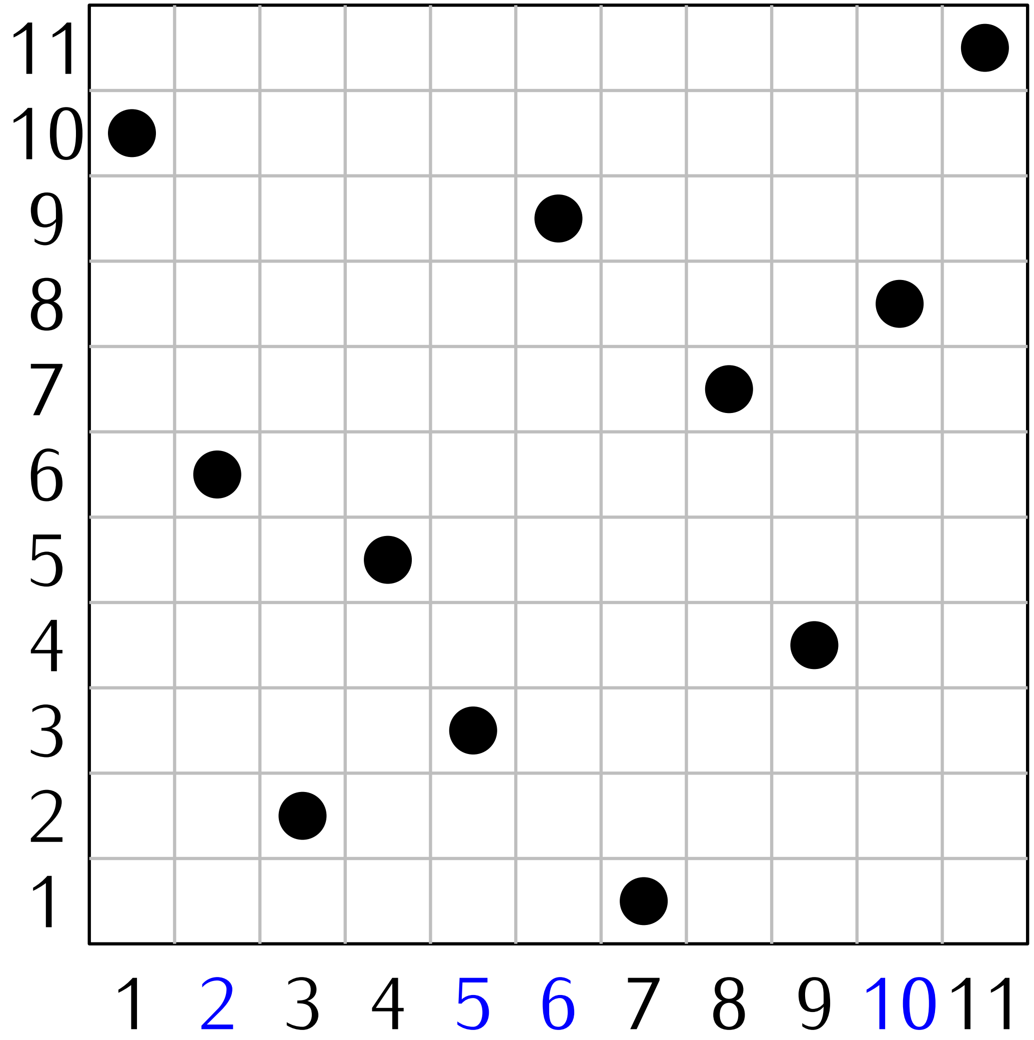
Permutation patterns

$\sigma = (10, 6, 2, 5, 3, 9, 1, 7, 4, 8, 11) \in \mathfrak{S}_{11}$



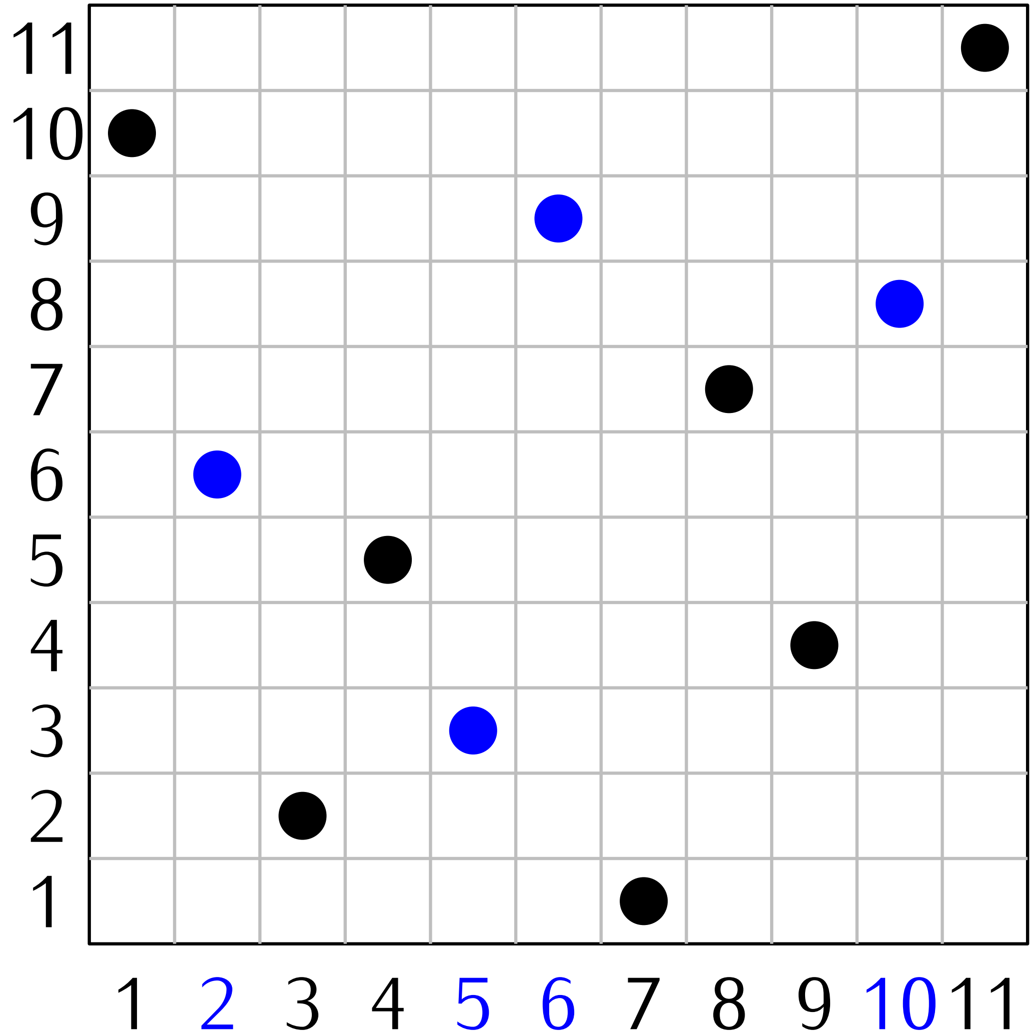
Permutation patterns

$\sigma = (10, 6, 2, 5, 3, 9, 1, 7, 4, 8, 11) \in \mathfrak{S}_{11}$



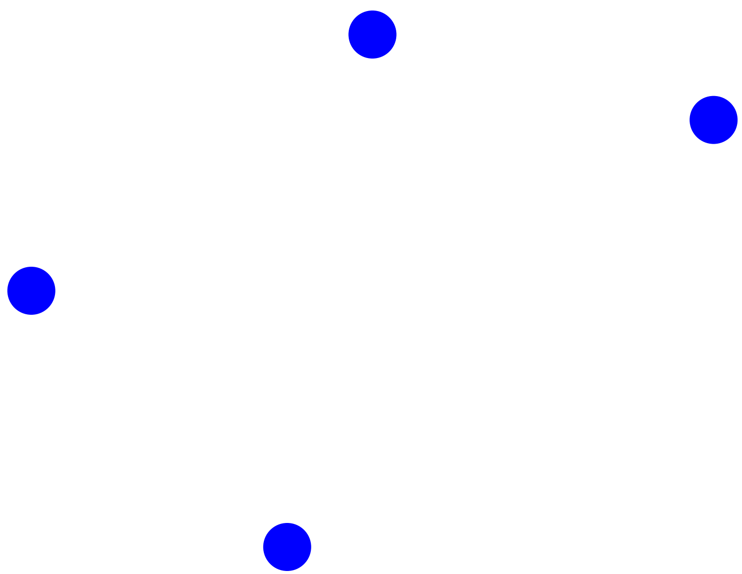
Permutation patterns

$\sigma = (10, 6, 2, 5, 3, 9, 1, 7, 4, 8, 11) \in \mathfrak{S}_{11}$



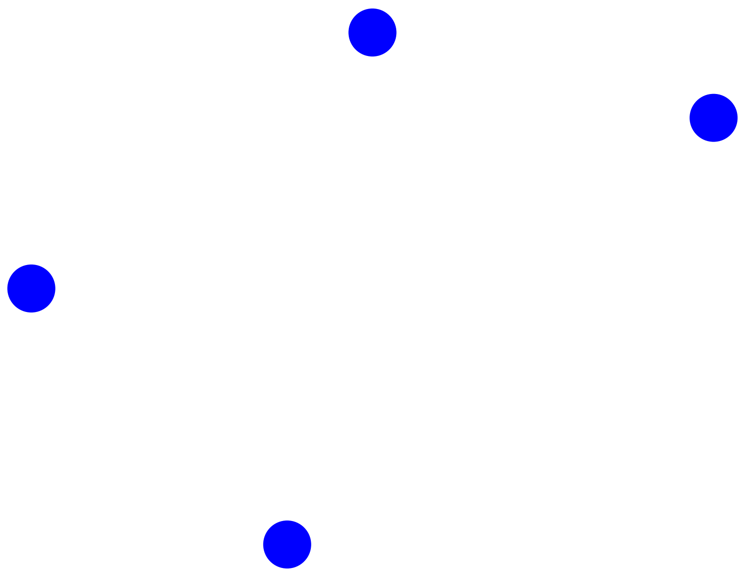
Permutation patterns

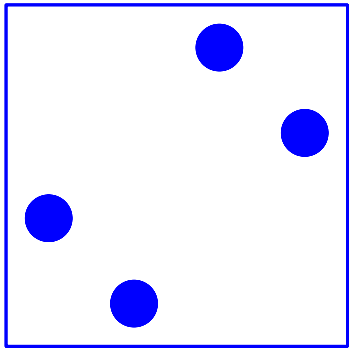
$$\sigma = (10, 6, 2, 5, 3, 9, 1, 7, 4, 8, 11) \in \mathfrak{S}_{11}$$



Permutation patterns

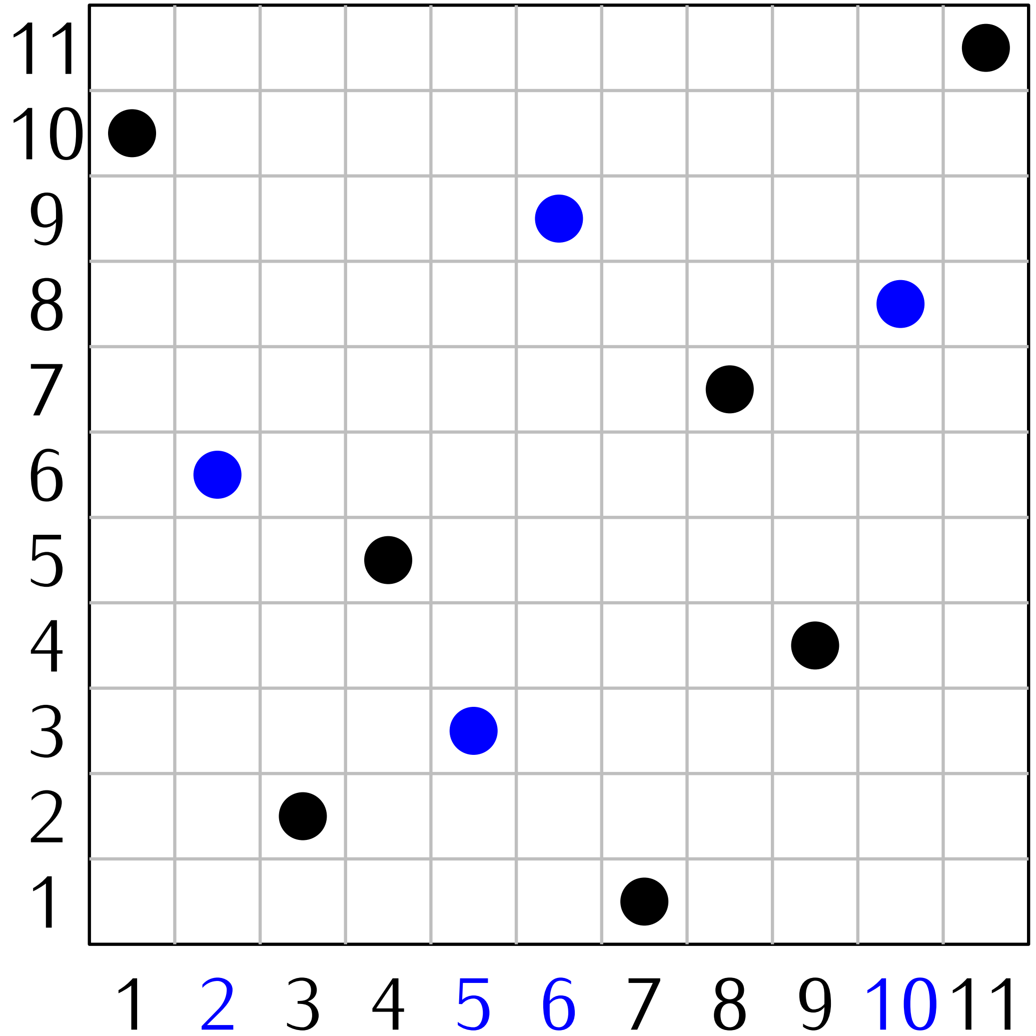
$\sigma = (10, 6, 2, 5, 3, 9, 1, 7, 4, 8, 11) \in \mathfrak{S}_{11}$

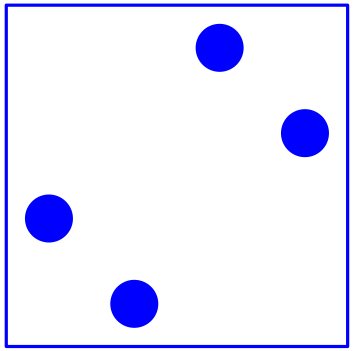




Permutation patterns

$\sigma = (10, 6, 2, 5, 3, 9, 1, 7, 4, 8, 11) \in \mathfrak{S}_{11}$

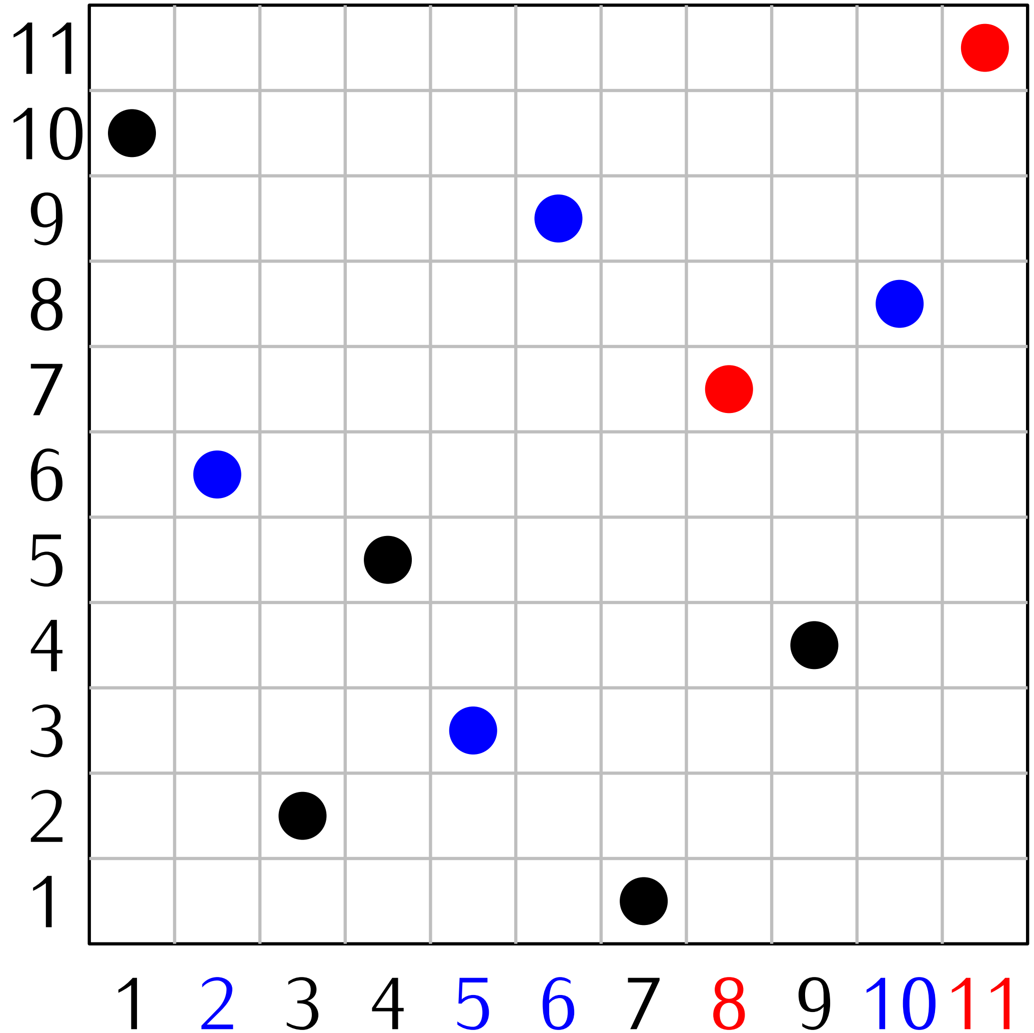


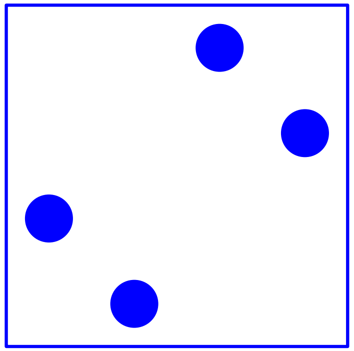


$\mathrm{pat}_{\{2,5,6,10\}}(\sigma) = (2143)$

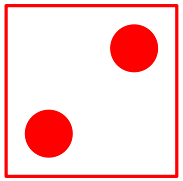
Permutation patterns

$\sigma = (10, 6, 2, 5, 3, 9, 1, 7, 4, 8, 11) \in \mathfrak{S}_{11}$





$\mathrm{pat}_{\{2,5,6,10\}}(\sigma) = (2143)$



$\mathrm{pat}_{\{8,11\}}(\sigma) = (12)$

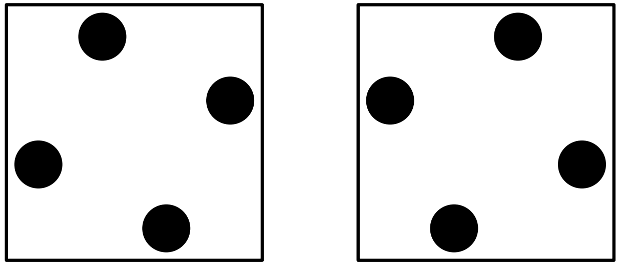
Classes of permutation and pattern-avoidance

Permutation class: set of permutations closed under pattern extraction. Can always be written as $\mathrm{Av}(B)$, the set of permutations that avoid patterns in some *basis* B.

Classes of permutation and pattern-avoidance

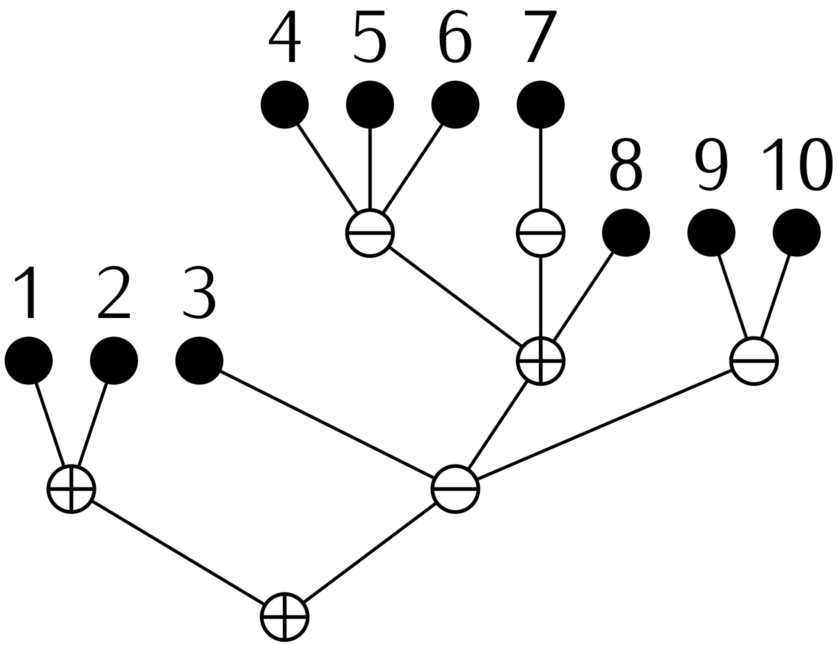
Permutation class: set of permutations closed under pattern extraction. Can always be written as $\mathrm{Av}(B)$, the set of permutations that avoid patterns in some *basis* B.

Separable permutations: $\mathrm{Av}(3142, 2413)$



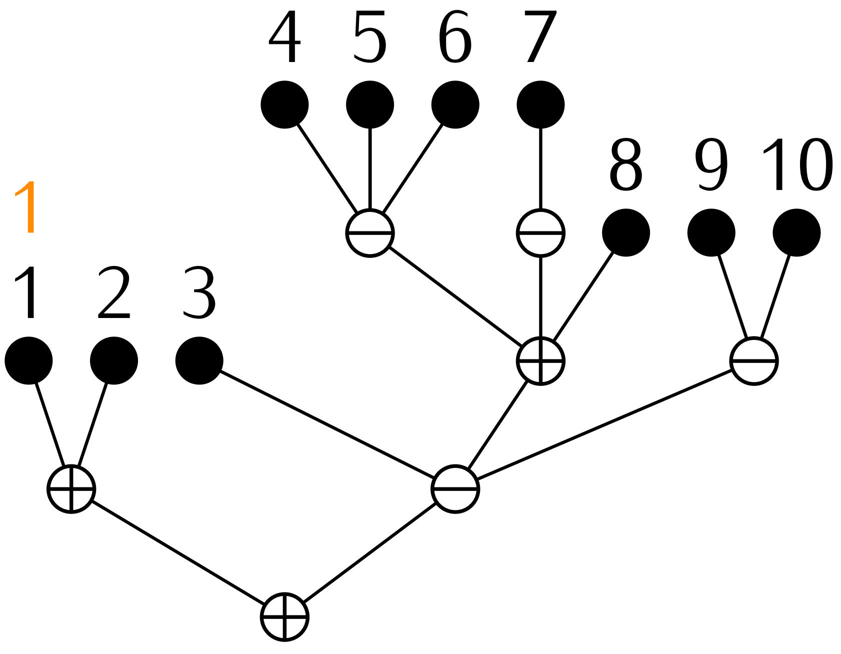
(Avis–Newborn '80, Bose–Buss–Lubiw '93)

Separable permutations



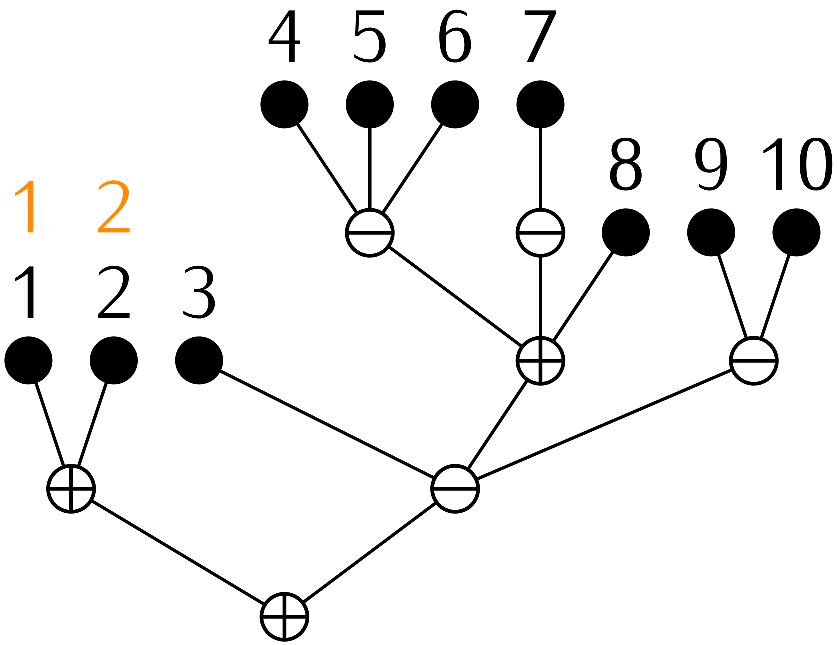
Signed tree τ

Separable permutations



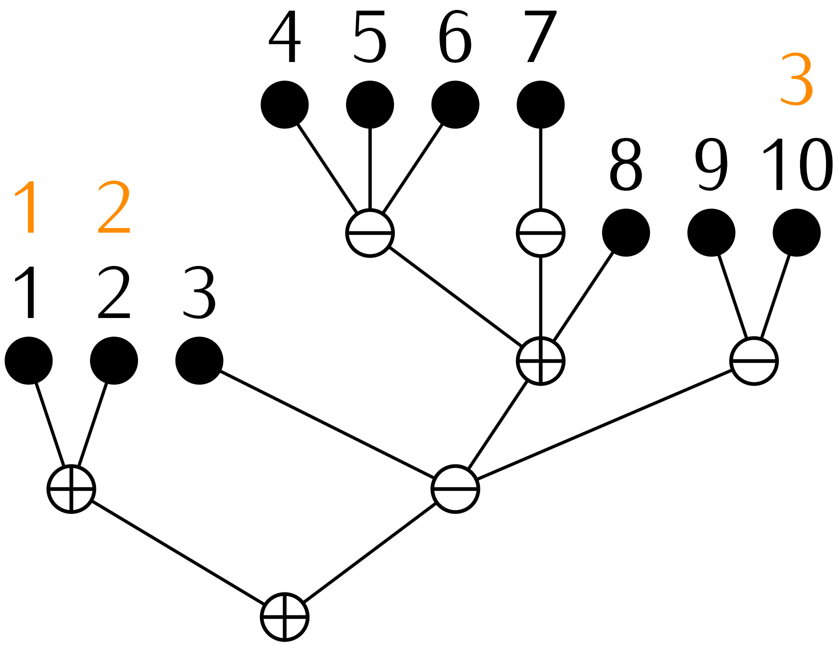
Signed tree τ

Separable permutations



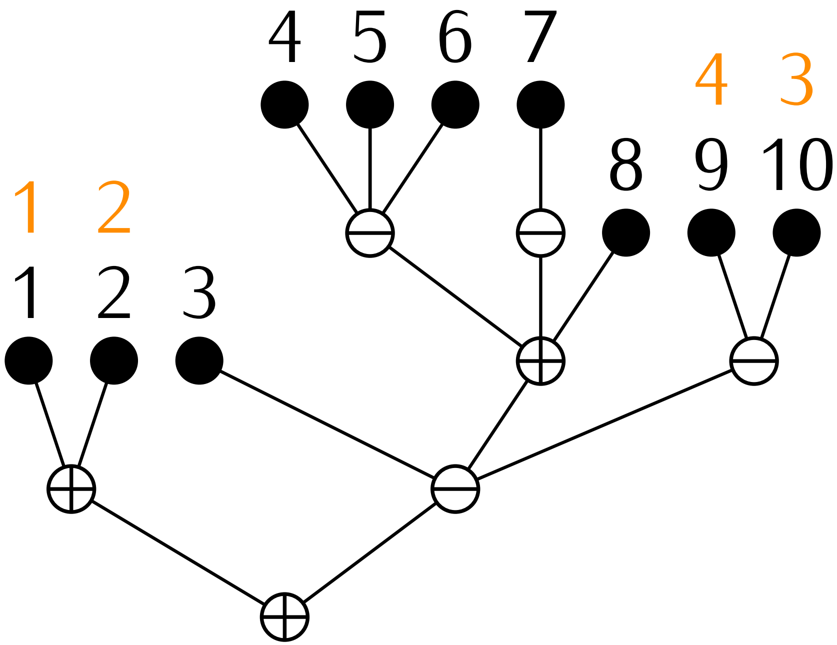
Signed tree τ

Separable permutations



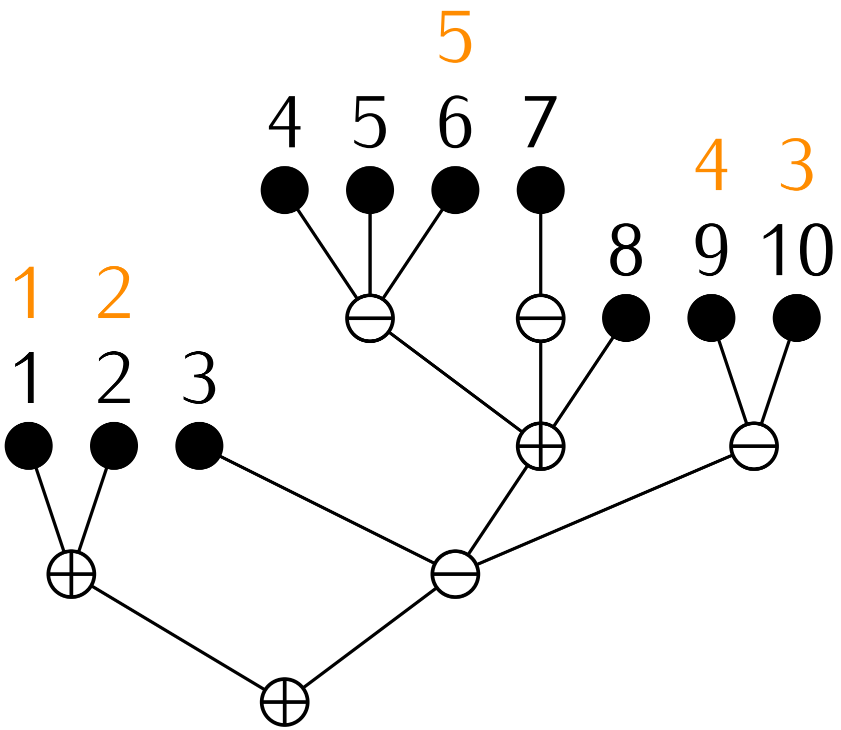
Signed tree τ

Separable permutations



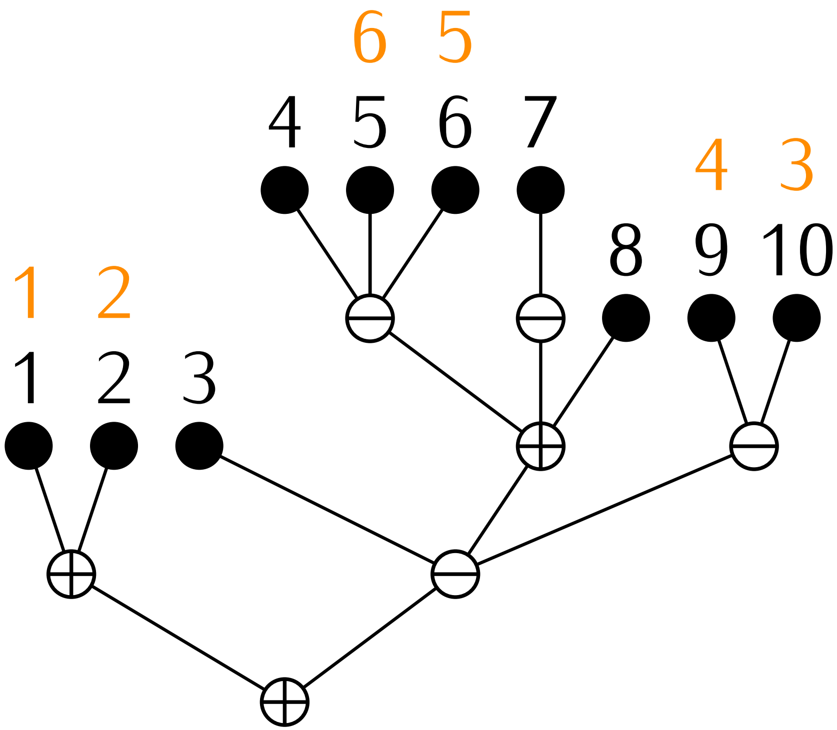
Signed tree τ

Separable permutations



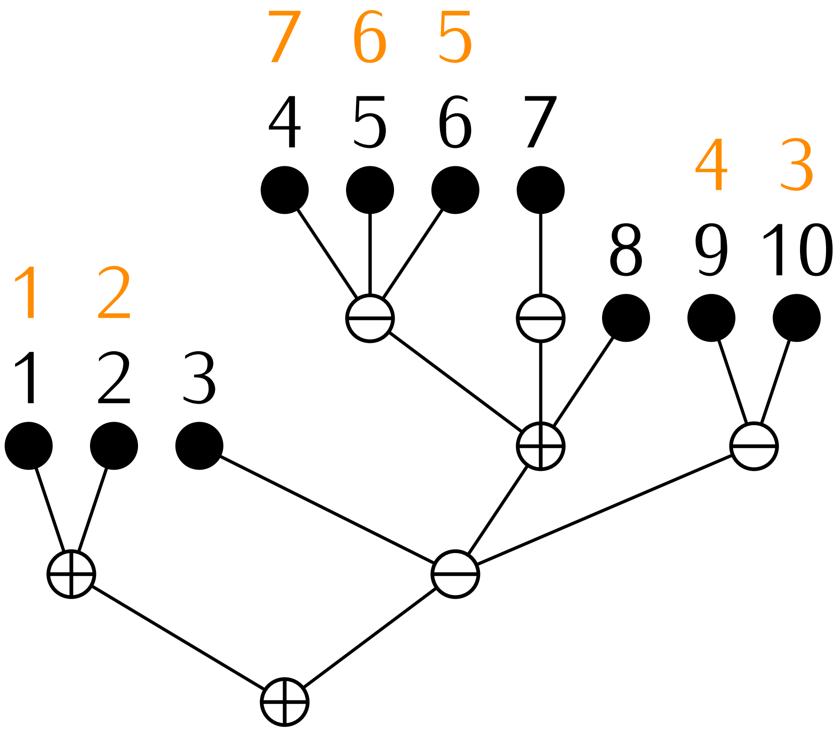
Signed tree τ

Separable permutations



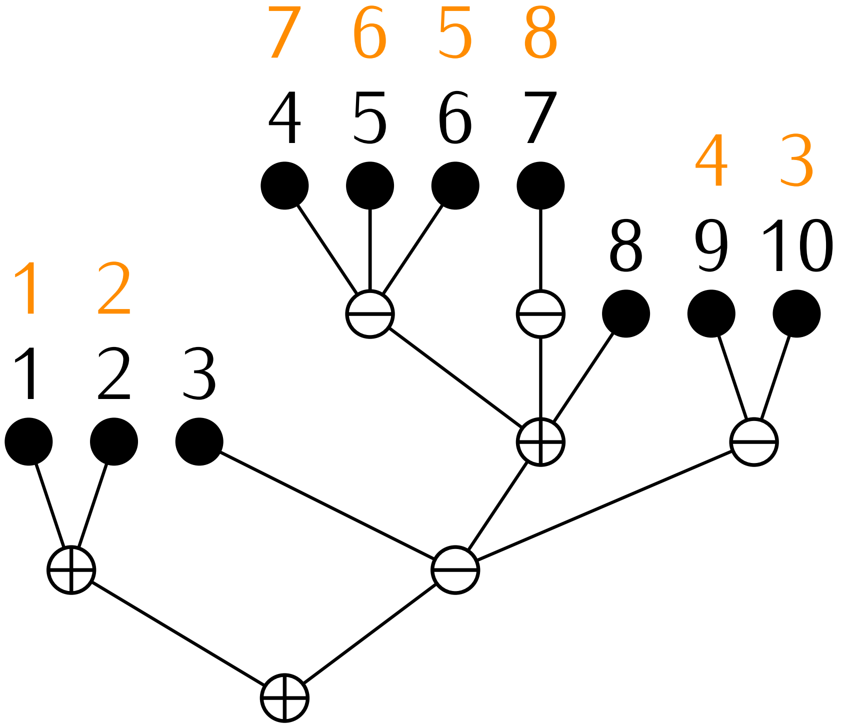
Signed tree τ

Separable permutations



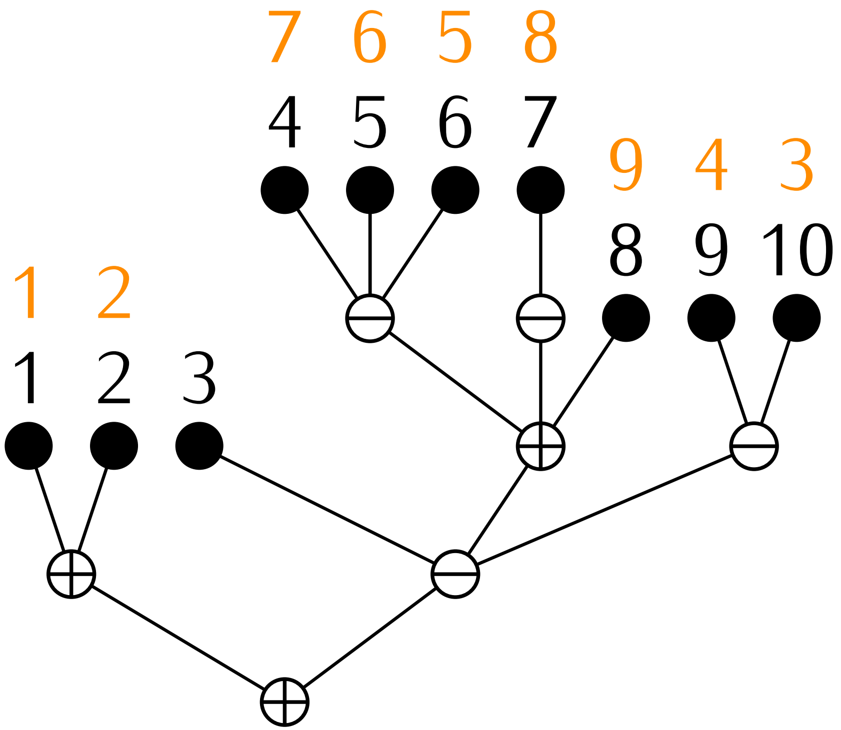
Signed tree τ

Separable permutations



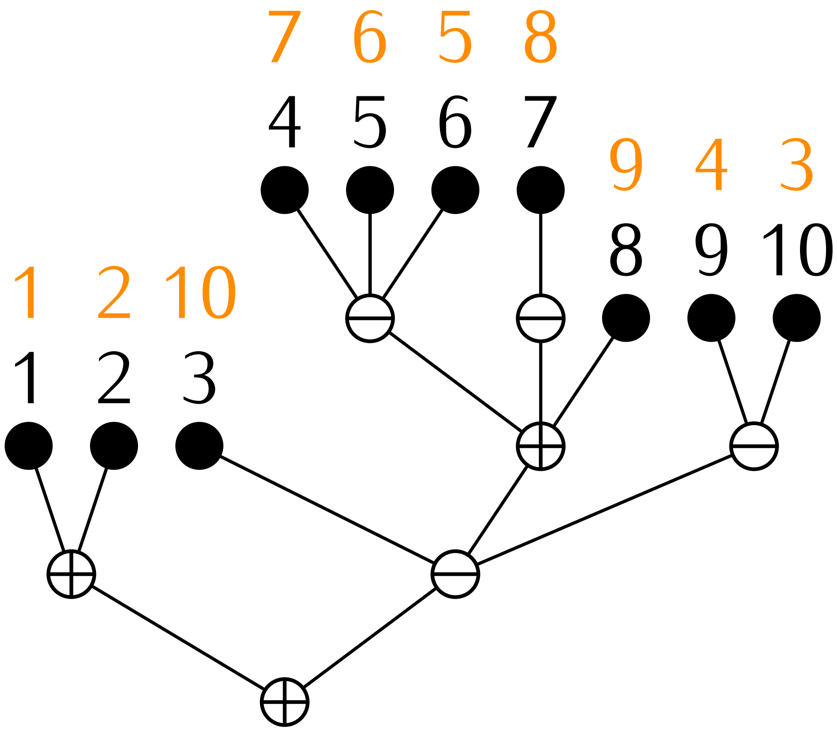
Signed tree τ

Separable permutations



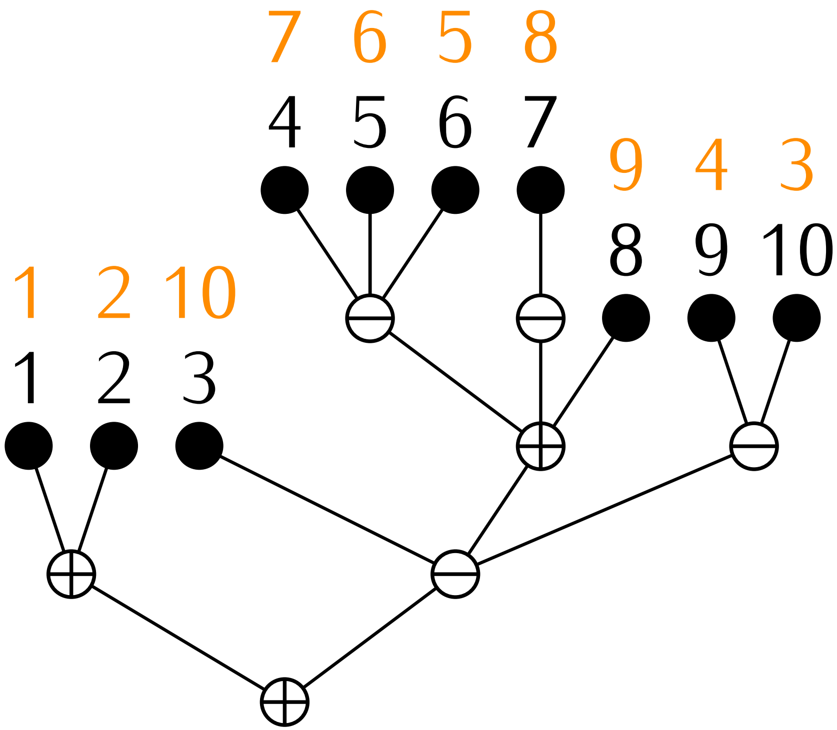
Signed tree τ

Separable permutations

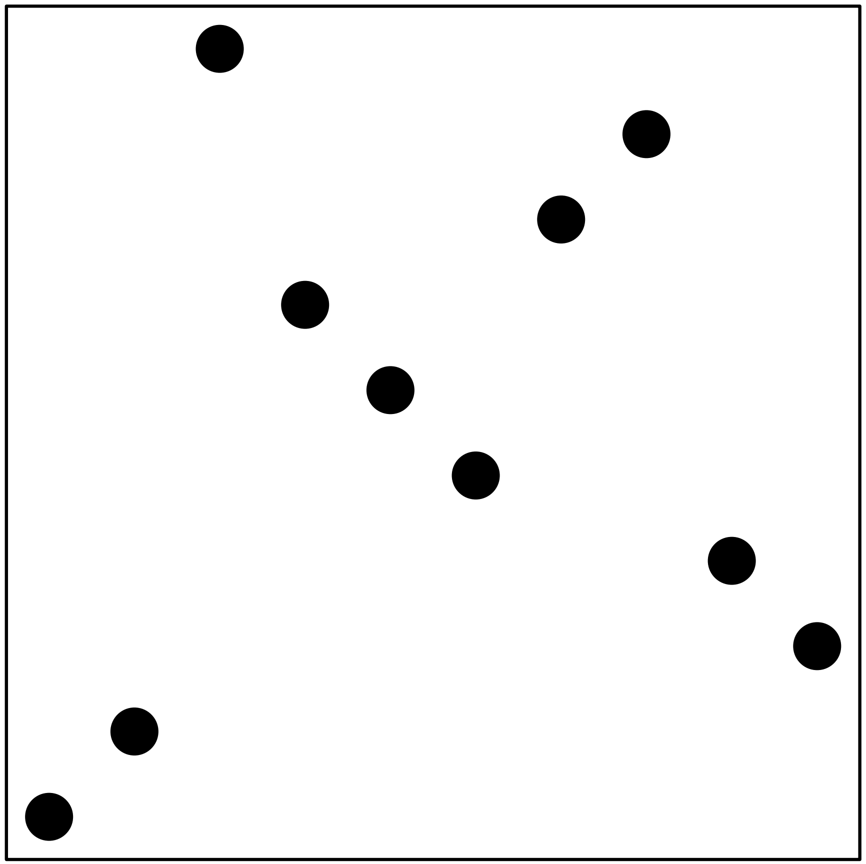


Signed tree τ

Separable permutations



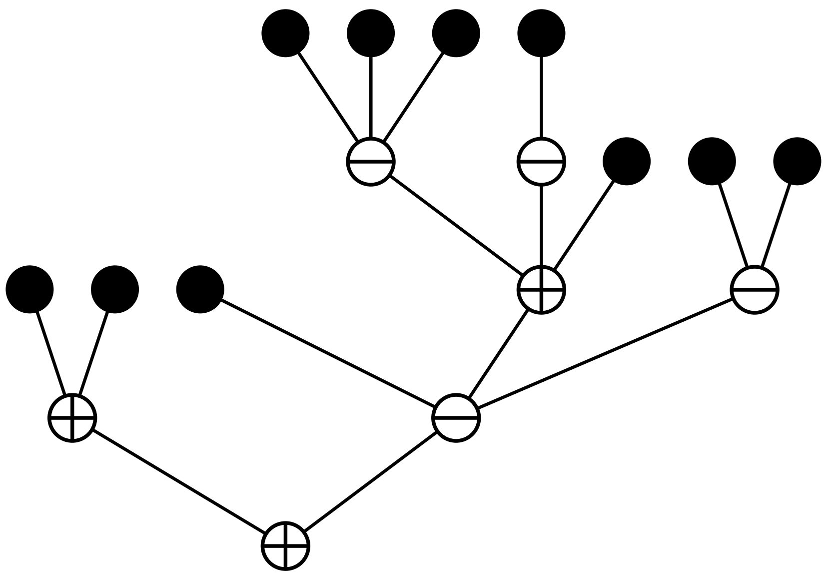
Signed tree τ



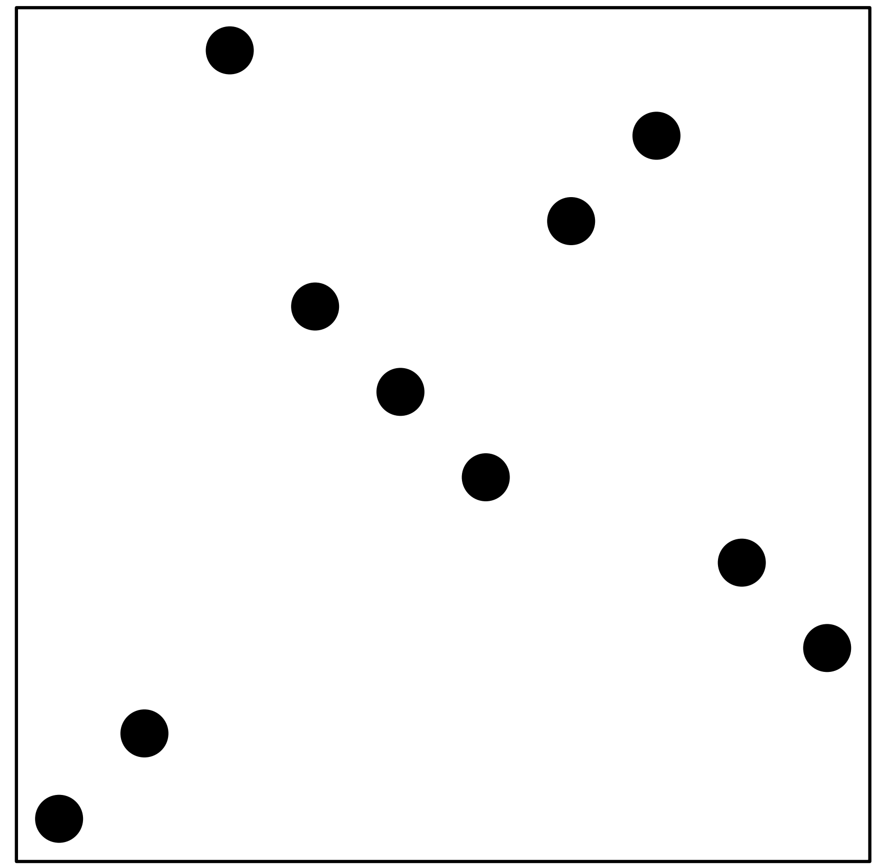
Separable permutation

$\mathrm{perm}(\tau) = (1\,2\,10\,7\,6\,5\,8\,9\,4\,3)$

Separable permutations

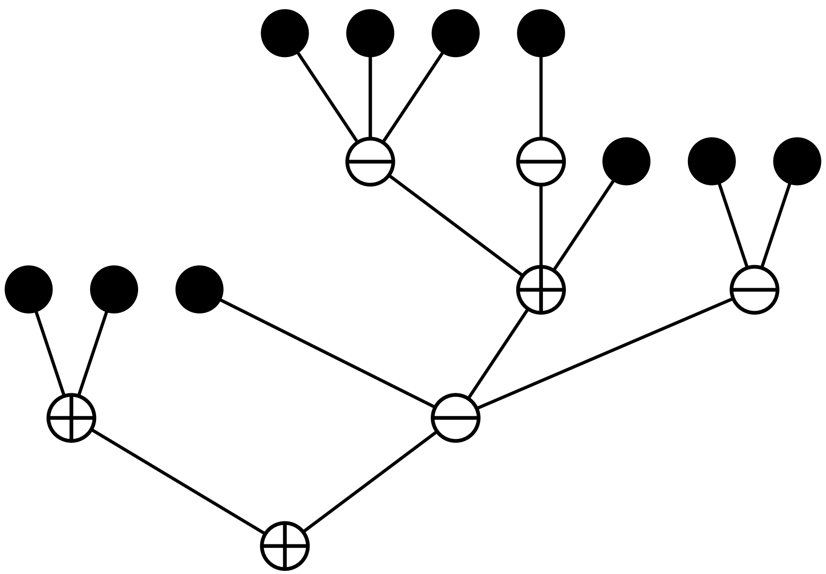


Signed tree τ

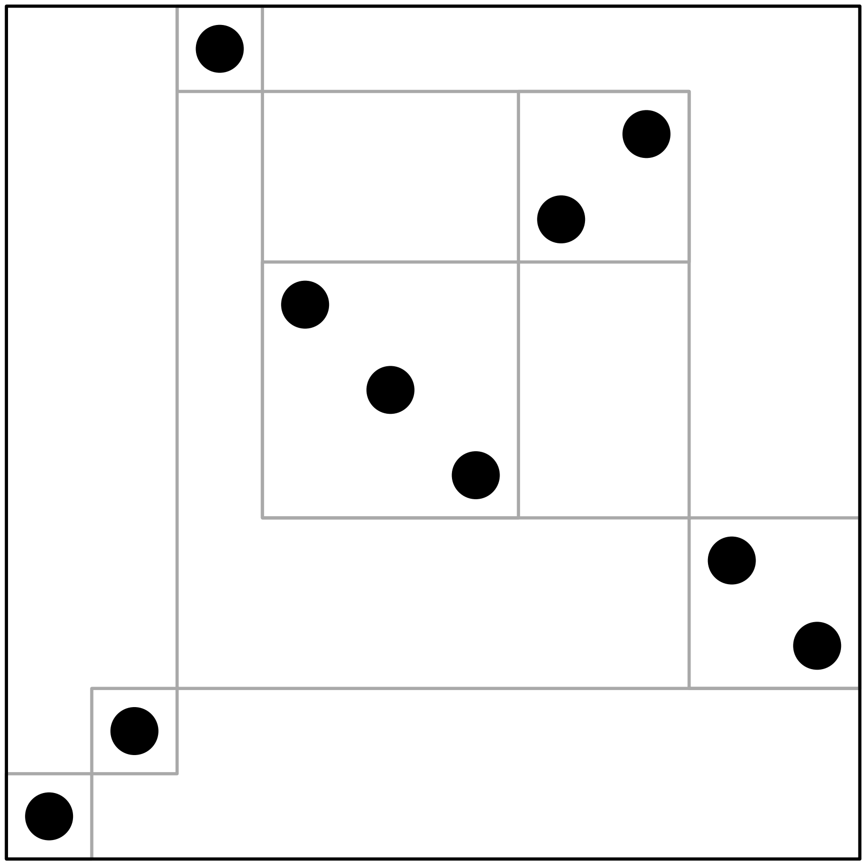


Separable permutation
$\mathrm{perm}(\tau) = (1\,2\,10\,7\,6\,5\,8\,9\,4\,3)$

Separable permutations

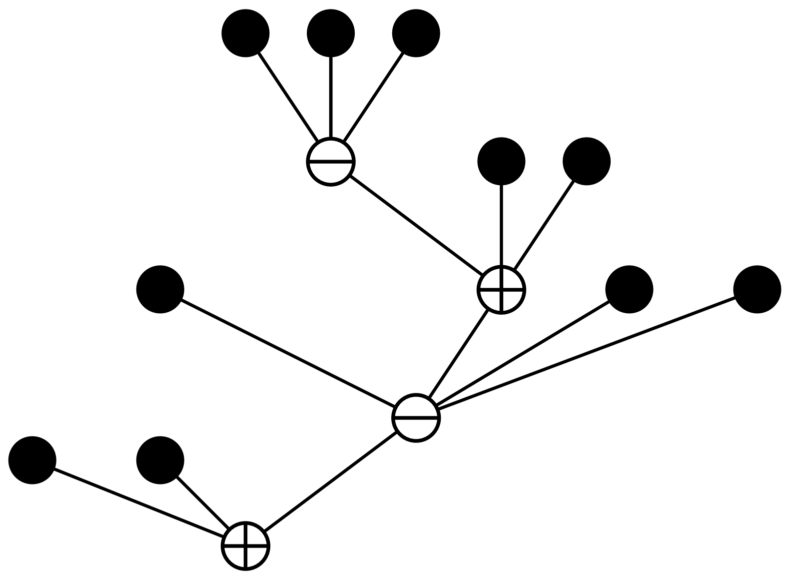


Signed tree τ

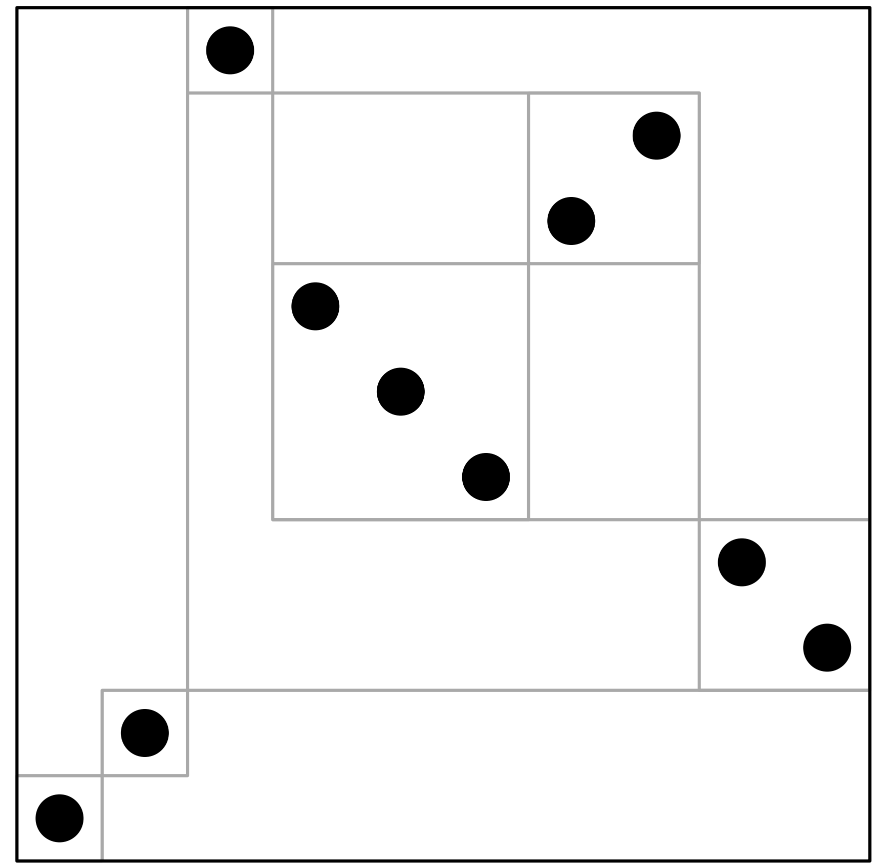


Separable permutation
$\mathrm{perm}(\tau) = (1\ 2\ 10\ 7\ 6\ 5\ 8\ 9\ 4\ 3)$

Separable permutations



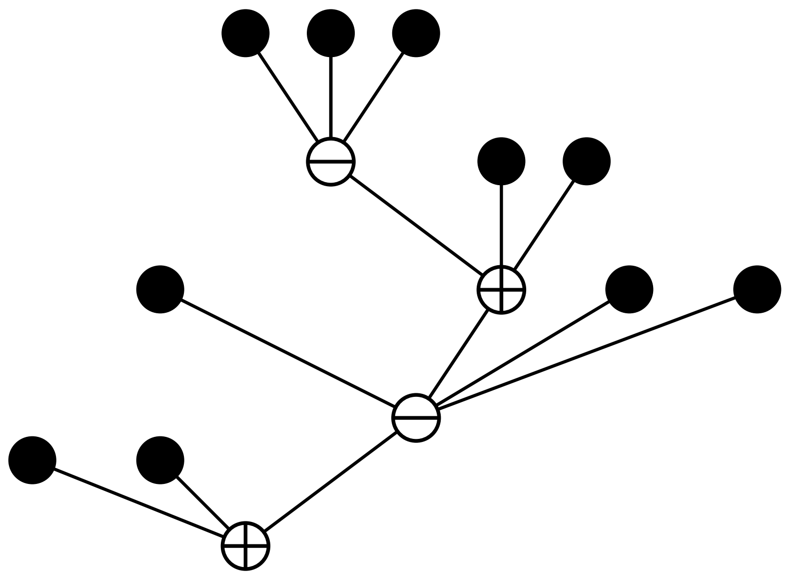
Alternating-signs Schröder tree



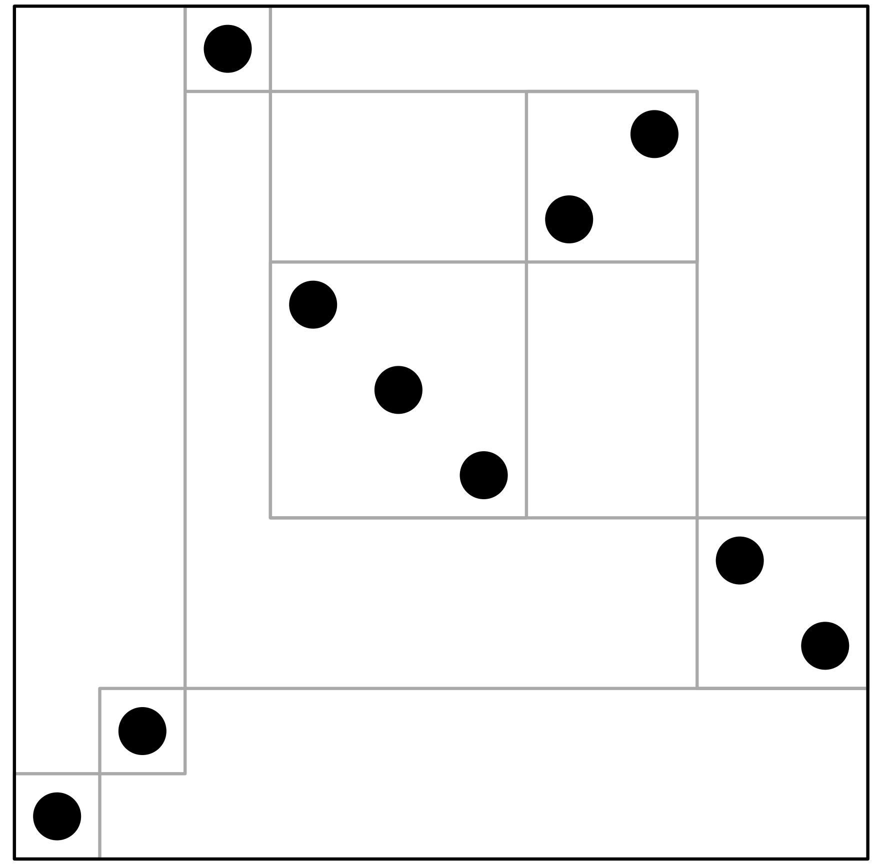
Separable permutation

$\mathrm{perm}(\tau) = (1\;2\;10\;7\;6\;5\;8\;9\;4\;3)$

Separable permutations



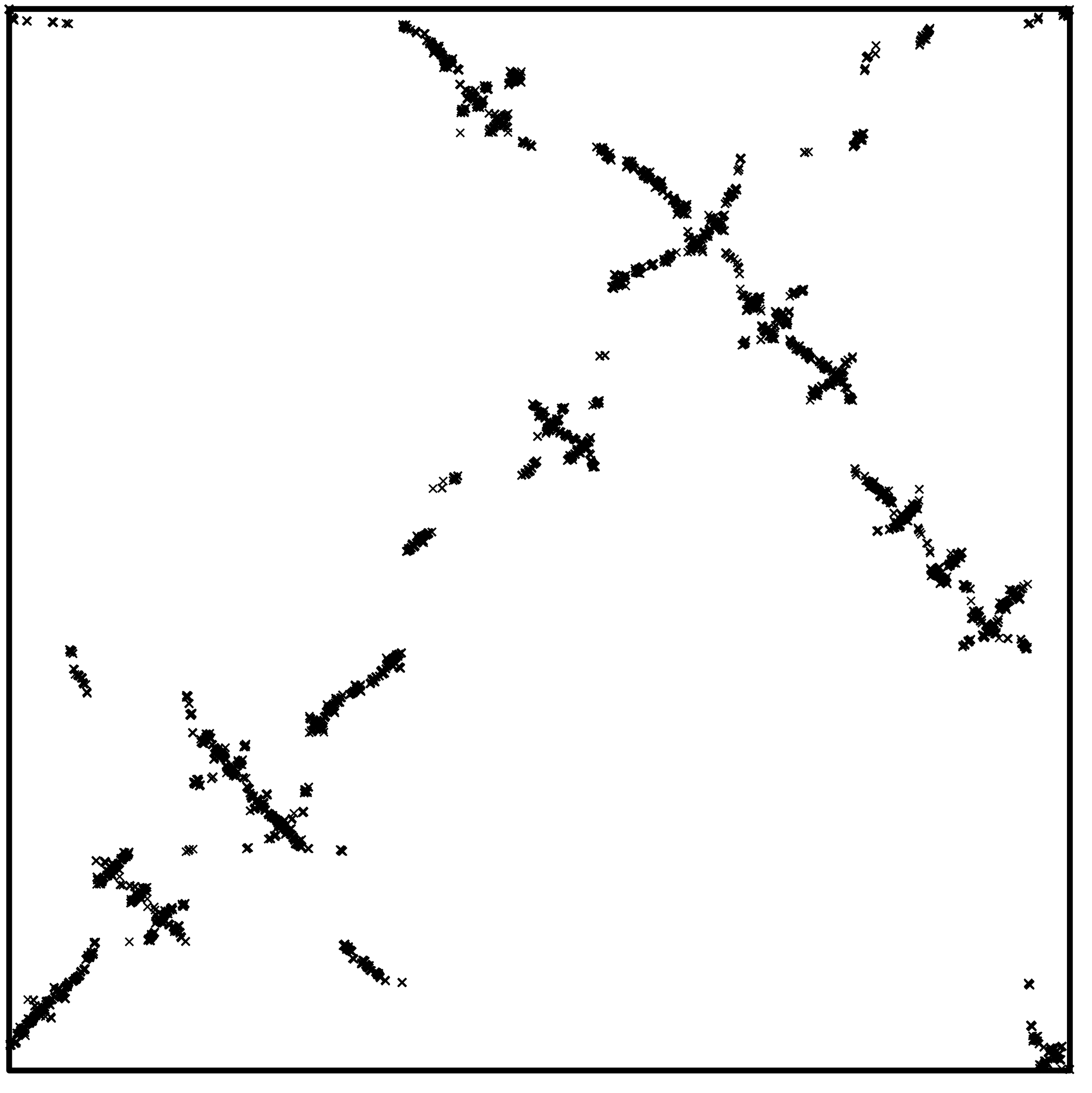
Alternating-signs Schröder tree



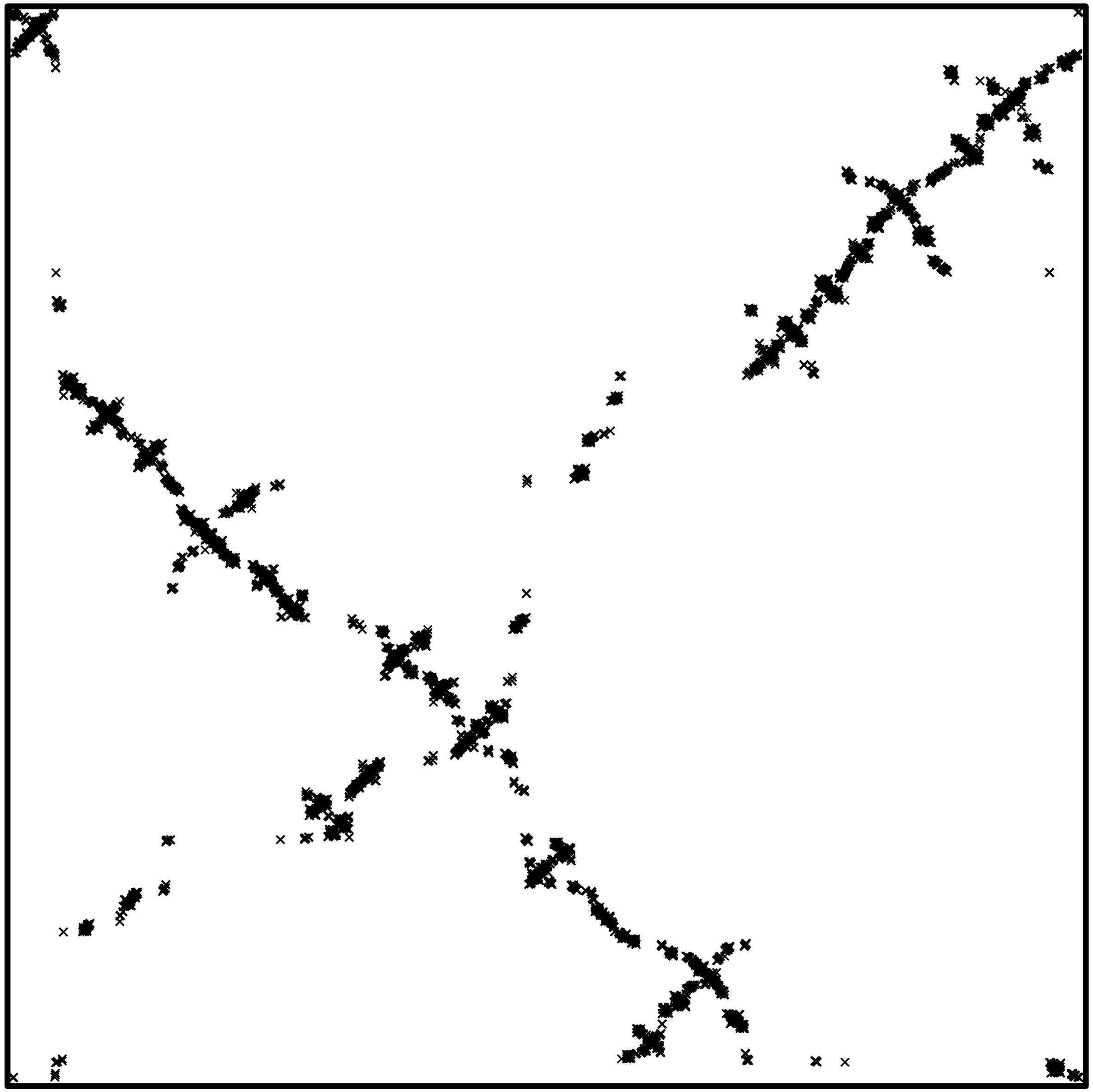
Separable permutation
$\mathrm{perm}(\tau) = (1\,2\,10\,7\,6\,5\,8\,9\,4\,3)$

Counted by large Schröder numbers
$1, 2, 6, 22, 90, 394, 1806, 8558, \ldots \asymp (3+\sqrt{8})^n n^{-3/2}$

A large uniform separable permutation

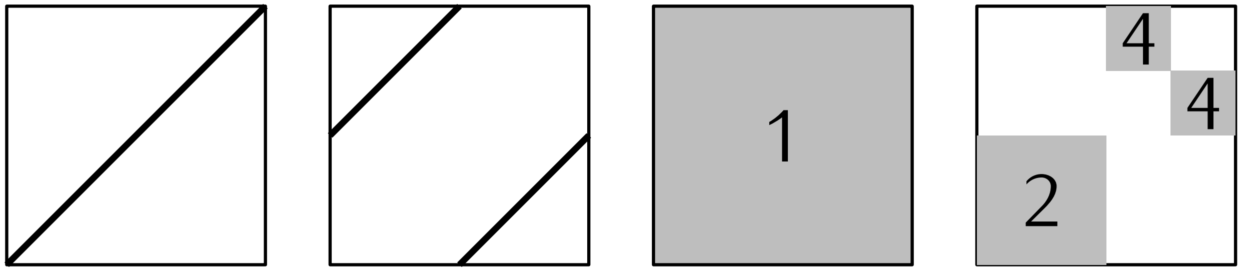


A large uniform separable permutation



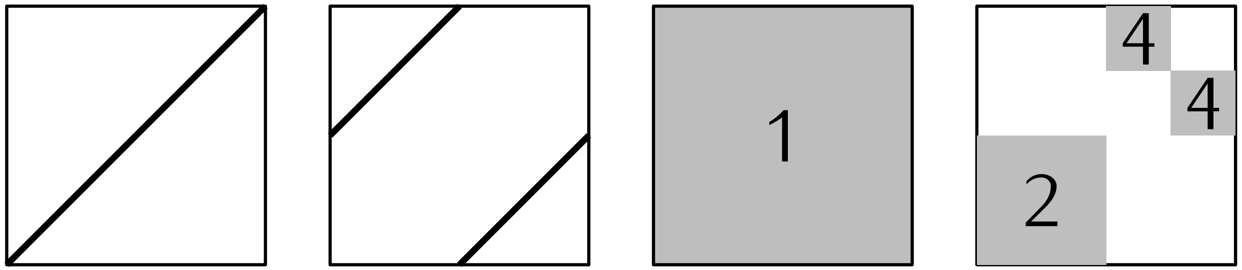
Permutons

A permuton is a probability measure on $[0,1]^2$ with both marginals uniform.



Permutons

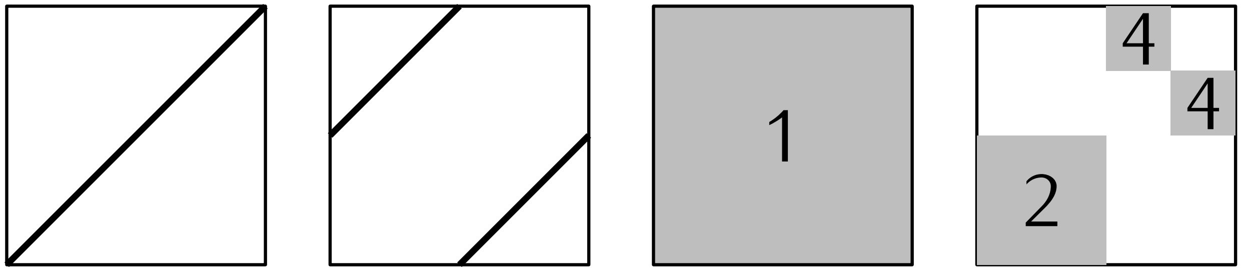
A permuton is a probability measure on $[0,1]^2$ with both marginals uniform.



$\implies$ compact metric space (with weak convergence).

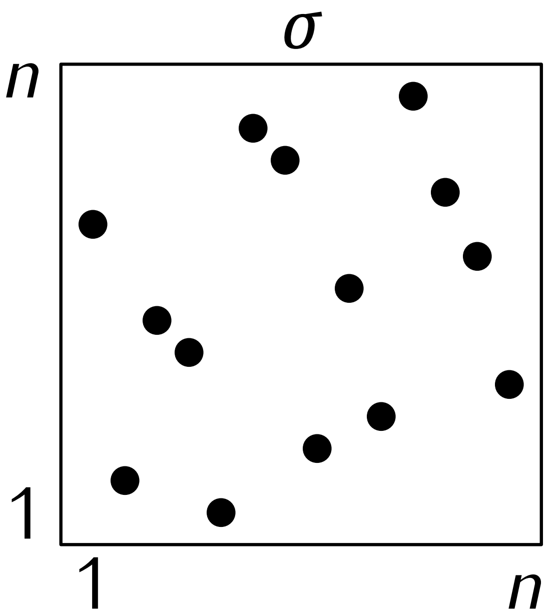
Permutons

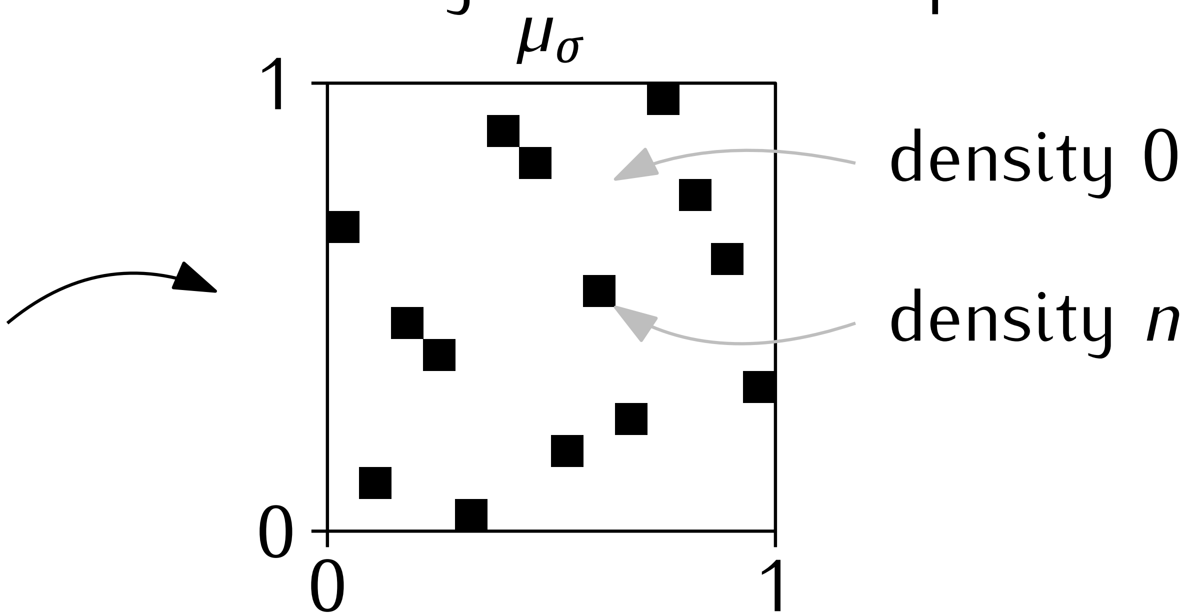
A permuton is a probability measure on $[0,1]^2$ with both marginals uniform.



$\implies$ compact metric space (with weak convergence).

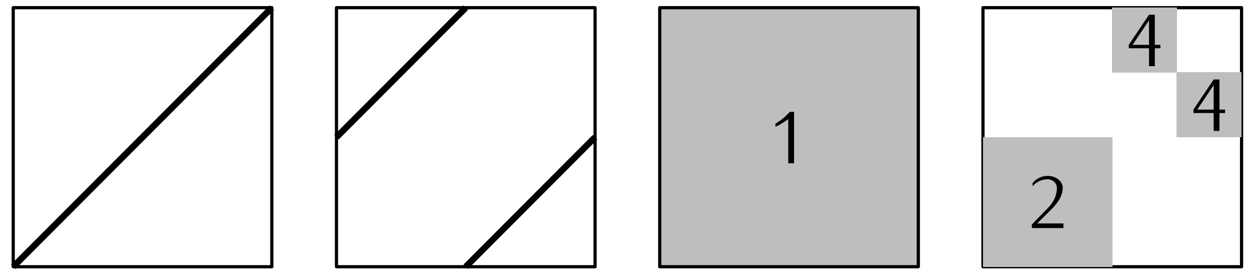
Permutations of all sizes are densely embedded in permutons.





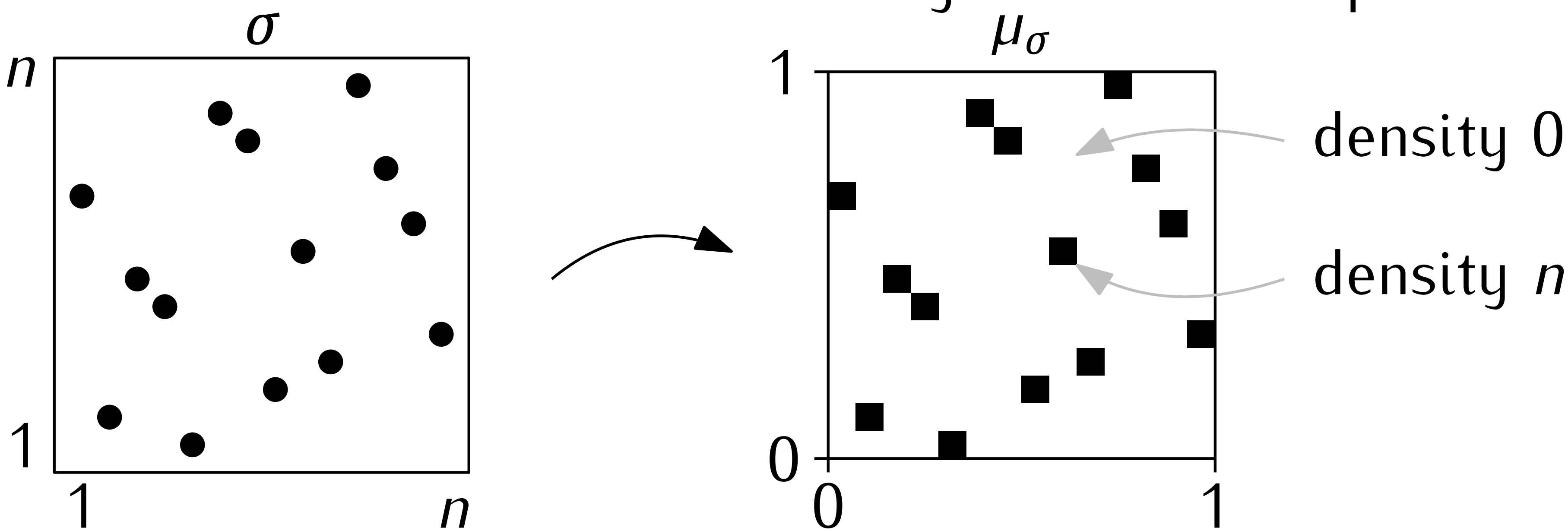
Permutons

A permuton is a probability measure on $[0,1]^2$ with both marginals uniform.



$\implies$ compact metric space (with weak convergence).

Permutations of all sizes are densely embedded in permutons.



We say that a sequence (σ_n) converges to μ when $\mu_{\sigma_n} \xrightarrow[n\to\infty]{w} \mu$.

Permuton convergence and subpermutations

For $\sigma \in \mathfrak{S}_n$ and $k \leq n$, $\mathrm{subperm}_k(\sigma)$ is a uniform subpermutation of length k in σ.

Permuton convergence and subpermutations

For $\sigma \in \mathfrak{S}_n$ and $k \leq n$, $\mathrm{subperm}_k(\sigma)$ is a uniform subpermutation of length k in σ.

This notion is extended to permutons: $\mathrm{subperm}_k(\mu)$ is the random permutation that is order-isomorphic to an i.i.d. pick according to μ.

Permuton convergence and subpermutations

For $\sigma \in \mathfrak{S}_n$ and $k \leq n$, $\mathrm{subperm}_k(\sigma)$ is a uniform subpermutation of length k in σ.

This notion is extended to permutons: $\mathrm{subperm}_k(\mu)$ is the random permutation that is order-isomorphic to an i.i.d. pick according to μ.

Theorem (Hoppen *et. al.*, 2013)
The sequence (σ_n) converges to μ iff for every k, $\mathrm{subperm}_k(\sigma_n) \xrightarrow[n\to\infty]{d} \mathrm{subperm}_k(\mu)$.

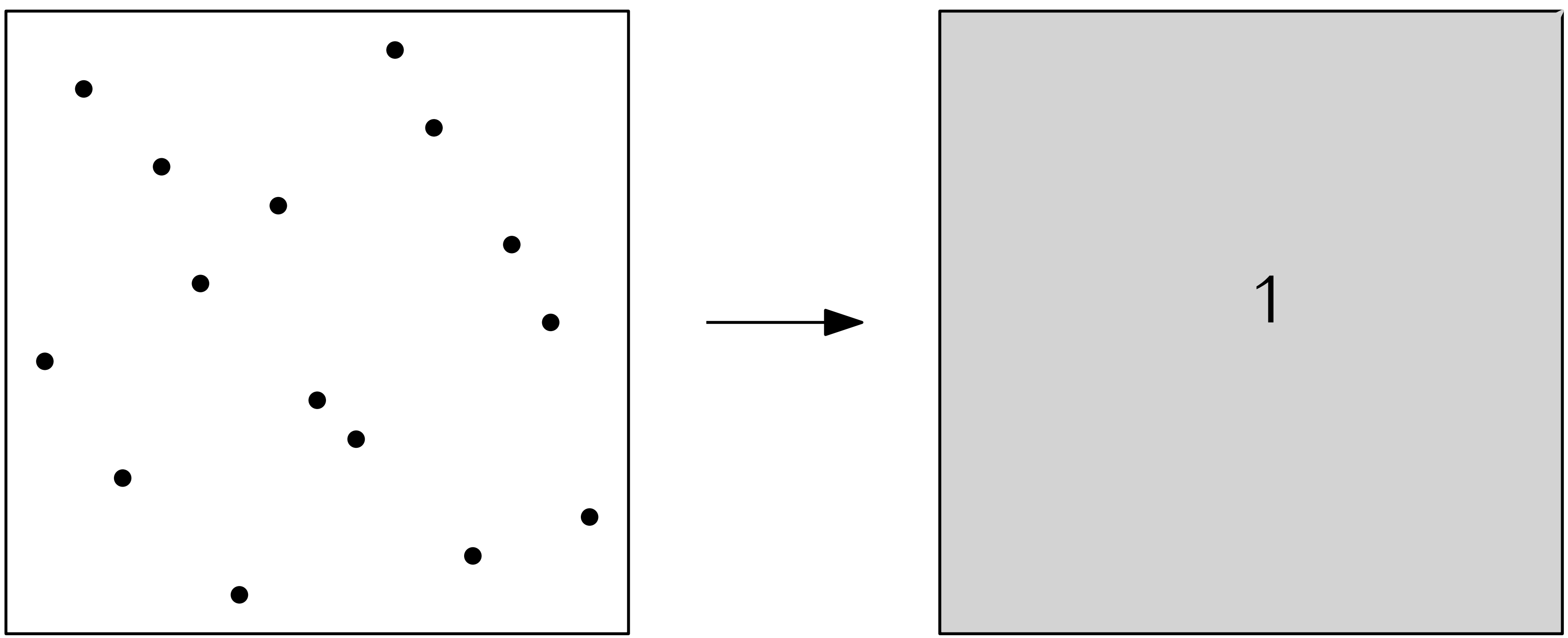
Sequences of random permutations

If σ_n is a sequence of random permutations, we can consider the convergence in distribution of the random permutons μ_{σ_n}.
Let $\sigma_n =$ uniform of size n in some class $\mathcal{C}$.

Sequences of random permutations

If σ_n is a sequence of random permutations, we can consider the convergence in distribution of the random permutons μ_{σ_n}.
Let $\sigma_n =$ uniform of size n in some class $\mathcal{C}$.

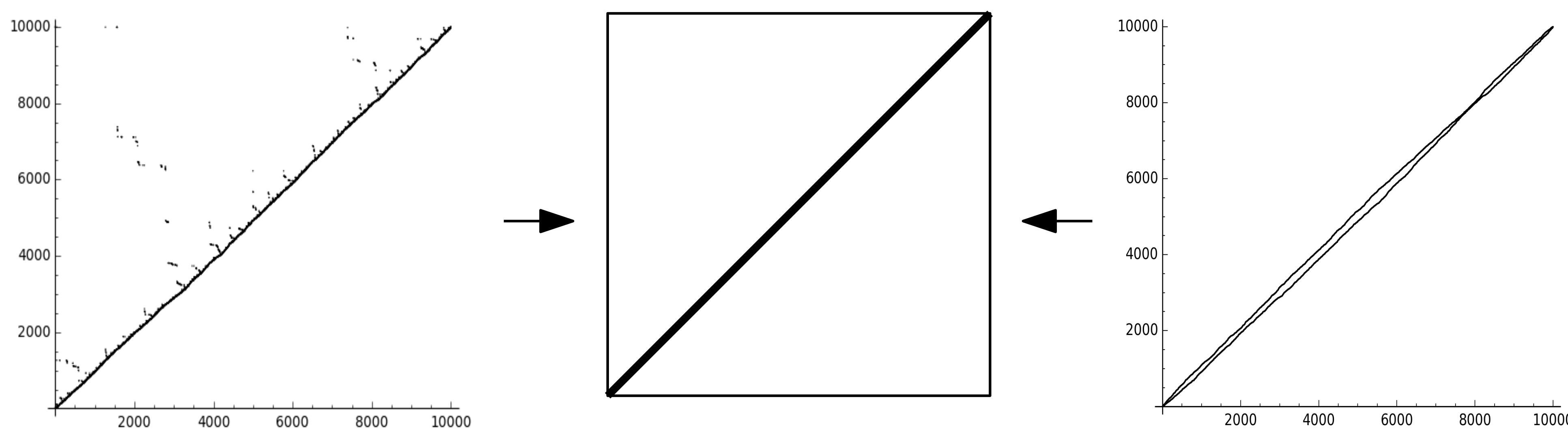
$\mathcal{C} = \mathfrak{S}$: $\sigma_n \xrightarrow{\mathbb{P}} \mathrm{Leb}_{[0,1]^2}$.



Sequences of random permutations

If σ_n is a sequence of random permutations, we can consider the convergence in distribution of the random permutons μ_{σ_n}.
Let σ_n = uniform of size n in some class $\mathcal{C}$.

$$\mathcal{C} = \mathrm{Av}(231) \text{ or } \mathrm{Av}(321) : \ \sigma_n \xrightarrow{\mathbb{P}} (\mathrm{id}, \mathrm{id})_* \mathrm{Leb}_{[0,1]}.$$

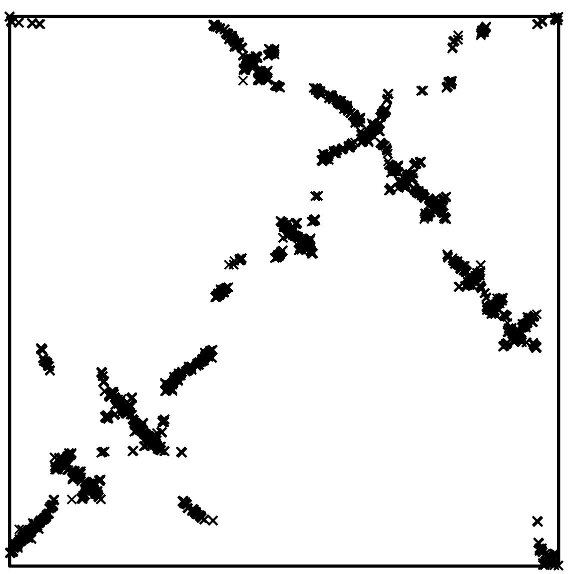


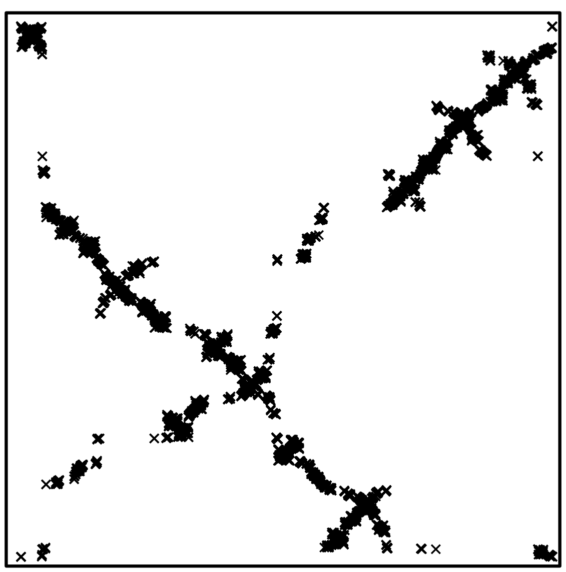
Fluctuations: Miner-Pak, Hoffman-Rizzolo-Slivken... Pictures from the latter.

Sequences of random permutations

If σ_n is a sequence of random permutations, we can consider the convergence in distribution of the random permutons μ_{σ_n}.
Let σ_n = uniform of size n in some class $\mathcal{C}$.

$\mathcal{C} = \mathrm{Av}(2413, 3142) = \{\text{separables}\}$:
Theorem (Bassino, Bouvel, Féray, Gerin, Pierrot 2016)
σ_n converges in distribution to some random permuton μ, called the Brownian separable permuton.





A portmanteau theorem for random permutons

Theorem (Bassino, Bouvel, Feray, Gerin, M., Pierrot. 2017)
The following are equivalent:

1. The random measure μ_{σ_n} converges in distribution to some random permuton μ.
2. $\mathbb{P}((\mathrm{subperm}_k(\sigma_n))_k \in \cdot \mid \sigma_n)$ converges in distribution,
3. $\mathrm{subperm}_k(\sigma_n) \xrightarrow[n\to\infty]{d} \beta_k$ random in $\mathfrak{S}_k$ for every k

Moreover, the law of μ is characterized by

$$\mathrm{subperm}_k(\mu) = \beta_k, k \geq 1.$$

A portmanteau theorem for random permutons

Theorem (Bassino, Bouvel, Feray, Gerin, M., Pierrot. 2017)
The following are equivalent:

1. The random measure μ_{σ_n} converges in distribution to some random permuton μ.
2. $\mathbb{P}((\mathrm{subperm}_k(\sigma_n))_k \in \cdot \mid \sigma_n)$ converges in distribution,
3. $\mathrm{subperm}_k(\sigma_n) \xrightarrow[n\to\infty]{d} \beta_k$ random in $\mathfrak{S}_k$ for every k

Moreover, the law of μ is characterized by

$$\mathrm{subperm}_k(\mu) = \beta_k, k \geq 1.$$

Idea of proof

Use the bijection with signed Schröder trees: $\sigma_n = \mathrm{perm}(t_n)$, where t_n is a uniform signed Schröder tree with n leaves.

Idea of proof

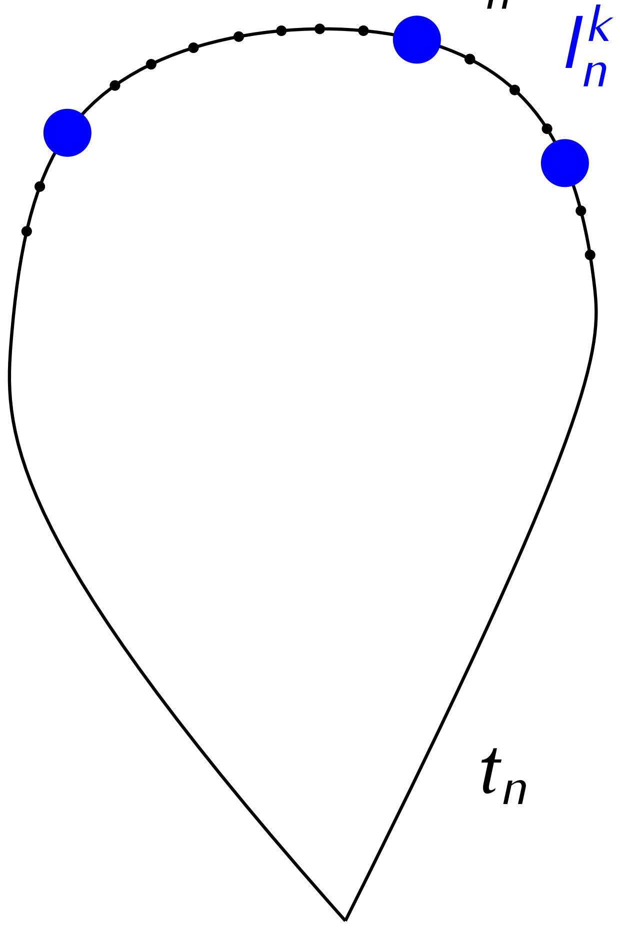
Use the bijection with signed Schröder trees: $\sigma_n = \mathrm{perm}(t_n)$, where t_n is a uniform signed Schröder tree with n leaves.

Fix $k(=3)$. Then $\mathrm{subperm}_k(\sigma_n) = \mathrm{pat}_{I_n^k}(\sigma_n) = \mathrm{perm}(t_{n|I_n^k})$, where $t_{n|I_n^k}$ is the reduced subtree of t_n induced by the leaves with indexes in I_n^k.

Idea of proof

Use the bijection with signed Schröder trees: $\sigma_n = \mathrm{perm}(t_n)$, where t_n is a uniform signed Schröder tree with n leaves.

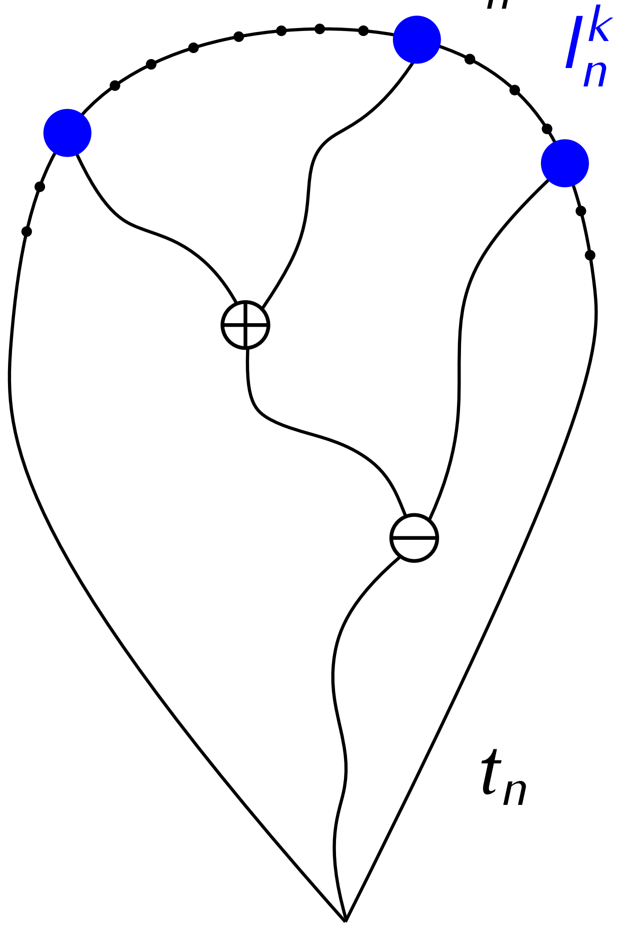
Fix $k(=3)$. Then $\mathrm{subperm}_k(\sigma_n) = \mathrm{pat}_{I_n^k}(\sigma_n) = \mathrm{perm}(t_{n|I_n^k})$, where $t_{n|I_n^k}$ is the reduced subtree of t_n induced by the leaves with indexes in I_n^k.



Idea of proof

Use the bijection with signed Schröder trees: $\sigma_n = \mathrm{perm}(t_n)$, where t_n is a uniform signed Schröder tree with n leaves.

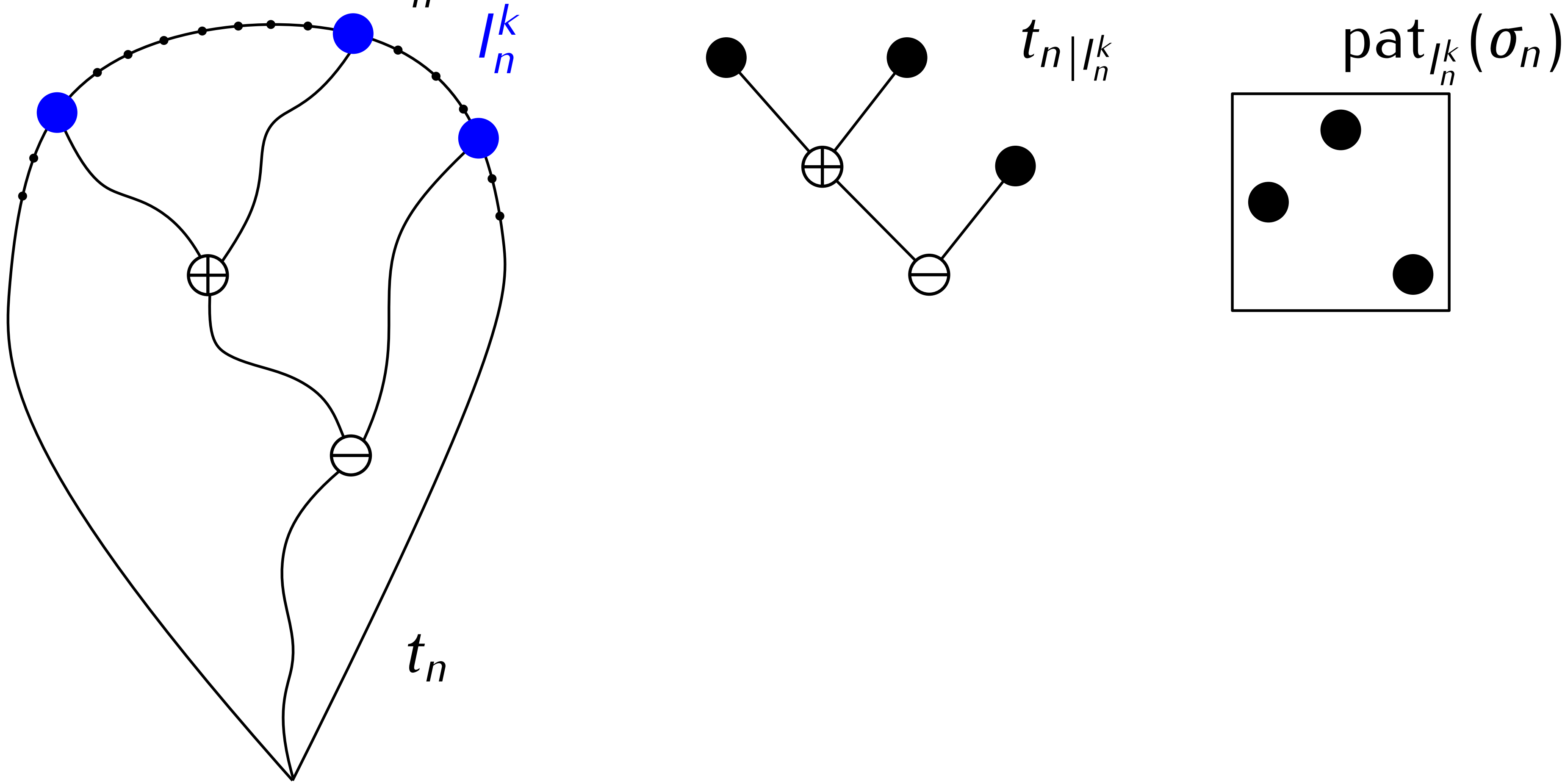
Fix $k(=3)$. Then $\mathrm{subperm}_k(\sigma_n) = \mathrm{pat}_{I_n^k}(\sigma_n) = \mathrm{perm}(t_{n|I_n^k})$, where $t_{n|I_n^k}$ is the reduced subtree of t_n induced by the leaves with indexes in I_n^k.



Idea of proof

Use the bijection with signed Schröder trees: $\sigma_n = \mathrm{perm}(t_n)$, where t_n is a uniform signed Schröder tree with n leaves.

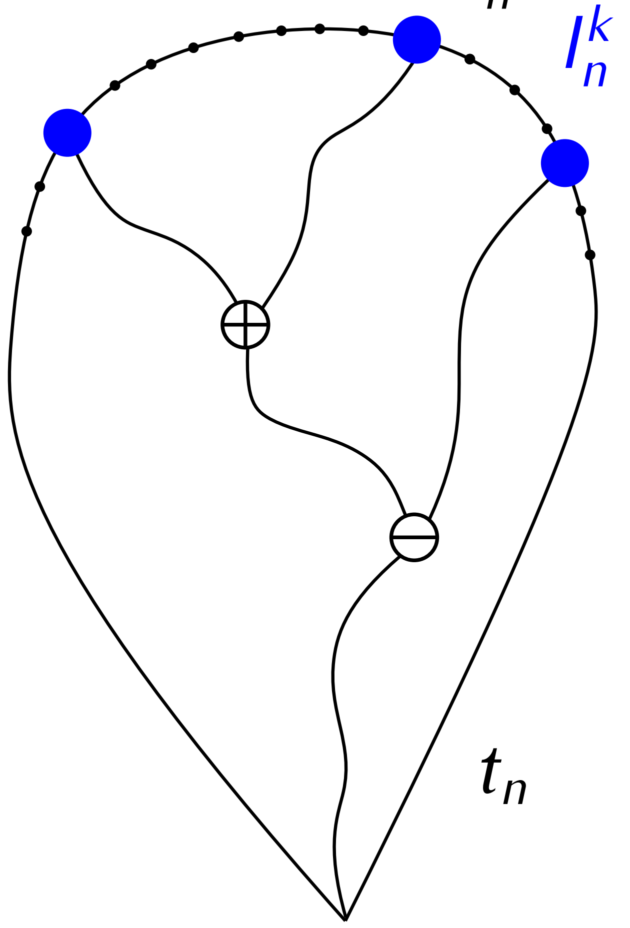
Fix $k(=3)$. Then $\mathrm{subperm}_k(\sigma_n) = \mathrm{pat}_{I_n^k}(\sigma_n) = \mathrm{perm}(t_{n|I_n^k})$, where $t_{n|I_n^k}$ is the reduced subtree of t_n induced by the leaves with indexes in I_n^k.

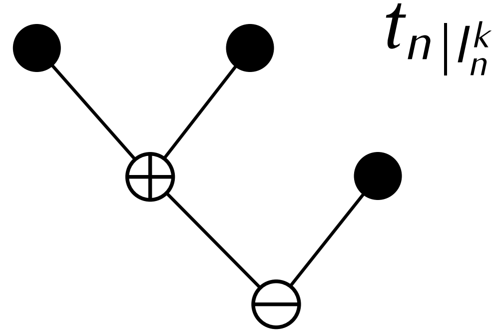


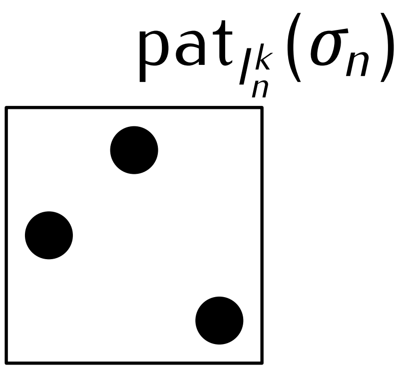
Idea of proof

Use the bijection with signed Schröder trees: $\sigma_n = \mathrm{perm}(t_n)$, where t_n is a uniform signed Schröder tree with n leaves.

Fix $k(=3)$. Then $\mathrm{subperm}_k(\sigma_n) = \mathrm{pat}_{I_n^k}(\sigma_n) = \mathrm{perm}(t_{n|I_n^k})$, where $t_{n|I_n^k}$ is the reduced subtree of t_n induced by the leaves with indexes in I_n^k.





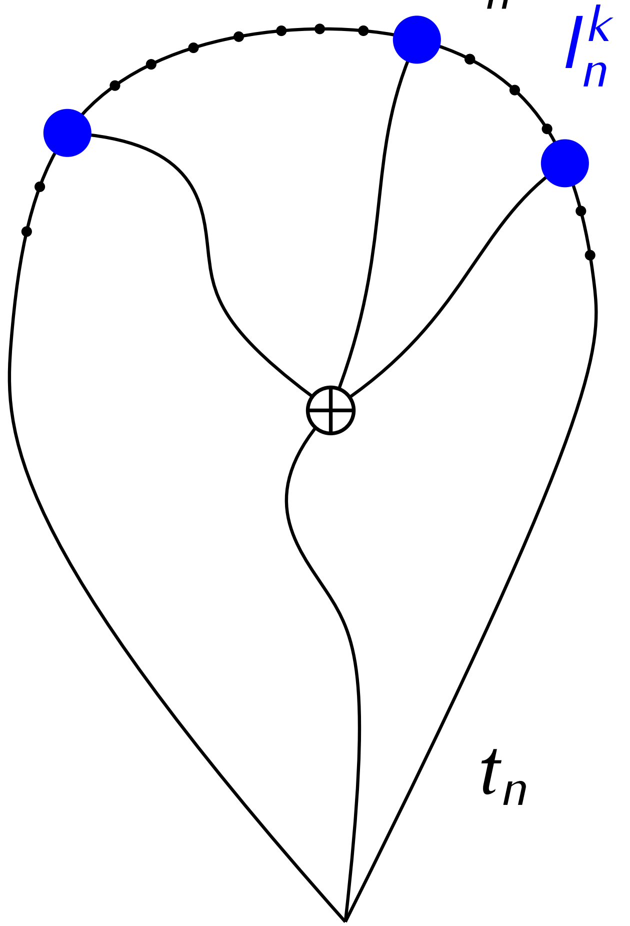


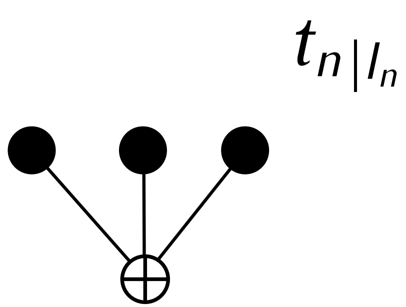
What does it look like as $n \to \infty$?

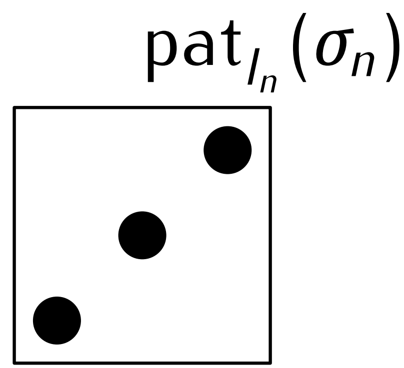
Idea of proof

Use the bijection with signed Schröder trees: $\sigma_n = \mathrm{perm}(t_n)$, where t_n is a uniform signed Schröder tree with n leaves.

Fix $k(=3)$. Then $\mathrm{subperm}_k(\sigma_n) = \mathrm{pat}_{I_n^k}(\sigma_n) = \mathrm{perm}(t_{n|I_n^k})$, where $t_{n|I_n^k}$ is the reduced subtree of t_n induced by the leaves with indexes in I_n^k.







What does it look like as $n \to \infty$?

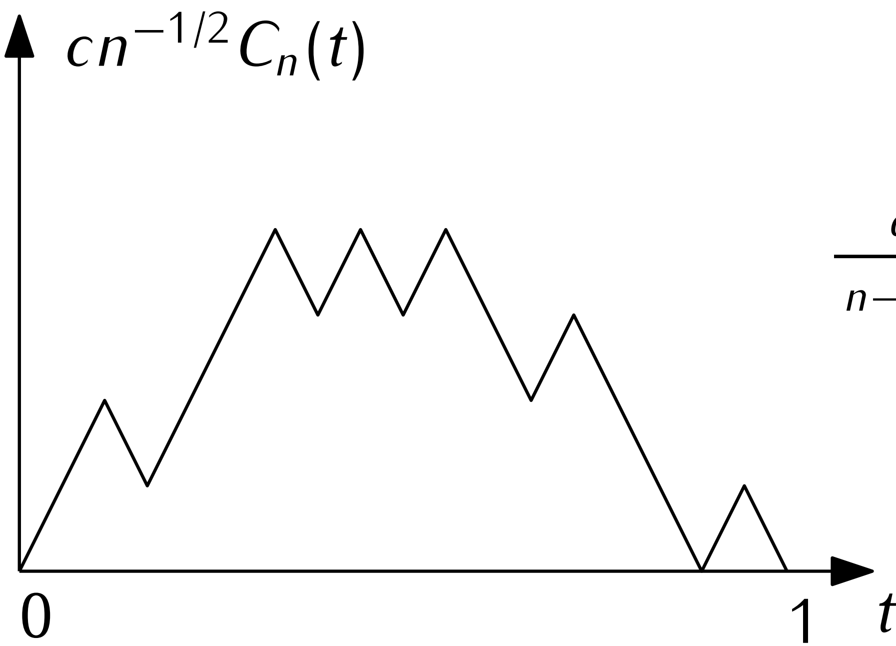
Scaling limits of trees

Scaling limits of trees

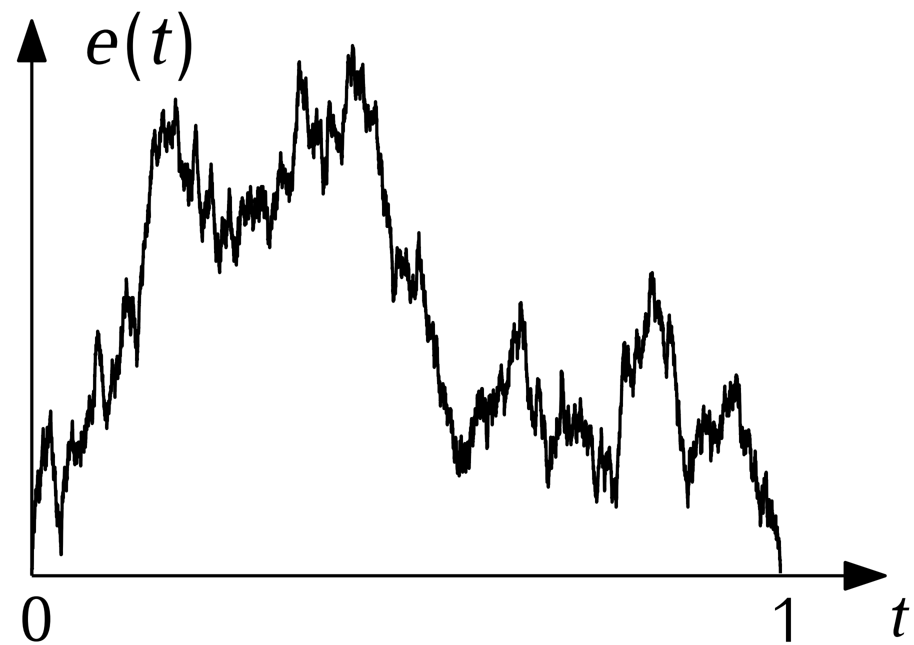
Many "nice" models of random trees $(t_n)_n$ where n is some size parameter, converge to (a multiple of) the Brownian CRT at $\sqrt{n}$. More precisely, if C_n is the contour function of t_n, for some constant $c > 0$, $cn^{-1/2}C_n$ converges in distribution to the normalized Brownian excursion.

Scaling limits of trees

Many "nice" models of random trees $(t_n)_n$ where n is some size parameter, converge to (a multiple of) the Brownian CRT at $\sqrt{n}$. More precisely, if C_n is the contour function of t_n, for some constant $c > 0$, $cn^{-1/2}C_n$ converges in distribution to the normalized Brownian excursion.

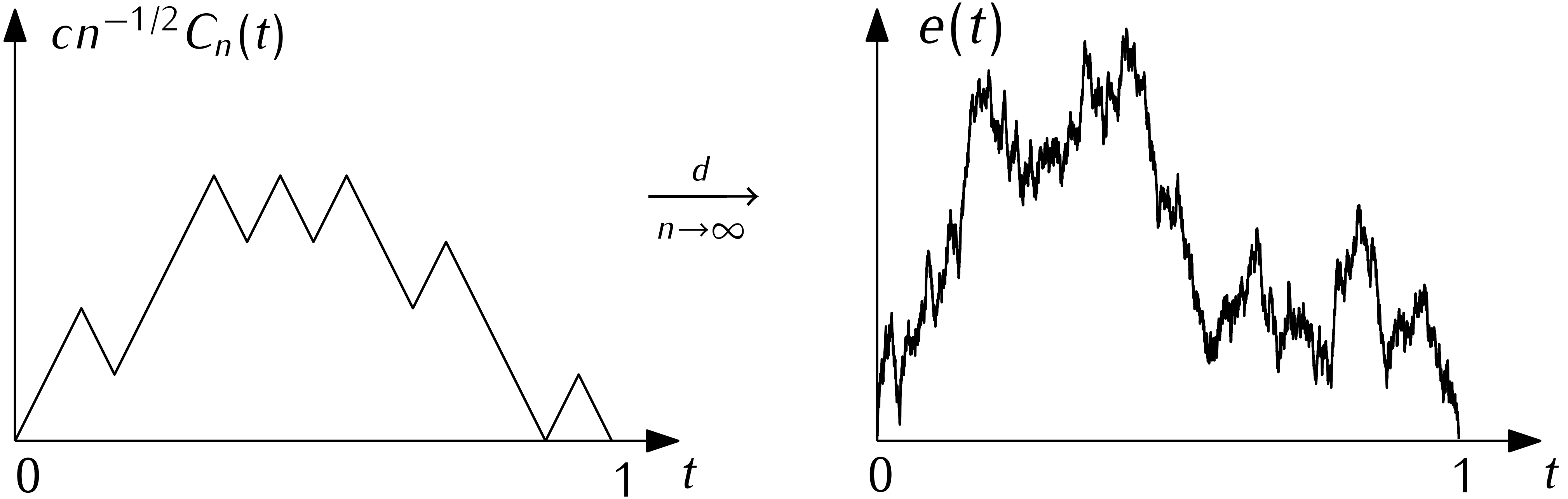


$$\xrightarrow[n\to\infty]{d}$$



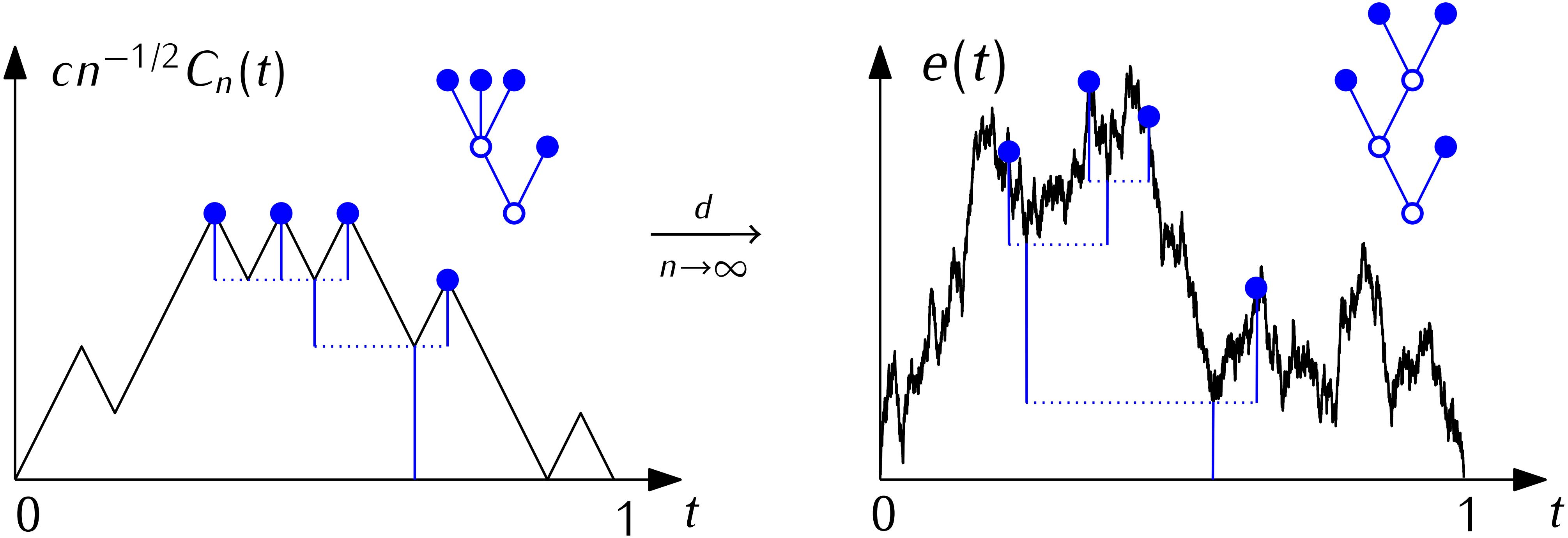
Scaling limits of trees

Leaf-counted Schröder trees are (critical, finite-variance) BGW trees conditioned on the number of leaves and fall in this category (Kortchemski ’12)



Scaling limits of trees

So uniform extracted subtrees from t_n converge to uniform extracted subtrees from the Brownian excursion, which are uniform binary trees (Aldous '93, Le Gall '93)



Signs in extracted subtrees

Since Schröder trees are alternating-signs, this boils down to parity of branches lengths in the extracted subtree.
(CLT in the previous slide $\implies$ lengths of order $\sqrt{n}$, but we need microscopic information)

Signs in extracted subtrees

Since Schröder trees are alternating-signs, this boils down to parity of branches lengths in the extracted subtree.
(CLT in the previous slide $\implies$ lengths of order $\sqrt{n}$, but we need microscopic information)

We can show that signs are **asymptotically balanced and independent** using

- a neat trick (as done in the 2016 paper)

Signs in extracted subtrees

Since Schröder trees are alternating-signs, this boils down to parity of branches lengths in the extracted subtree.
(CLT in the previous slide $\implies$ lengths of order $\sqrt{n}$, but we need microscopic information)

We can show that signs are **asymptotically balanced and independent** using

- a neat trick (as done in the 2016 paper)
- exact combinatorial formulas for trees with a marked leaf at a certain height

Signs in extracted subtrees

Since Schröder trees are alternating-signs, this boils down to parity of branches lengths in the extracted subtree.
(CLT in the previous slide $\implies$ lengths of order $\sqrt{n}$, but we need microscopic information)

We can show that signs are **asymptotically balanced and independent** using

- a neat trick (as done in the 2016 paper)
- exact combinatorial formulas for trees with a marked leaf at a certain height
- analytic combinatorics (used for our subsequent generalization, see part 2)

Summing up...

We have shown that if σ_n is a uniform separable permutation of size n, $\text{subperm}_k(\sigma_n)$ converges in distribution to $\text{perm}(\tau_k)$, where τ_k is a uniform signed binary tree with k leaves.

Summing up...

We have shown that if σ_n is a uniform separable permutation of size n, $\text{subperm}_k(\sigma_n)$ converges in distribution to $\text{perm}(\tau_k)$, where τ_k is a uniform signed binary tree with k leaves.

This shows convergence in distribution of σ_n to some permuton μ, called the Brownian separable permuton

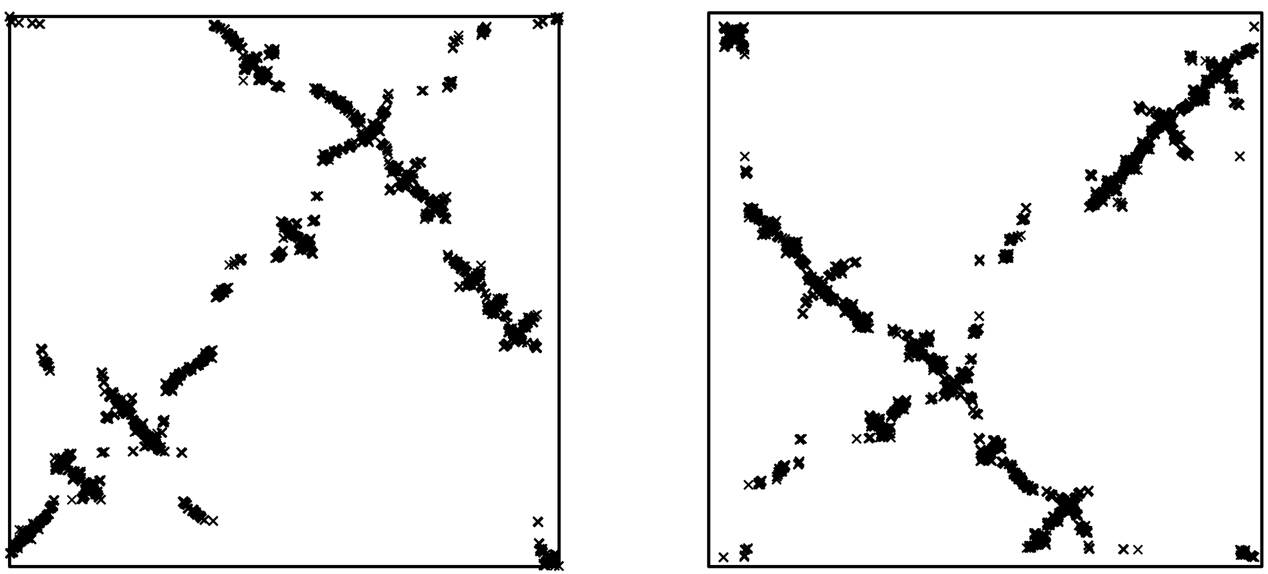
Summing up...

We have shown that if σ_n is a uniform separable permutation of size n, $\mathrm{subperm}_k(\sigma_n)$ converges in distribution to $\mathrm{perm}(\tau_k)$, where τ_k is a uniform signed binary tree with k leaves.

This shows convergence in distribution of σ_n to some permuton μ, called the Brownian separable permuton

This random permuton is characterized in distribution by

$$\forall k \geq 1, \mathrm{subperm}_k(\mu) \stackrel{d}{=} \mathrm{perm}(\tau_k).$$



3 – Universality of permuton limits in substitution-closed classes.

Joint work with F. Bassino, M. Bouvel, V. Féray, L. Gerin and A. Pierrot
[arXiv:1706.08333]

Substitution decomposition

Generalizing $\oplus$ and $\ominus$?

Substitution decomposition

Generalizing $\oplus$ and $\ominus$?

For $\sigma \in \mathfrak{S}_k$, $\rho_1, \ldots, \rho_k \in \mathfrak{S}$, define $\sigma[\rho_1, \ldots, \rho_k]$ by replacing the i-th dot in σ by π_i.

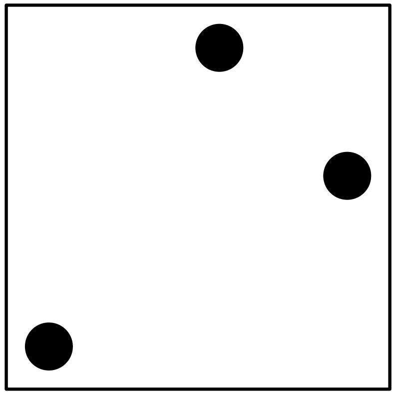
Example : $132[21, 312, 2413] = 219784635$.

Substitution decomposition

Generalizing $\oplus$ and $\ominus$?

For $\sigma \in \mathfrak{S}_k$, $\rho_1, \dots, \rho_k \in \mathfrak{S}$, define $\sigma[\rho_1, \dots, \rho_k]$ by replacing the i-th dot in σ by π_i.

Example : $132[21, 312, 2413] = 219784635$.

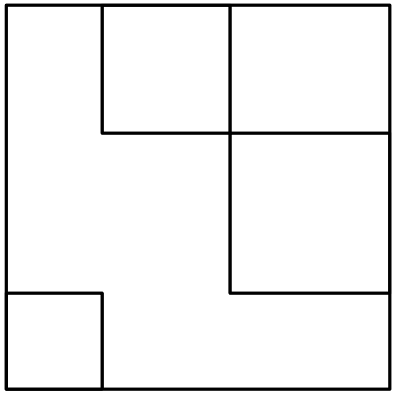


Substitution decomposition

Generalizing $\oplus$ and $\ominus$?

For $\sigma \in \mathfrak{S}_k$, $\rho_1, \ldots, \rho_k \in \mathfrak{S}$, define $\sigma[\rho_1, \ldots, \rho_k]$ by replacing the i–th dot in σ by π_i.

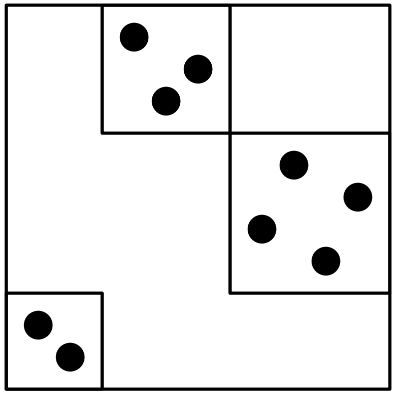
Example : $132[21, 312, 2413] = 219784635$.



Substitution decomposition

Generalizing $\oplus$ and $\ominus$?

For $\sigma \in \mathfrak{S}_k$, $\rho_1, \ldots, \rho_k \in \mathfrak{S}$, define $\sigma[\rho_1, \ldots, \rho_k]$ by replacing the i-th dot in σ by π_i.



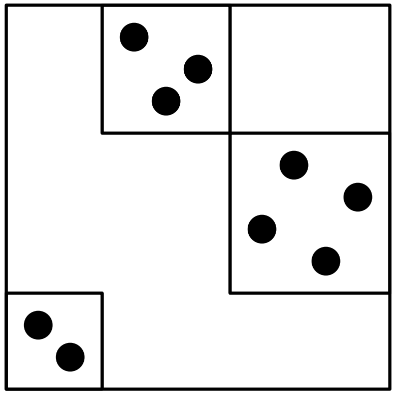
Example : $132[21, 312, 2413] = 219784635$.

Substitution decomposition

Generalizing $\oplus$ and $\ominus$?

For $\sigma \in \mathfrak{S}_k$, $\rho_1, \dots, \rho_k \in \mathfrak{S}$, define $\sigma[\rho_1, \dots, \rho_k]$ by replacing the i-th dot in σ by π_i.

Example : $132[21, 312, 2413] = 219784635$.



$\oplus$ (resp. $\ominus$) is just the substitution into $(12 \cdots r)$ (resp. $(r \cdots 21)$).

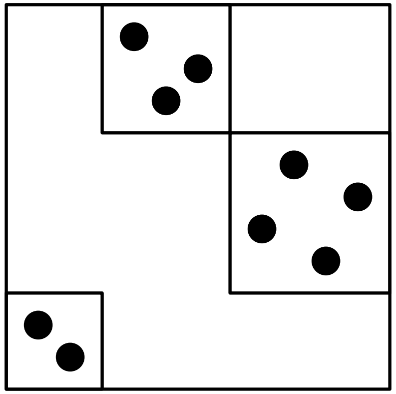
Given σ, either :

Substitution decomposition

Generalizing $\oplus$ and $\ominus$?

For $\sigma \in \mathfrak{S}_k$, $\rho_1, \ldots, \rho_k \in \mathfrak{S}$, define $\sigma[\rho_1, \ldots, \rho_k]$ by replacing the i-th dot in σ by π_i.

Example : $132[21, 312, 2413] = 219784635$.



$\oplus$ (resp. $\ominus$) is just the substitution into $(12 \cdots r)$ (resp. $(r \cdots 21)$).

Given σ, either :

- We can find a proper interval mapped to an interval, and then σ can be written as a substitution of smaller permutations

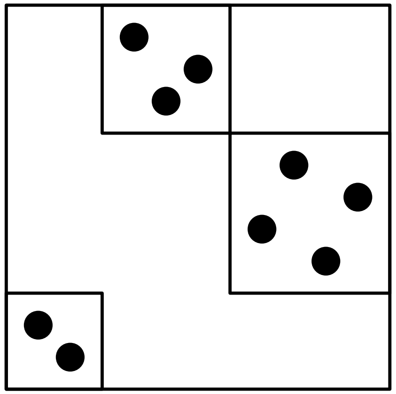
Substitution decomposition

Generalizing $\oplus$ and $\ominus$?

For $\sigma \in \mathfrak{S}_k$, $\rho_1, \ldots, \rho_k \in \mathfrak{S}$, define $\sigma[\rho_1, \ldots, \rho_k]$ by replacing the i-th dot in σ by π_i.

Example : $132[21, 312, 2413] = 219784635$.

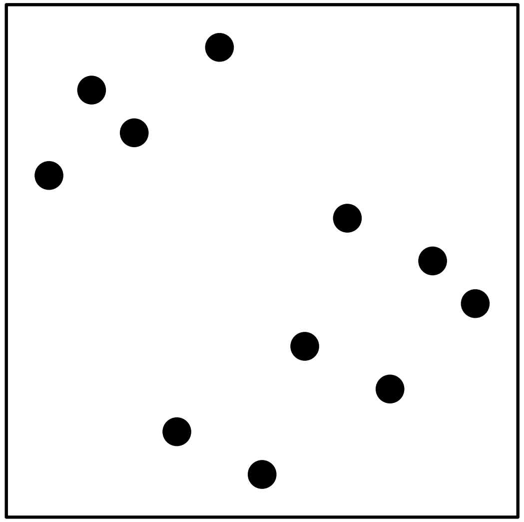
$\oplus$ (resp. $\ominus$) is just the substitution into $(12 \cdots r)$ (resp. $(r \cdots 21)$).



Given σ, either :

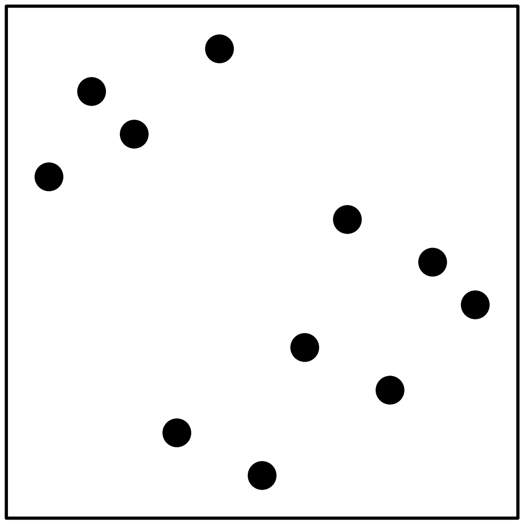
- We can find a proper interval mapped to an interval, and then σ can be written as a substitution of smaller permutations
- Or σ can't be decomposed by a nontrivial substitution : σ is a **simple permutation**. Ex : $1, 12, 21, 2413, 3142, 31524, \ldots \sim \frac{n!}{e^2}$.

Substitution decomposition

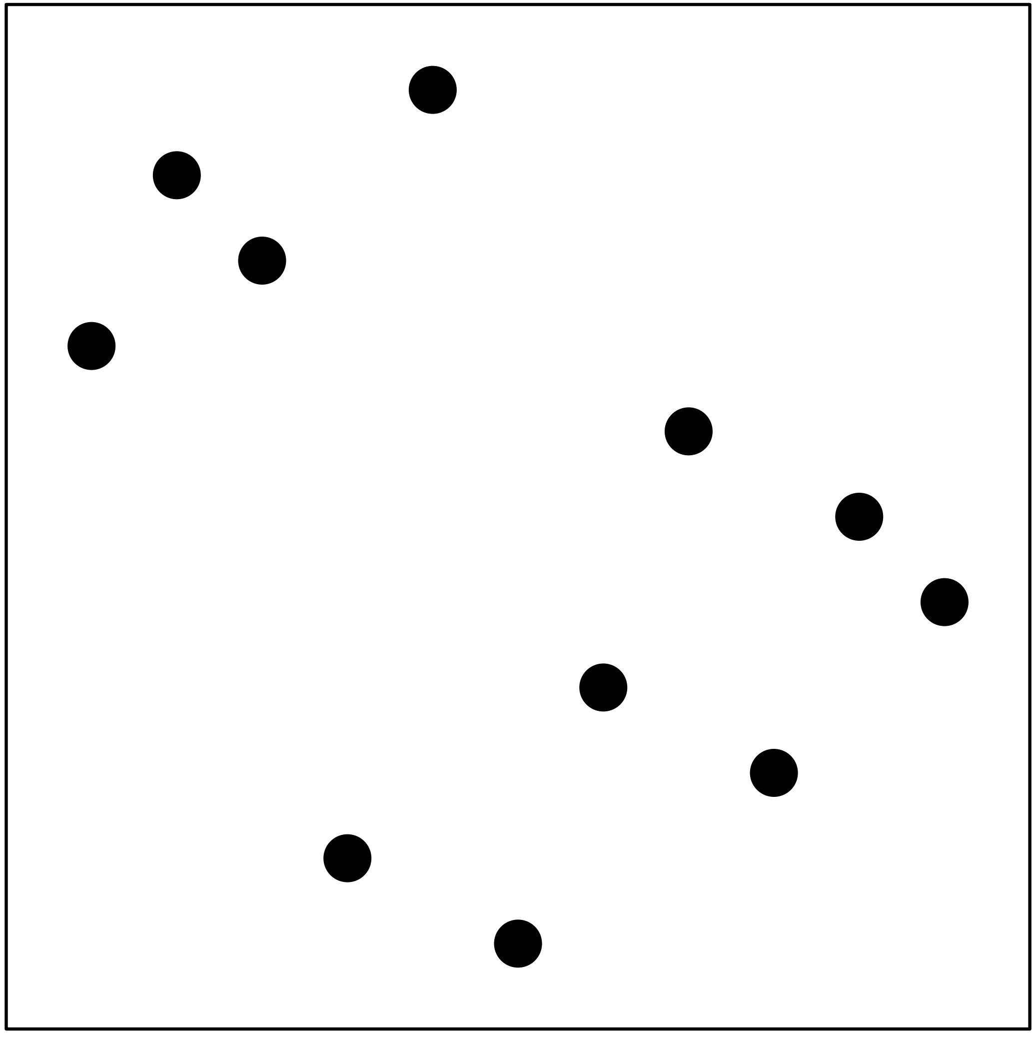


$(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)$

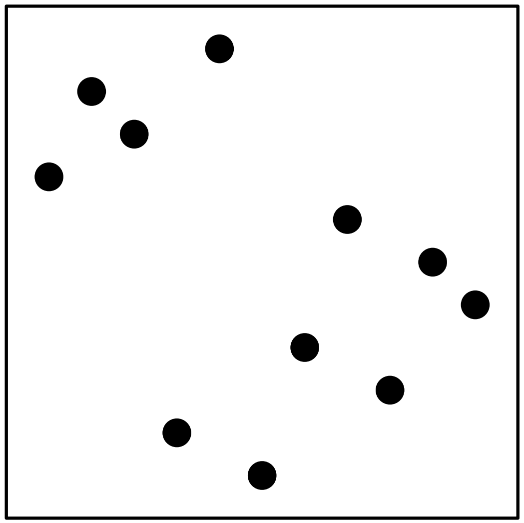
Substitution decomposition



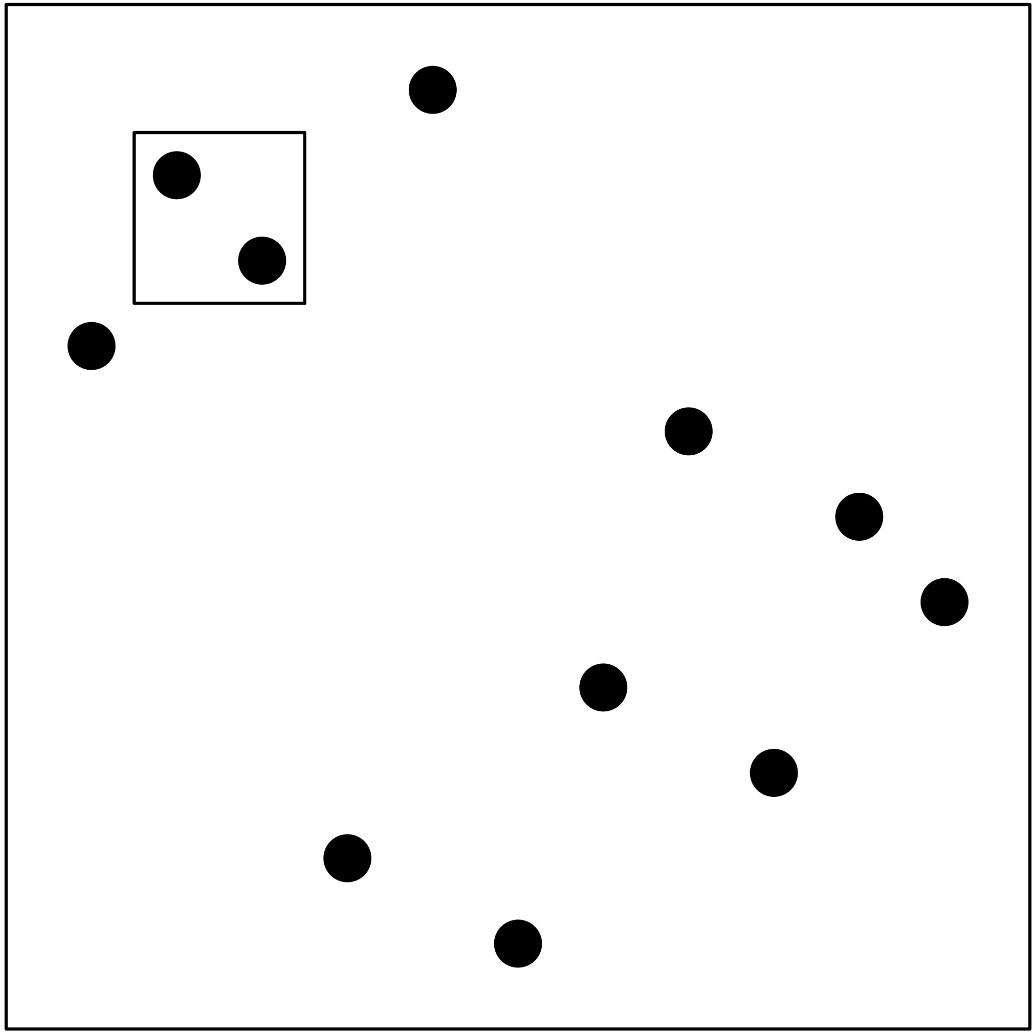
$(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)$



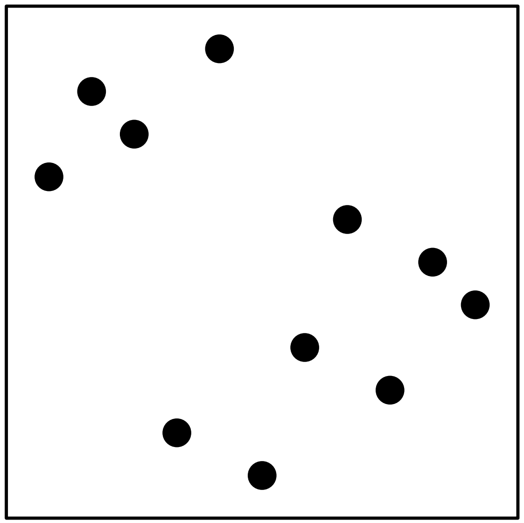
Substitution decomposition



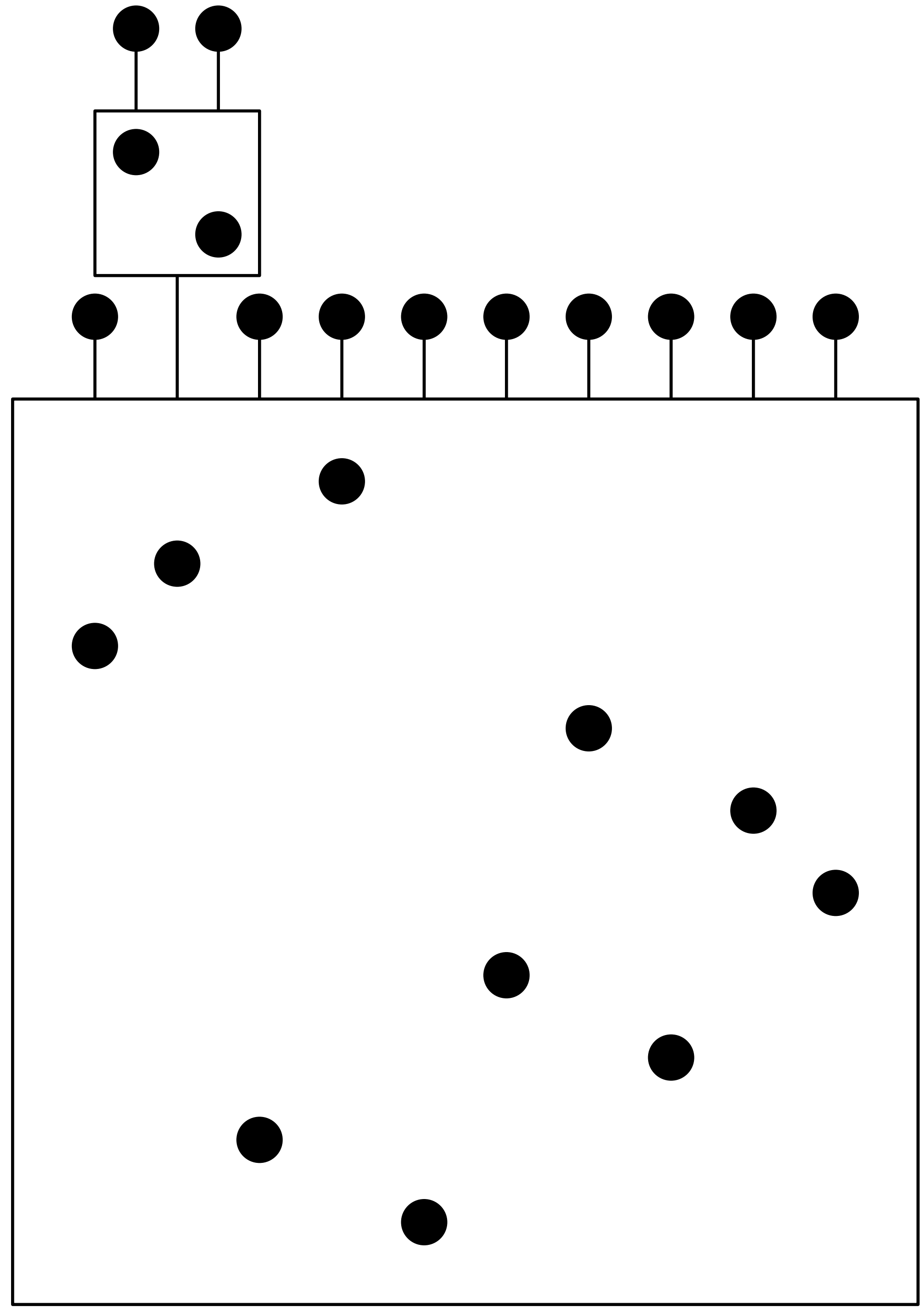
$(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)$



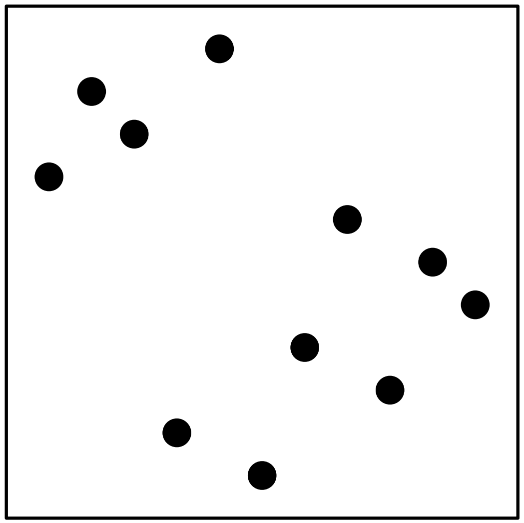
Substitution decomposition



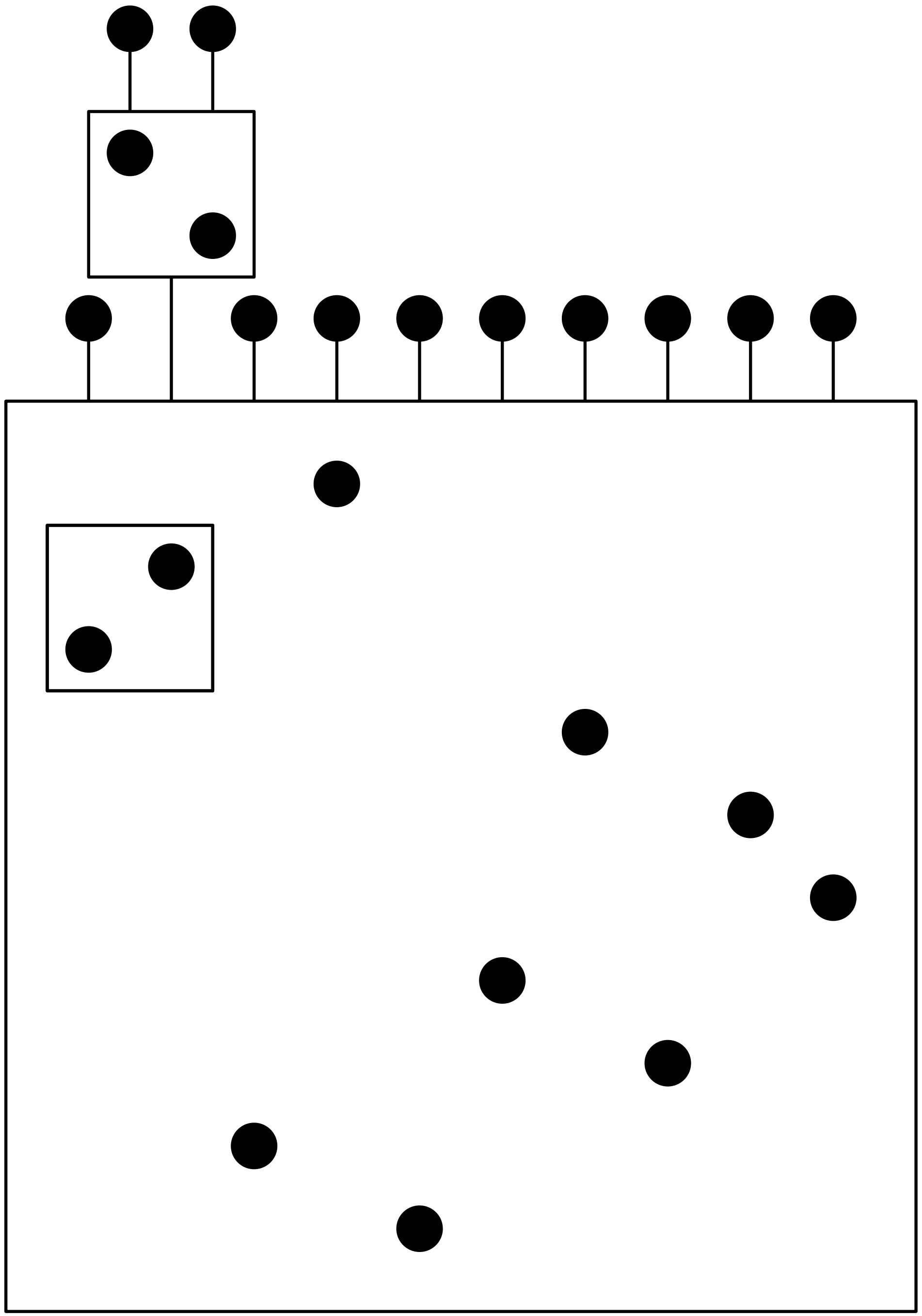
$(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)$



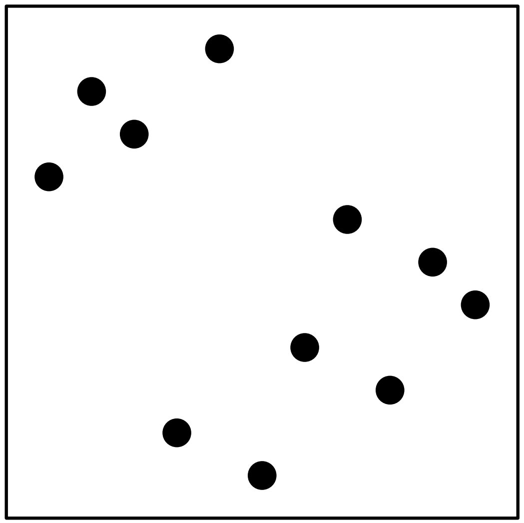
Substitution decomposition



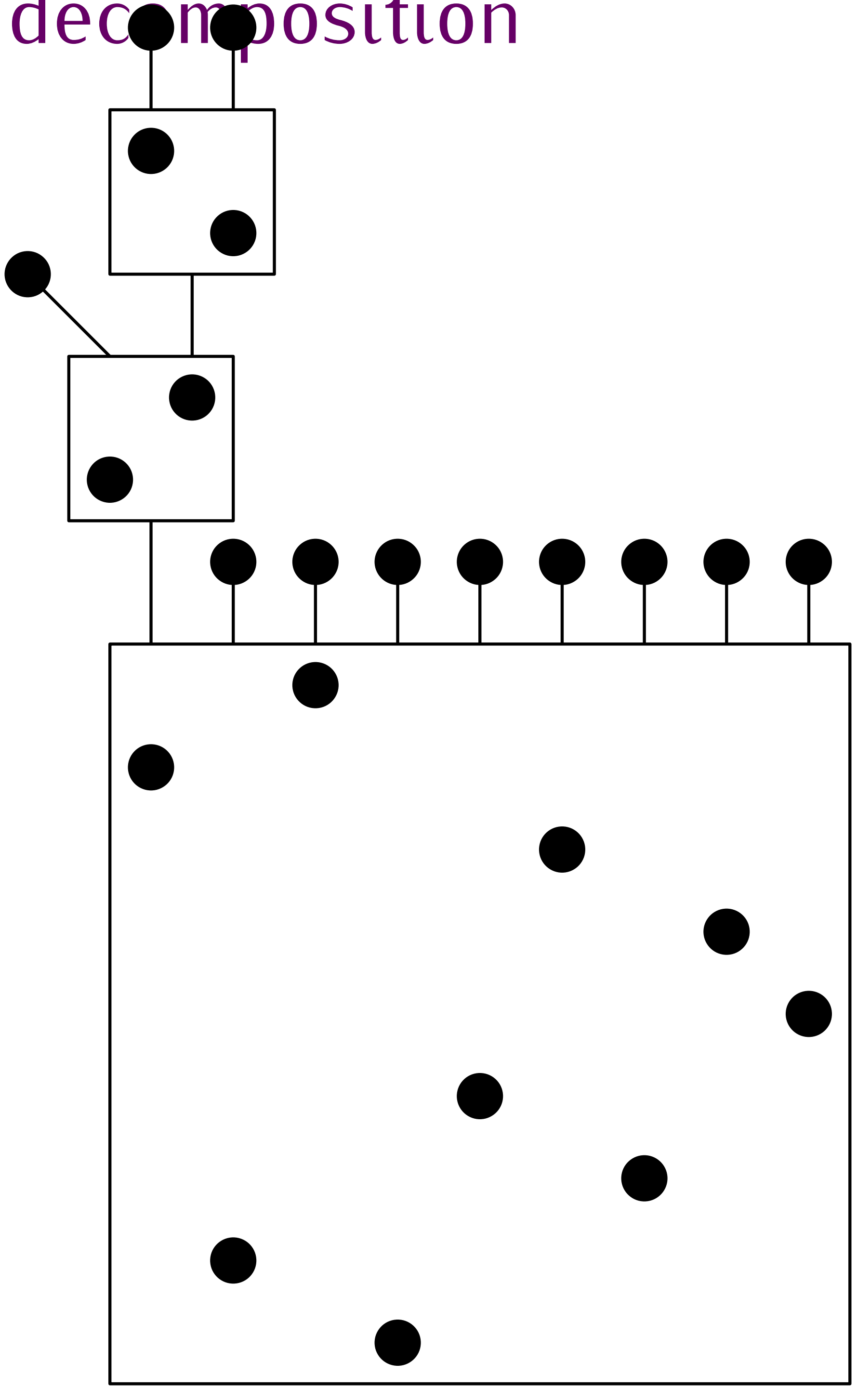
$(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)$



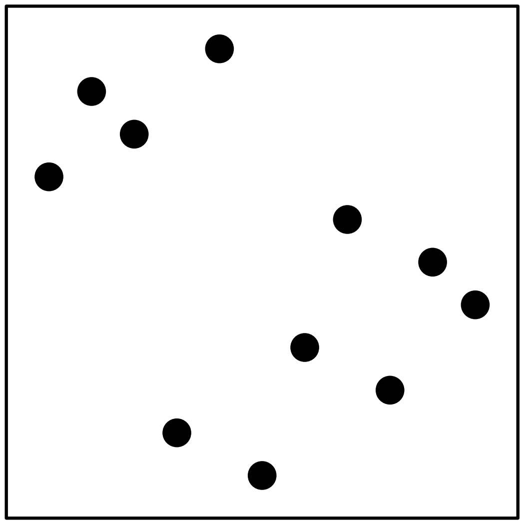
Substitution decomposition



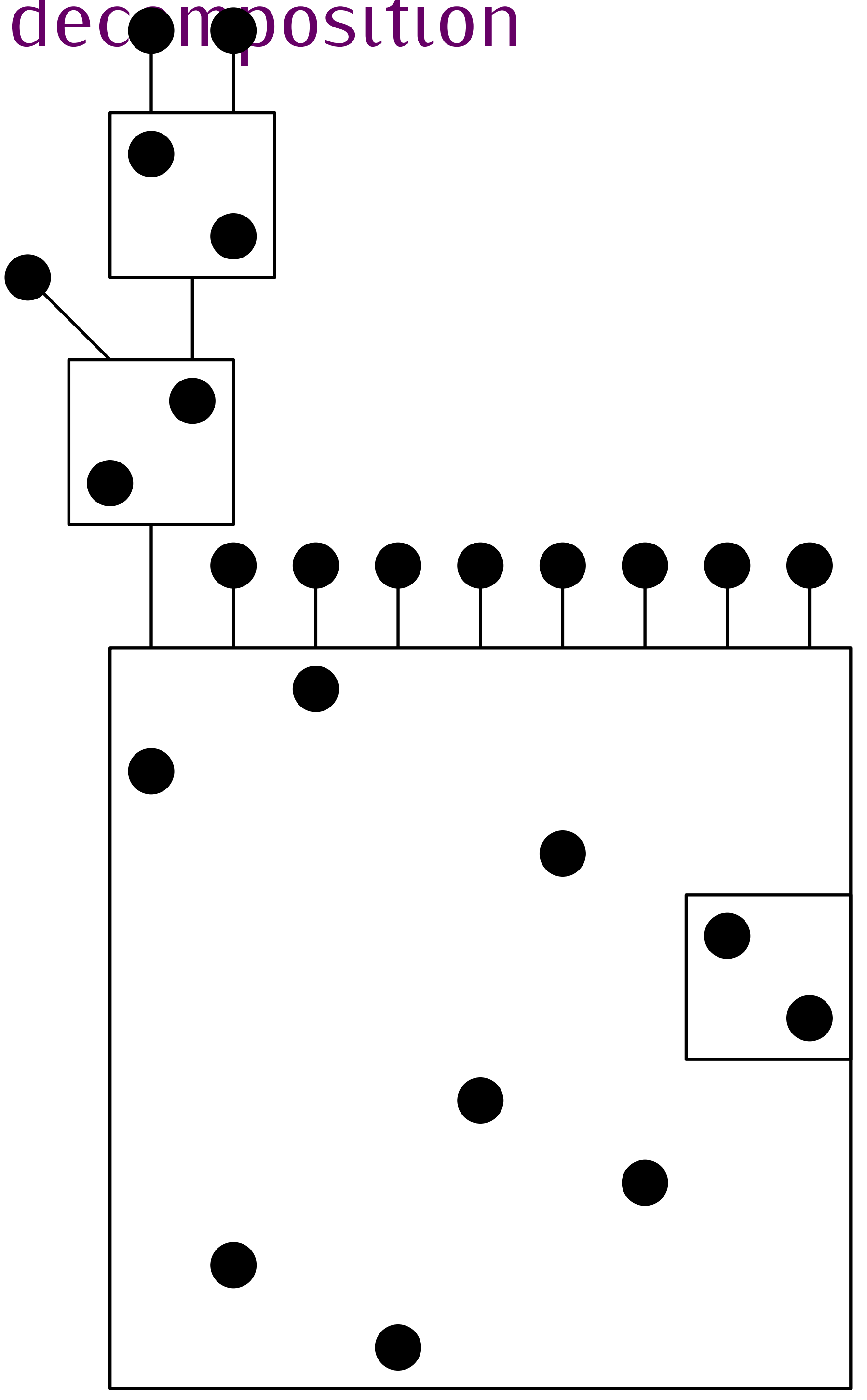
(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)



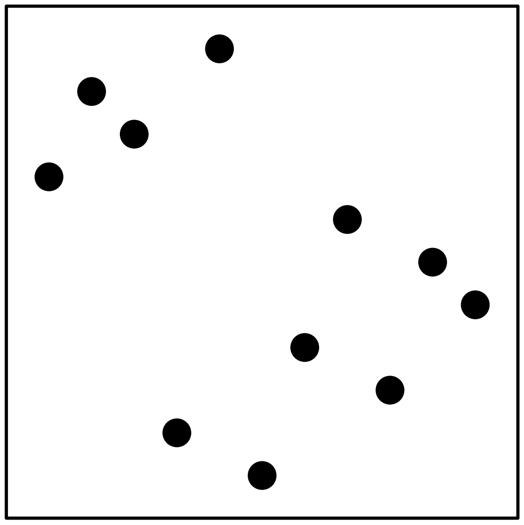
Substitution decomposition



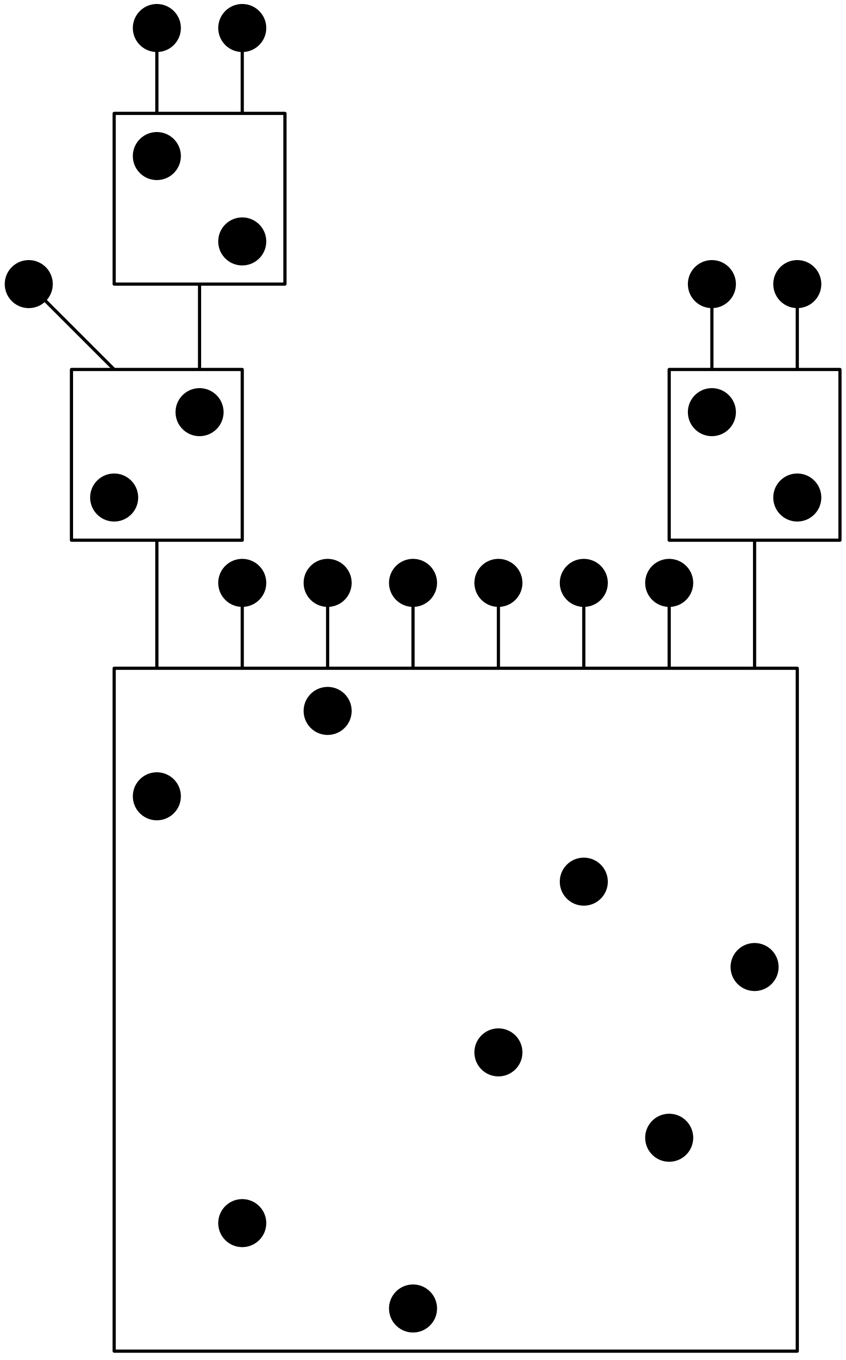
(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)



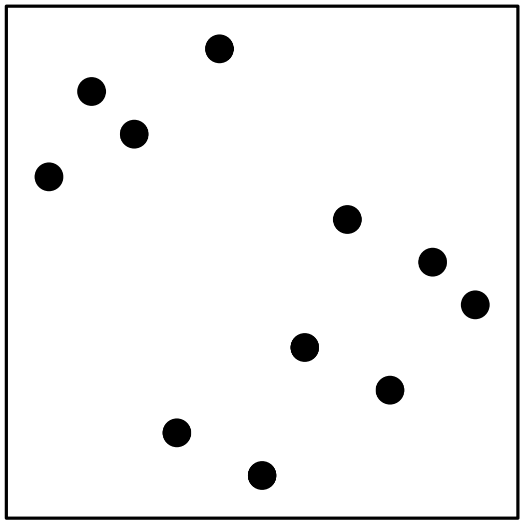
Substitution decomposition



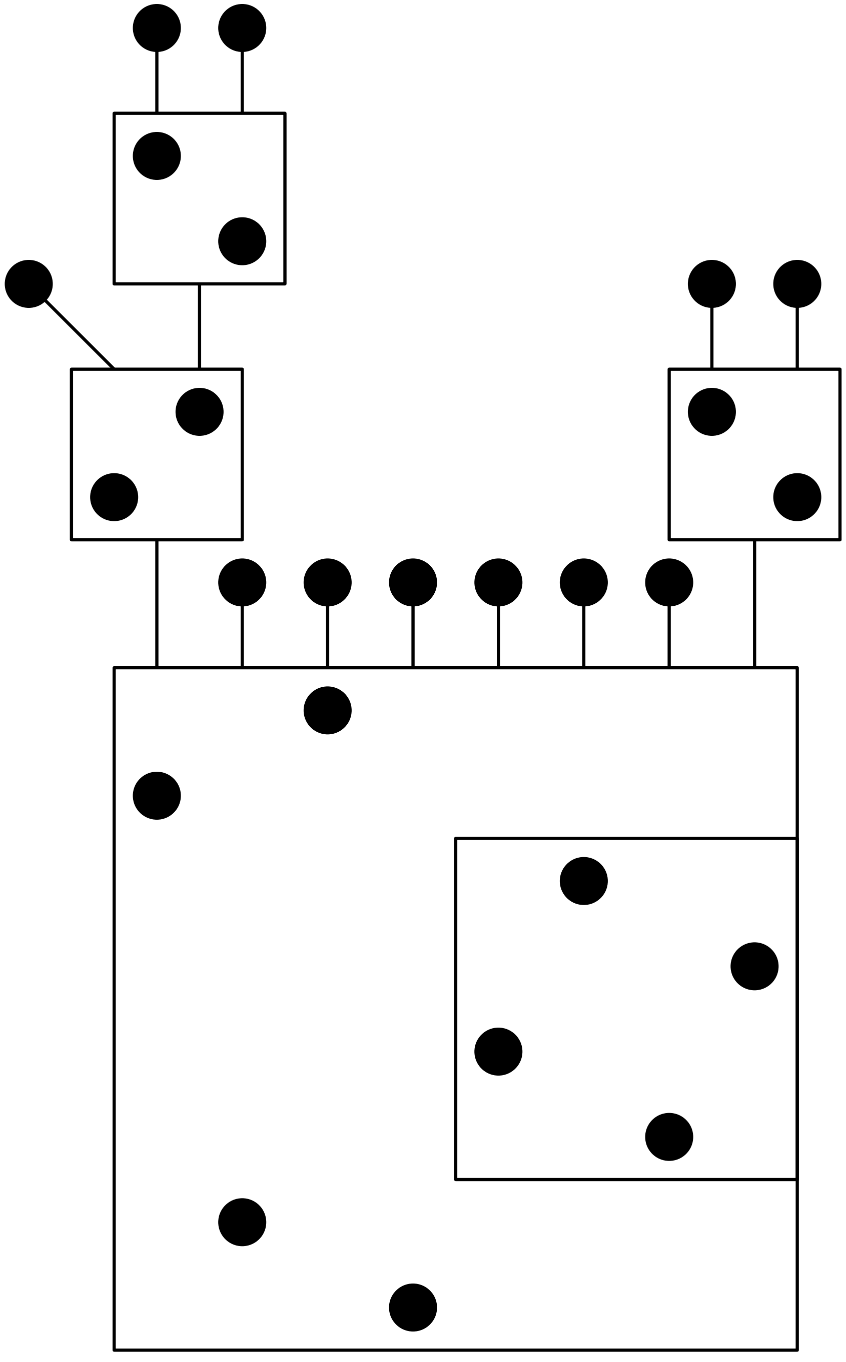
$(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)$



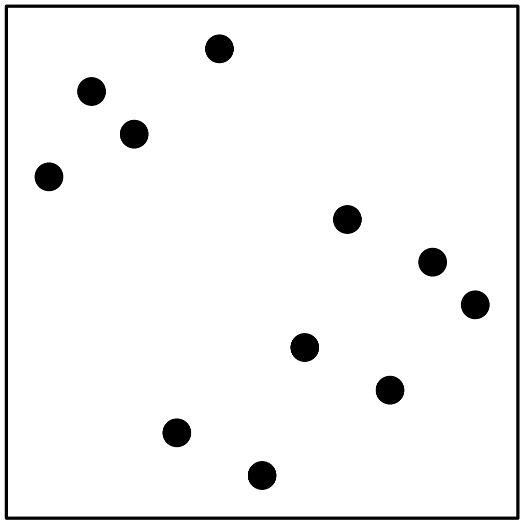
Substitution decomposition



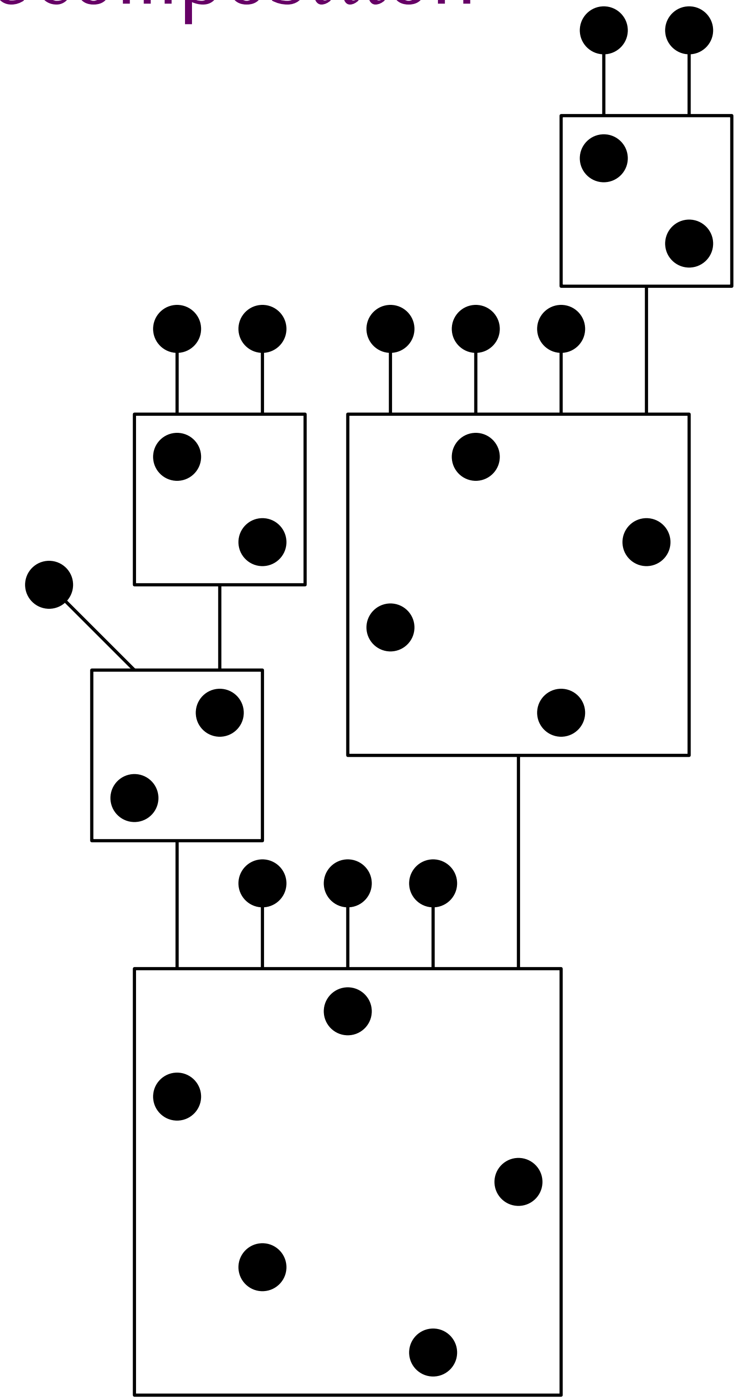
$(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)$



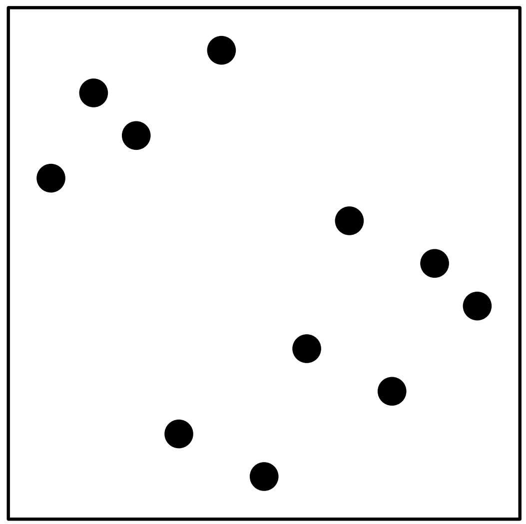
Substitution decomposition



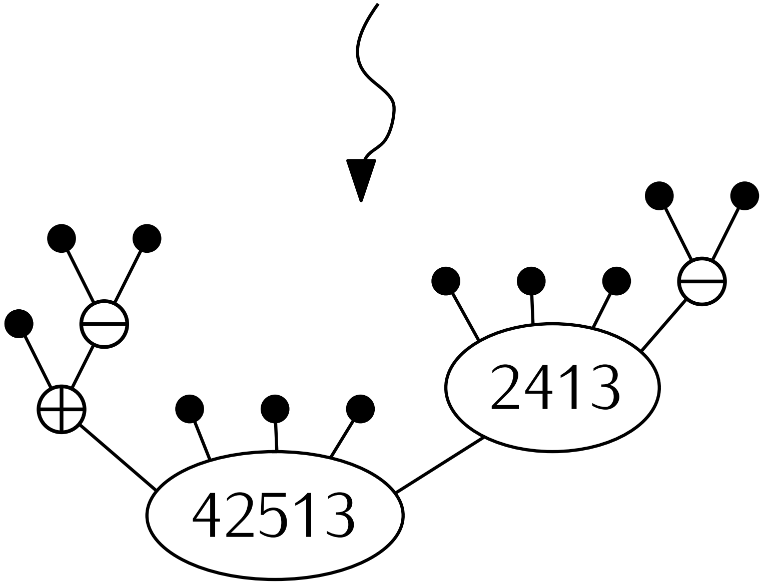
$(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)$

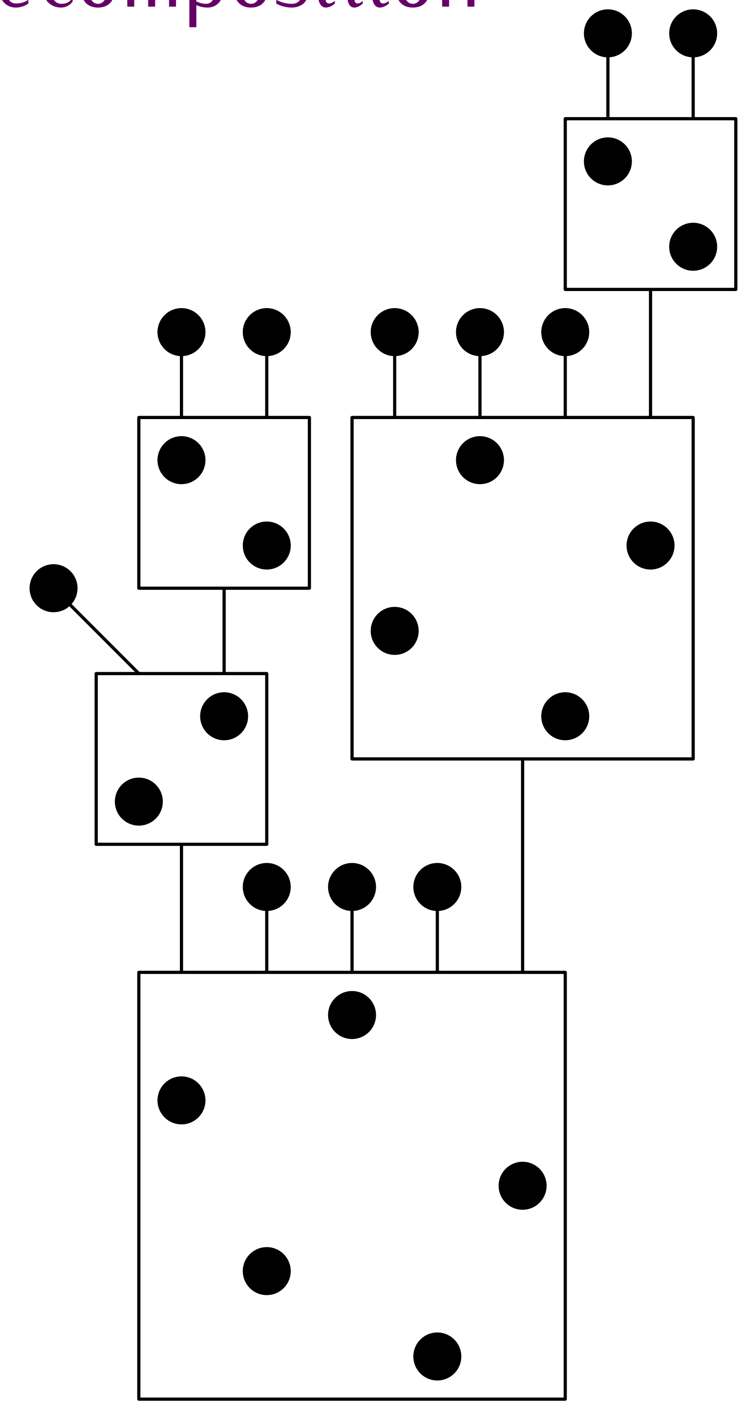


Substitution decomposition

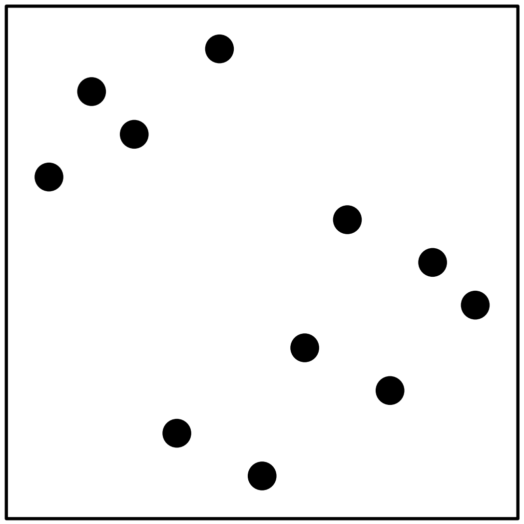


(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)

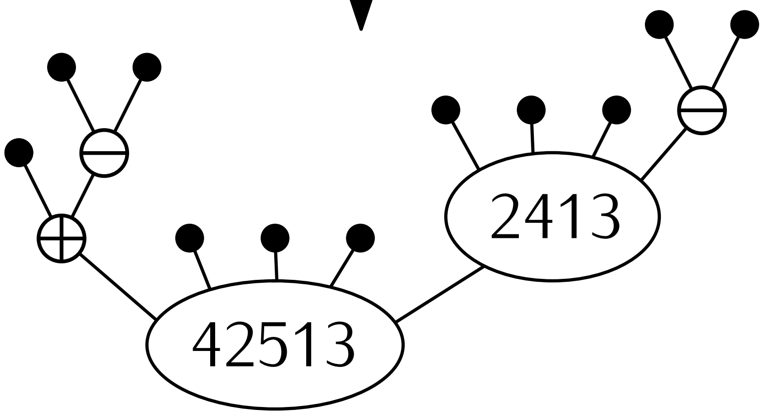




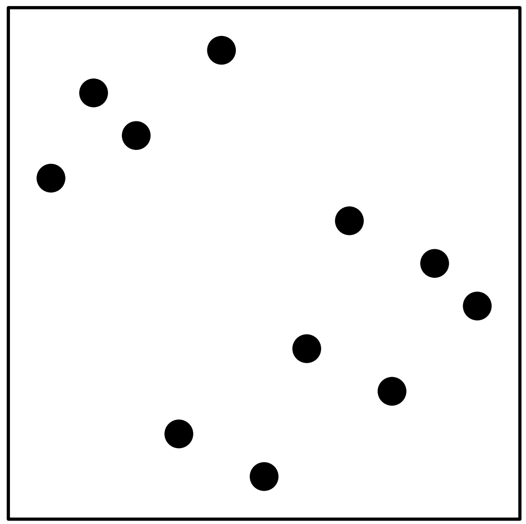
Substitution decomposition



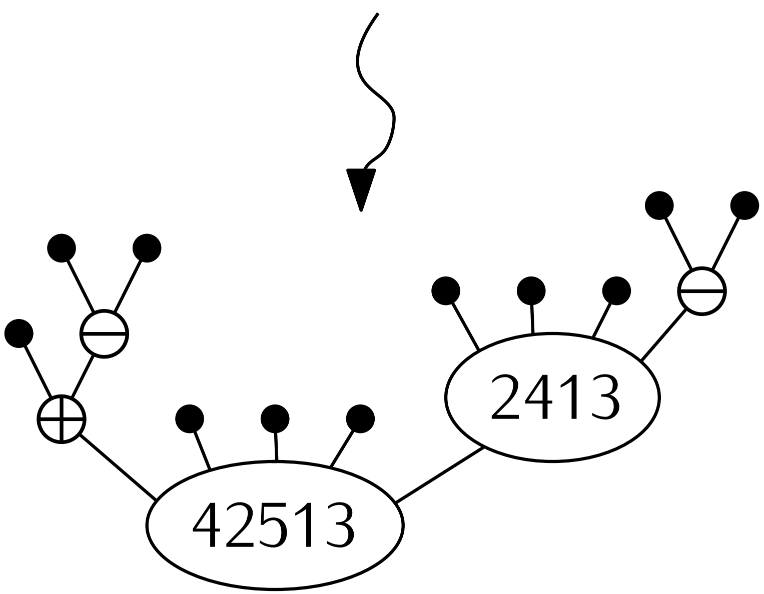
$(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)$



Substitution decomposition



$(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)$



Theorem (Albert, Atkinson 2005): Any permutation can be decomposed into a substitution tree with $\oplus$, $\ominus$ nodes, and simple nodes of length ≥ 4, unique as long as adjacent $\oplus$ and $\ominus$ are merged.

Substitution-closed families

$\mathcal{S} \subset \{\text{simple permutations of length} \geq 4\}$.

Substitution-closed families

$\mathcal{S} \subset \{$simple permutations of length $\geq 4\}$.

$\langle \mathcal{S} \rangle = \{$permutations built by substituting $\bullet, \oplus, \ominus$, and $\alpha \in \mathcal{S}\}$.

Substitution-closed families

$\mathcal{S} \subset \{$simple permutations of length $\geq 4\}$.

$\langle \mathcal{S} \rangle = \{$permutations built by substituting $\bullet, \oplus, \ominus$, and $\alpha \in \mathcal{S}\}$.

Examples : $\langle \emptyset \rangle = \{$ separables $\} = \mathrm{Av}(3142, 2413)$.

$\langle 3142 \rangle = \mathrm{Av}(2413, 41352, 415263, 531642)$.

More generally, any class whose basis contains only simples is as such (substitution-closed class)

Substitution-closed families

$\mathcal{S} \subset \{$simple permutations of length $\geq 4\}$.

$\langle \mathcal{S} \rangle = \{$permutations built by substituting $\bullet, \oplus, \ominus$, and $\alpha \in \mathcal{S}\}$.

Examples : $\langle \emptyset \rangle = \{$ separables $\} = \mathrm{Av}(3142, 2413)$.

$\langle 3142 \rangle = \mathrm{Av}(2413, 41352, 415263, 531642)$.

More generally, any class whose basis contains only simples is as such (substitution-closed class)

Let σ_n be a uniform permutation of size n in $\langle \mathcal{S} \rangle$.

$S(z) = \sum_{\alpha \in \mathcal{S}} z^{|\alpha|}$ generating function of the simples, radius R. Set $a = S'(R) - 2/(1+R)^2 + 1$ and $b = S''(R)$

Substitution-closed families

$\mathcal{S} \subset \{$simple permutations of length $\geq 4\}$.

$\langle \mathcal{S} \rangle = \{$permutations built by substituting $\bullet, \oplus, \ominus$, and $\alpha \in \mathcal{S}\}$.

Examples : $\langle \emptyset \rangle = \{$ separables $\} = \mathrm{Av}(3142, 2413)$.

$\langle 3142 \rangle = \mathrm{Av}(2413, 41352, 415263, 531642)$.

More generally, any class whose basis contains only simples is as such (substitution-closed class)

Let σ_n be a uniform permutation of size n in $\langle \mathcal{S} \rangle$.

$S(z) = \sum_{\alpha \in \mathcal{S}} z^{|\alpha|}$ generating function of the simples, radius R. Set $a = S'(R) - 2/(1+R)^2 + 1$ and $b = S''(R)$

Substitution-closed families

$\mathcal{S} \subset \{$simple permutations of length $\geq 4\}$.
$\langle \mathcal{S} \rangle = \{$permutations built by substituting $\bullet, \oplus, \ominus$, and $\alpha \in \mathcal{S}\}$.
Examples : $\langle \emptyset \rangle = \{$ separables $\} = \mathrm{Av}(3142, 2413)$.
$\langle 3142 \rangle = \mathrm{Av}(2413, 41352, 415263, 531642)$.
More generally, any class whose basis contains only simples is as such (substitution-closed class)

Let σ_n be a uniform permutation of size n in $\langle \mathcal{S} \rangle$.
$S(z) = \sum_{\alpha \in \mathcal{S}} z^{|\alpha|}$ generating function of the simples, radius R. Set $a = S'(R) - 2/(1+R)^2 + 1$ and $b = S''(R)$

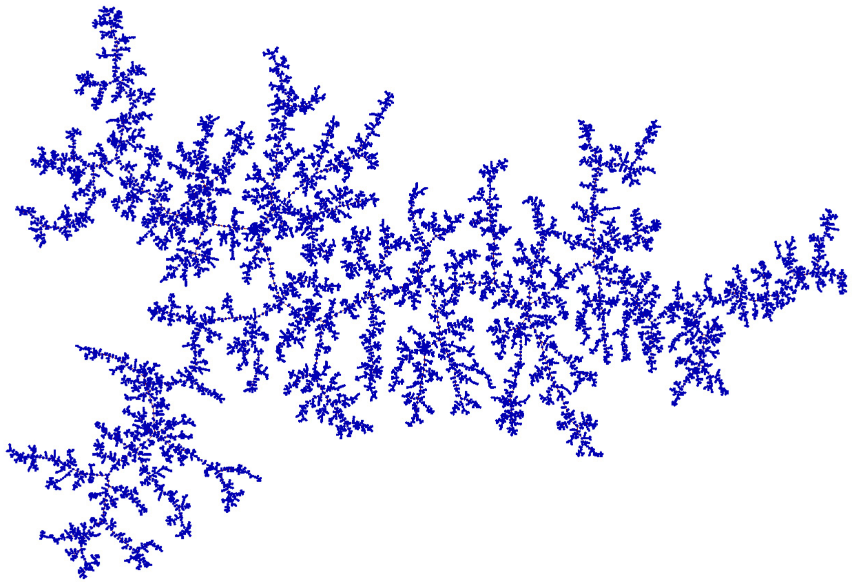
Theorem (Bassino, Bouvel, Féray, Gerin, M., Pierrot 2017)
The limit in distribution of σ_n is

- a **biased** Brownian separable permuton if $a > 0$ or $a = 0, b < \infty$,
- the same limit ν as an uniform simple permutation in $\mathcal{S}$ if $a < 0$,
- a **stable permuton** if $a = 0$, $b = \infty$.

When $a \leq 0$ additional regularity hypotheses on S near its singularity are needed.

Biased Brownian separable permuton

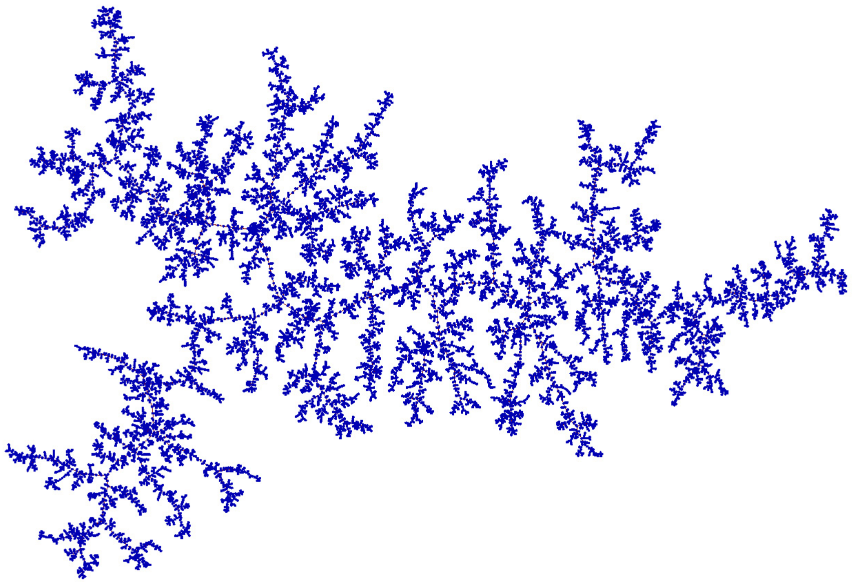
Regime where the decomposition tree converges to a Brownian CRT.



Picture by I. Kortchemski

Biased Brownian separable permuton

Regime where the decomposition tree converges to a Brownian CRT.

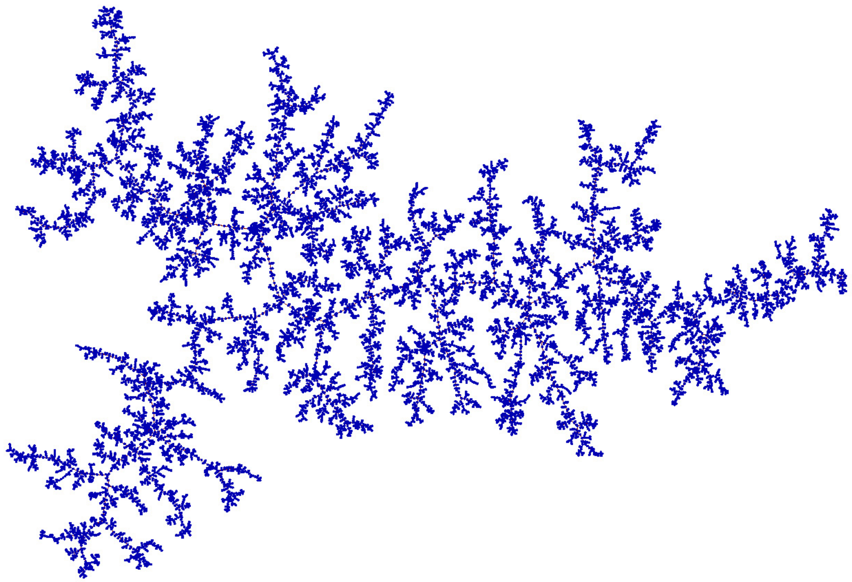


Picture by I. Kortchemski

The signs in a uniform subtree are biased: $\mathbb{P}(\oplus) = p$, and p depends explicitely on $\mathcal{S}$.
Here $p = 0.2$.

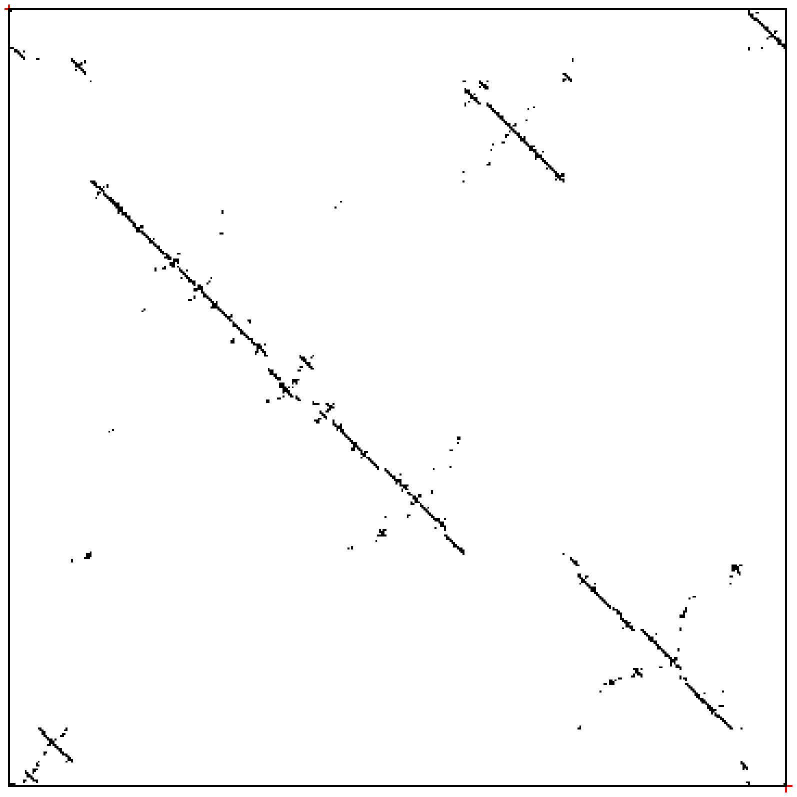
Biased Brownian separable permuton

Regime where the decomposition tree converges to a Brownian CRT.



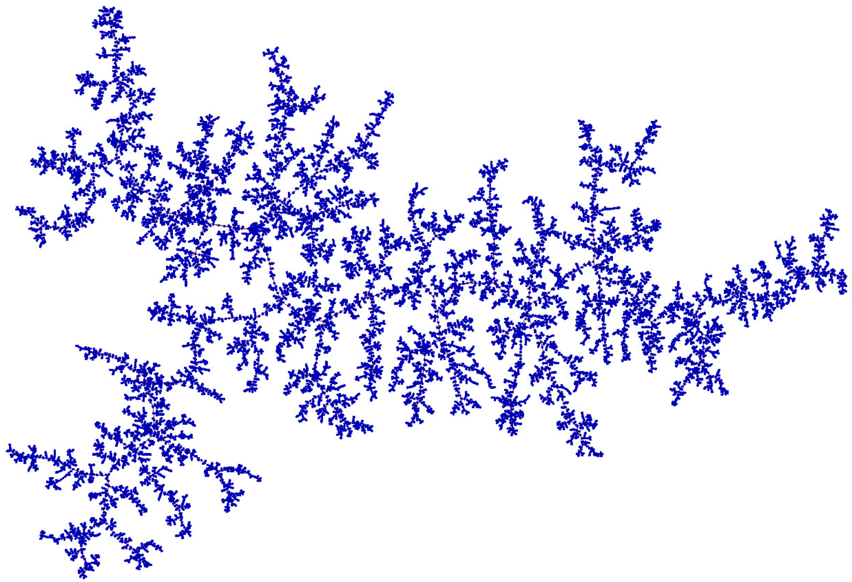
Picture by I. Kortchemski

The signs in a uniform subtree are biased: $\mathbb{P}(\oplus) = p$, and p depends explicitely on $\mathcal{S}$. Here $p = 0.2$.



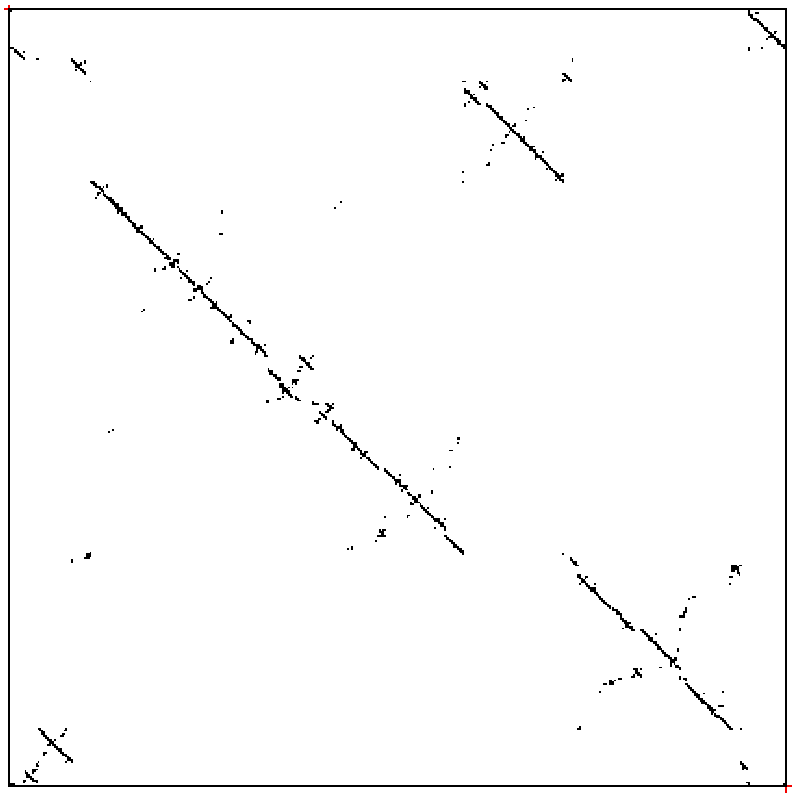
Biased Brownian separable permuton

Regime where the decomposition tree converges to a Brownian CRT.



Picture by I. Kortchemski

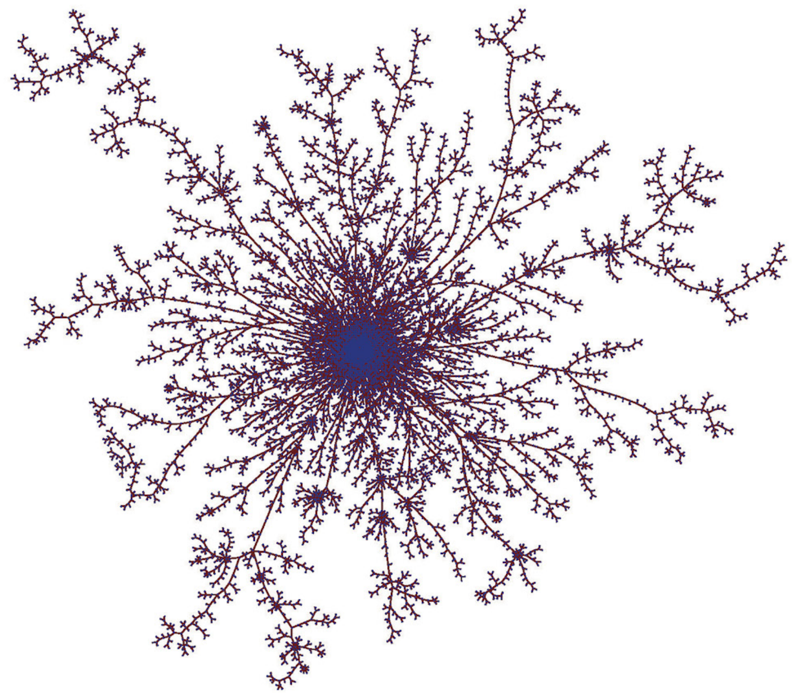
The signs in a uniform subtree are biased: $\mathbb{P}(\oplus) = p$, and p depends explicitely on $\mathcal{S}$. Here $p = 0.2$.



The regime $a > 0$ covers most known substitution-closed classes: $\mathcal{S}$ finite or subexponential, S rational,...

Degenerate case a<0

Regime where the decomposition tree exhibits a condensation phenomenon. Roughly, σ_n looks like a large uniform simple permutation in $\mathcal{S}$ and converges to the same limit ν.



Picture by I. Kortchemski

Example: Av(2413). We still need to understand the permuton limit of large simples in this class (+ technical hypotheses) to apply our theorem.

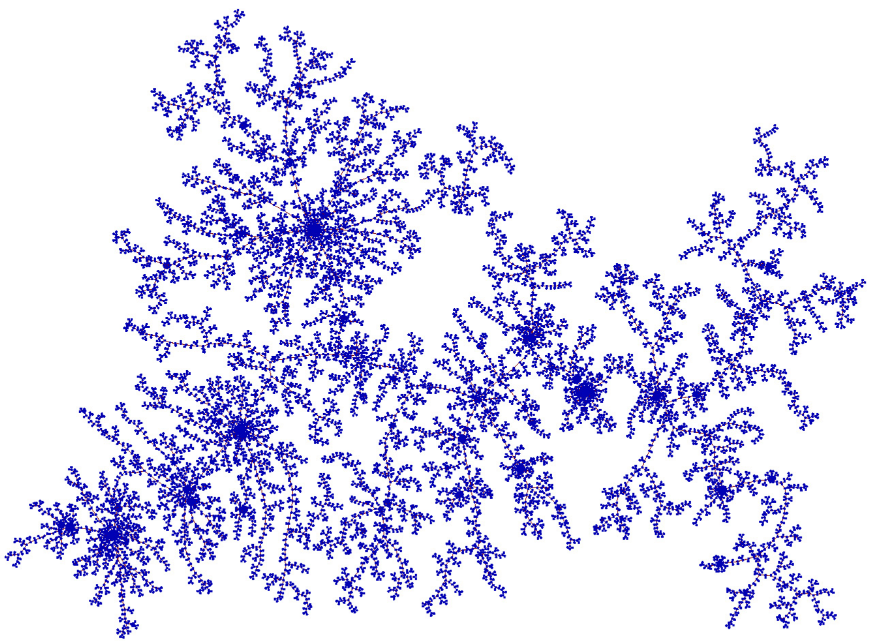
Stable permutons

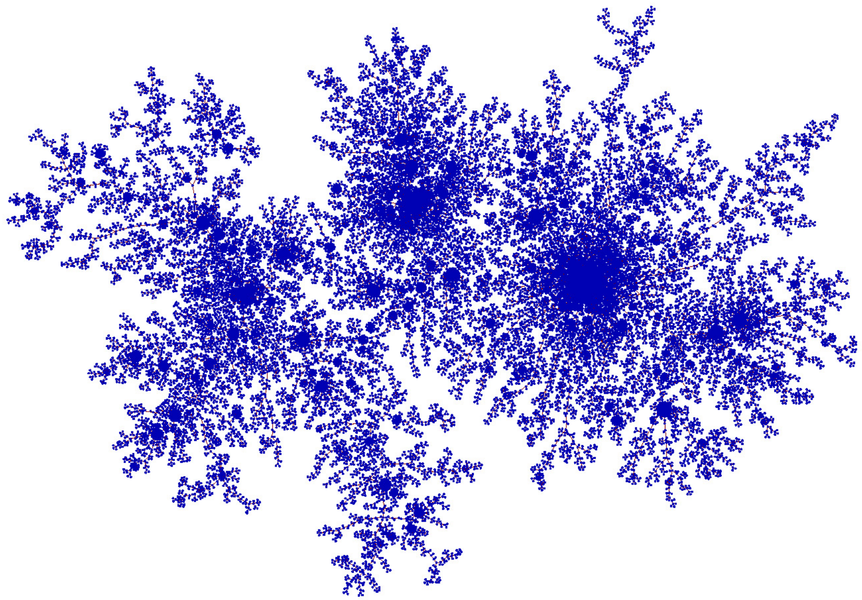
Regime where the decomposition tree converges to a α-stable tree, α explicit.

Stable permutons

Regime where the decomposition tree converges to a α-stable tree, α explicit.

$\alpha = 1.5$ $\alpha = 1.1$



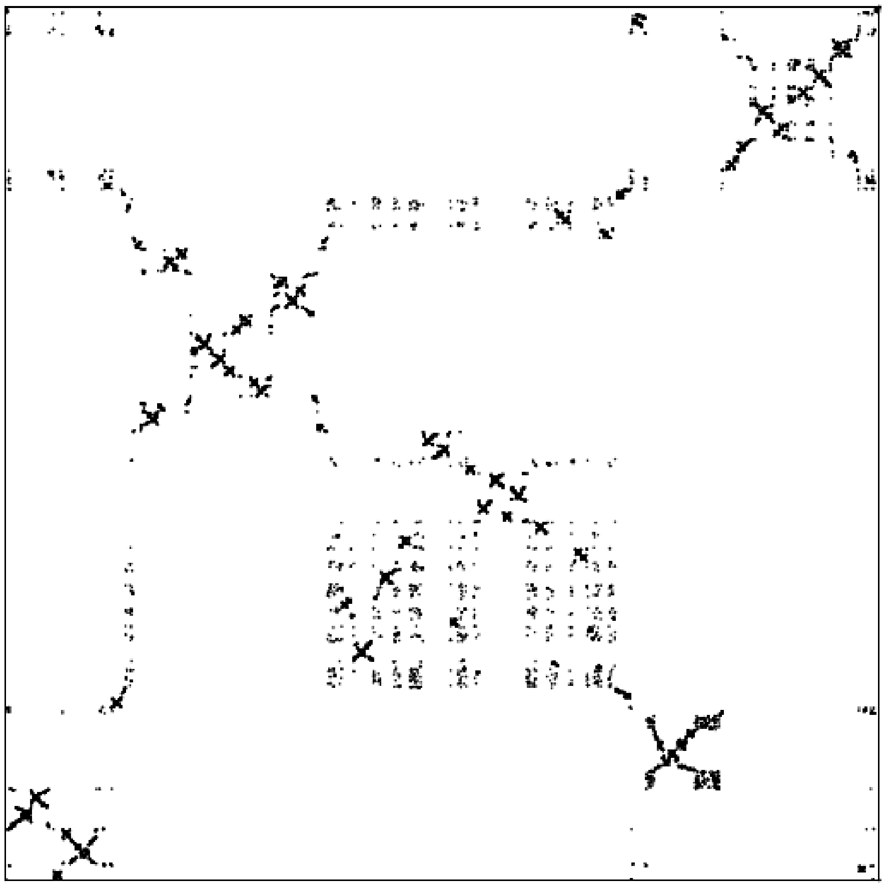


Pictures by I. Kortchemski

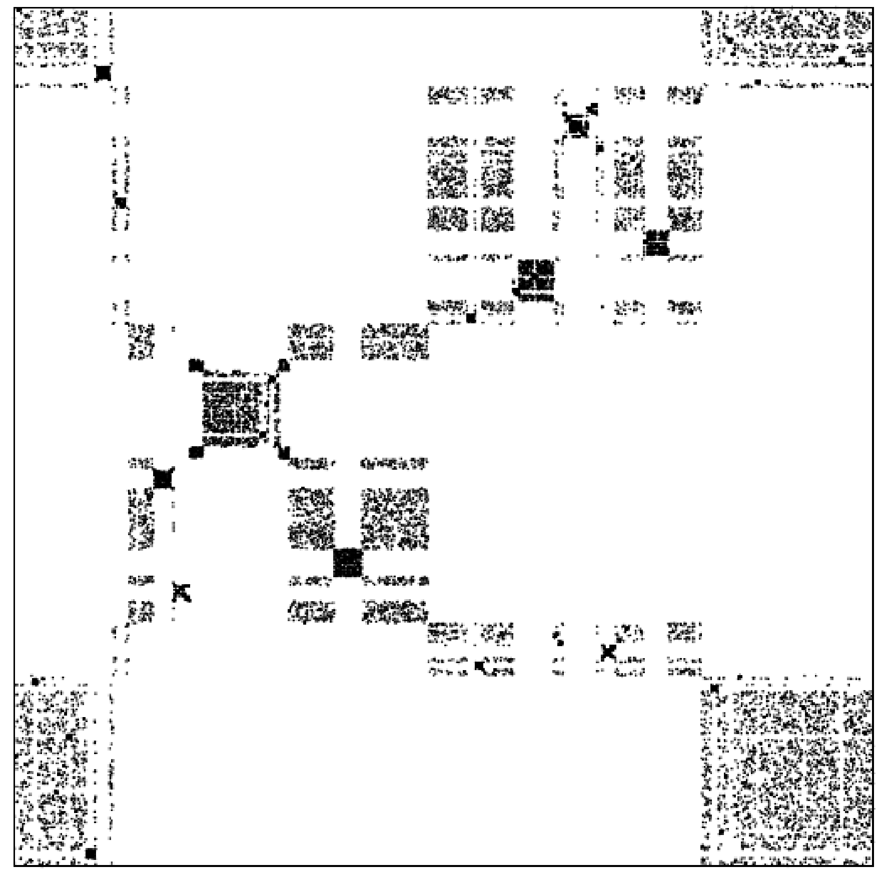
Stable permutons

Regime where the decomposition tree converges to a α-stable tree, α explicit.

$\alpha = 1.5$



$\alpha = 1.1$



Branches from each infinite-degree point are reordered according to an independent copy of ν (the limit of large simples in the class)

Idea of proof (first, separable permutations)

Analytic combinatorics

Let $(a_n)_n$ be a nonnegative sequence and $A(z) = \sum_n a_n z^n$ its generating function of radius ρ

Transfer Theorem (Flajolet & Odlyzko) If

- A is defined on a Δ-domain at $\rho > 0$
- $A(z) \underset{z \to \rho}{=} g(z) + (C + o(1))(\rho - z)^{\delta}$ with g analytic, $\delta \notin \mathbb{N}$,

then as , $a_n \underset{n \to \infty}{=} (\frac{C}{\Gamma(-\delta)} + o(1))\rho^{-n} n^{-1-\delta}$

Analytic combinatorics

Let $(a_n)_n$ be a nonnegative sequence and $A(z) = \sum_n a_n z^n$ its generating function of radius ρ

Transfer Theorem (Flajolet & Odlyzko) If

- A is defined on a Δ-domain at $\rho > 0$
- $A(z) \underset{z \to \rho}{=} g(z) + (C + o(1))(\rho - z)^\delta$ with g analytic, $\delta \notin \mathbb{N}$,

then as , $a_n \underset{n \to \infty}{=} (\frac{C}{\Gamma(-\delta)} + o(1))\rho^{-n} n^{-1-\delta}$

If $(a_n)_n$ counts a recursive structure, equations on A are easy to obtain from which the singular behavior can be inferred.

Analytic combinatorics for leaf-counted trees

For "nice" varieties of trees, the uniform k-leaf-subtree in a large tree converges to the uniform binary tree with k leaves.

Analytic combinatorics for leaf-counted trees

For "nice" varieties of trees, the uniform k-leaf-subtree in a large tree converges to the uniform binary tree with k leaves.

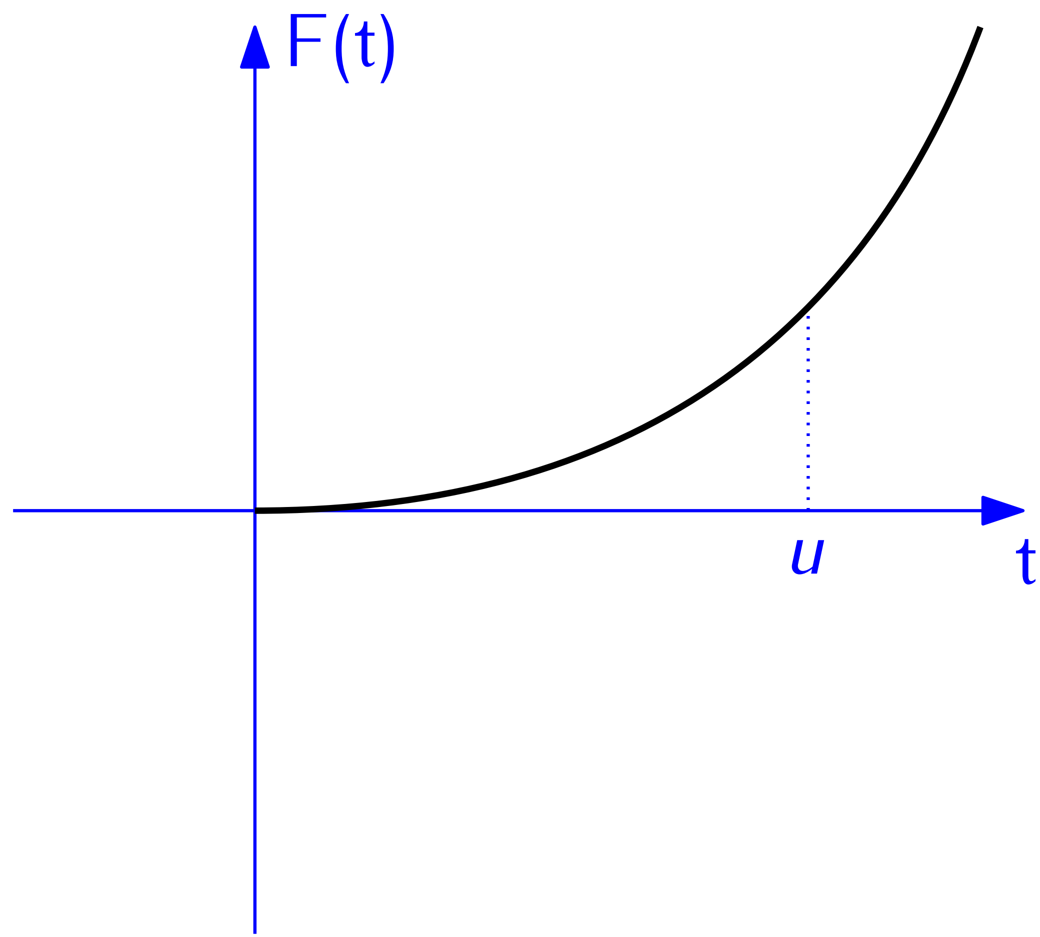
Recursive trees counted by number of leaves.
$T(z) = z + F(T(z))$ (Schröder: $F(t) = \sum_{k \geq 2} t^k$).

Analytic combinatorics for leaf-counted trees

For "nice" varieties of trees, the uniform k-leaf-subtree in a large tree converges to the uniform binary tree with k leaves.

Recursive trees counted by number of leaves.
$T(z) = z + F(T(z))$ (Schröder: $F(t) = \sum_{k\geq 2} t^k$).

In this case, "nice" $\overset{def}{\Longleftrightarrow}$
$\exists \quad 0 < u < R_F, F'(u) = 1.$
Then T is Δ-analytic at ρ with $T(\rho) = u$ and a square-root singularity (smooth implicit function schema).

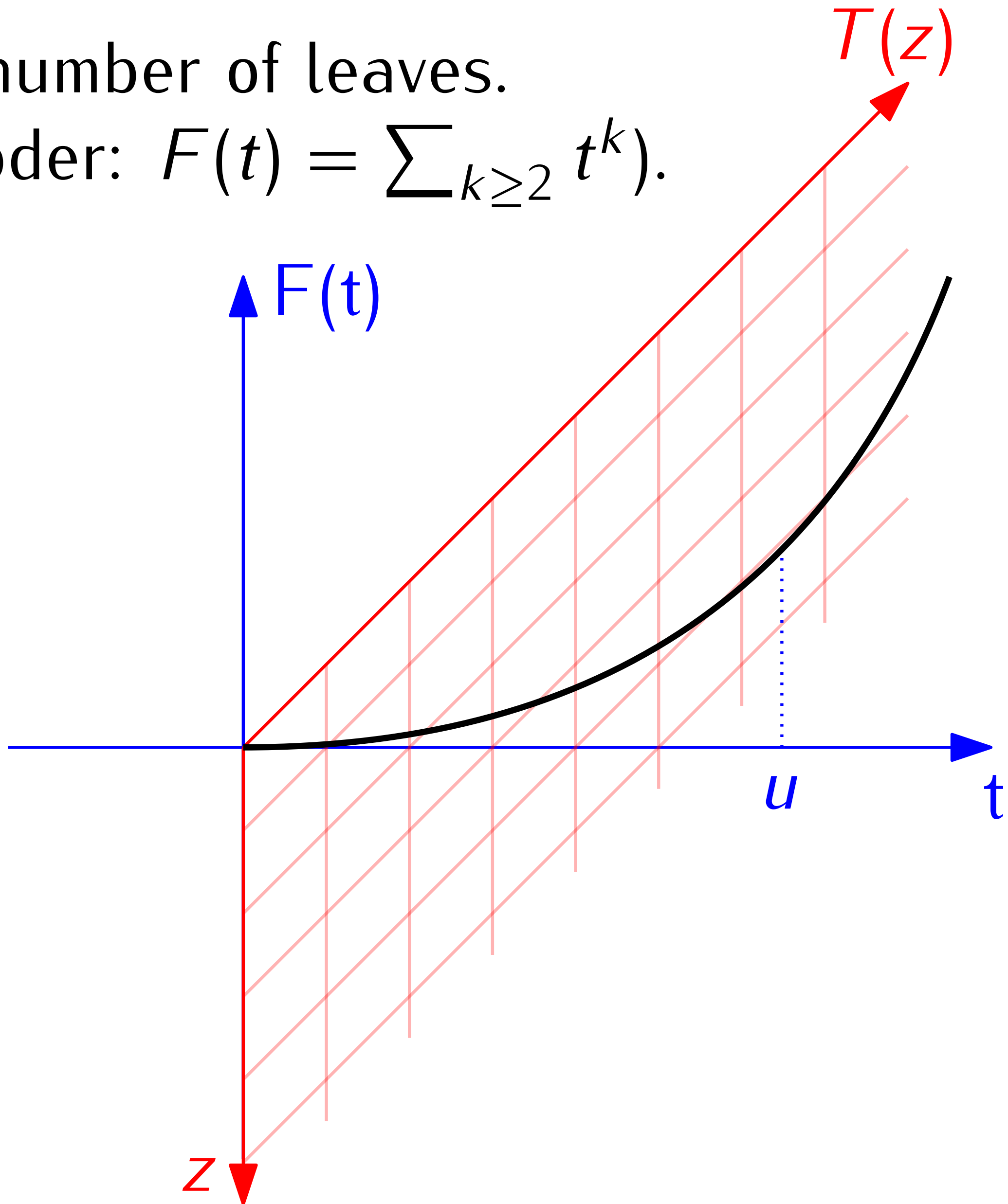


Analytic combinatorics for leaf-counted trees

For "nice" varieties of trees, the uniform k-leaf-subtree in a large tree converges to the uniform binary tree with k leaves.

Recursive trees counted by number of leaves.
$T(z) = z + F(T(z))$ (Schröder: $F(t) = \sum_{k\geq 2} t^k$).

In this case, "nice" $\overset{def}{\Longleftrightarrow}$
$\exists \quad 0 < u < R_F, F'(u) = 1$.
Then T is Δ-analytic at ρ with $T(\rho) = u$ and a square-root singularity (smooth implicit function schema).

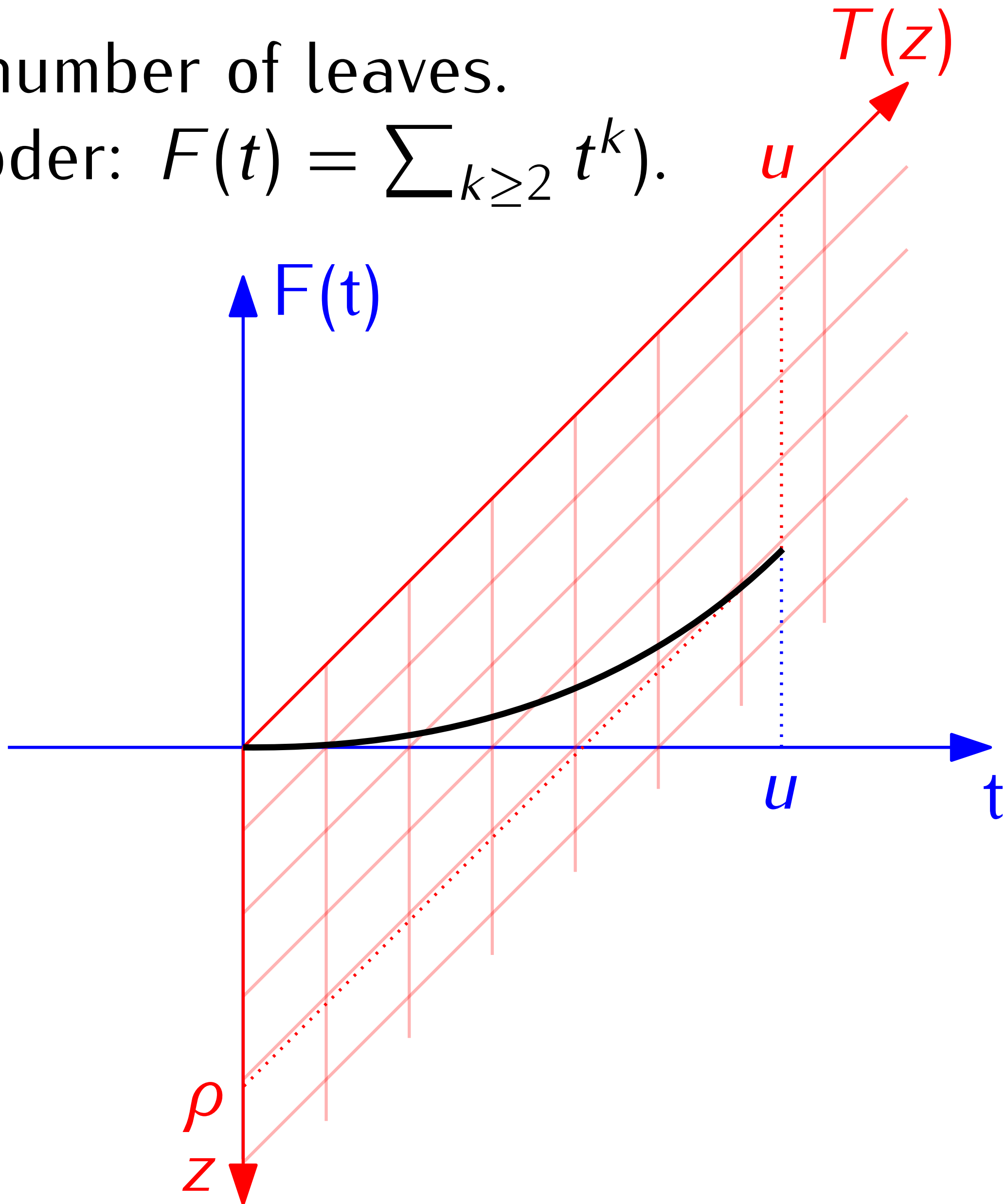


Analytic combinatorics for leaf-counted trees

For "nice" varieties of trees, the uniform k-leaf-subtree in a large tree converges to the uniform binary tree with k leaves.

Recursive trees counted by number of leaves.
$T(z) = z + F(T(z))$ (Schröder: $F(t) = \sum_{k\geq 2} t^k$).

In this case, "nice" $\overset{def}{\Longleftrightarrow}$
$\exists \quad 0 < u < R_F, F'(u) = 1.$
Then T is Δ-analytic at ρ with $T(\rho) = u$ and a square-root singularity (smooth implicit function schema).

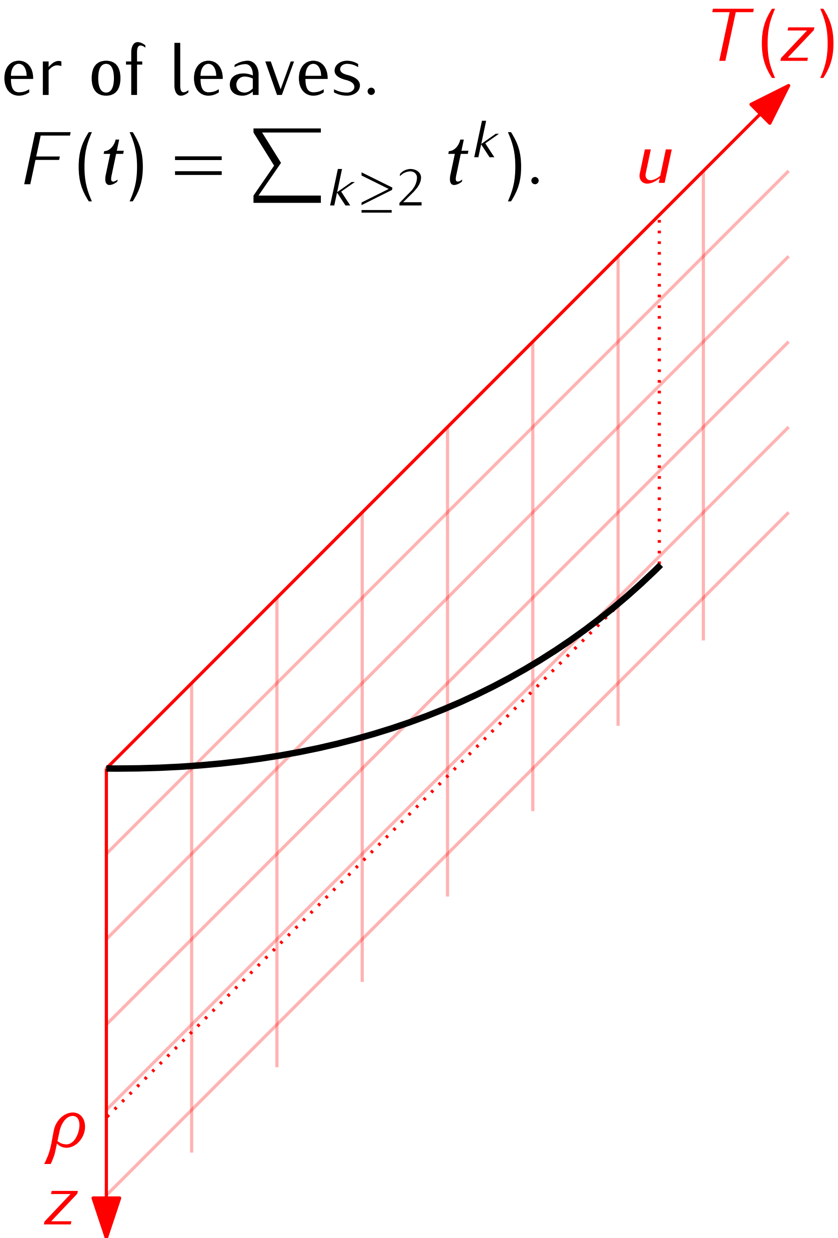


Analytic combinatorics for leaf-counted trees

For "nice" varieties of trees, the uniform k-leaf-subtree in a large tree converges to the uniform binary tree with k leaves.

Recursive trees counted by number of leaves.
$T(z) = z + F(T(z))$ (Schröder: $F(t) = \sum_{k\geq 2} t^k$).

In this case, "nice" $\overset{def}{\iff}$
$\exists \quad 0 < u < R_F, F'(u) = 1.$
Then T is Δ-analytic at ρ with $T(\rho) = u$ and a square-root singularity (smooth implicit function schema).



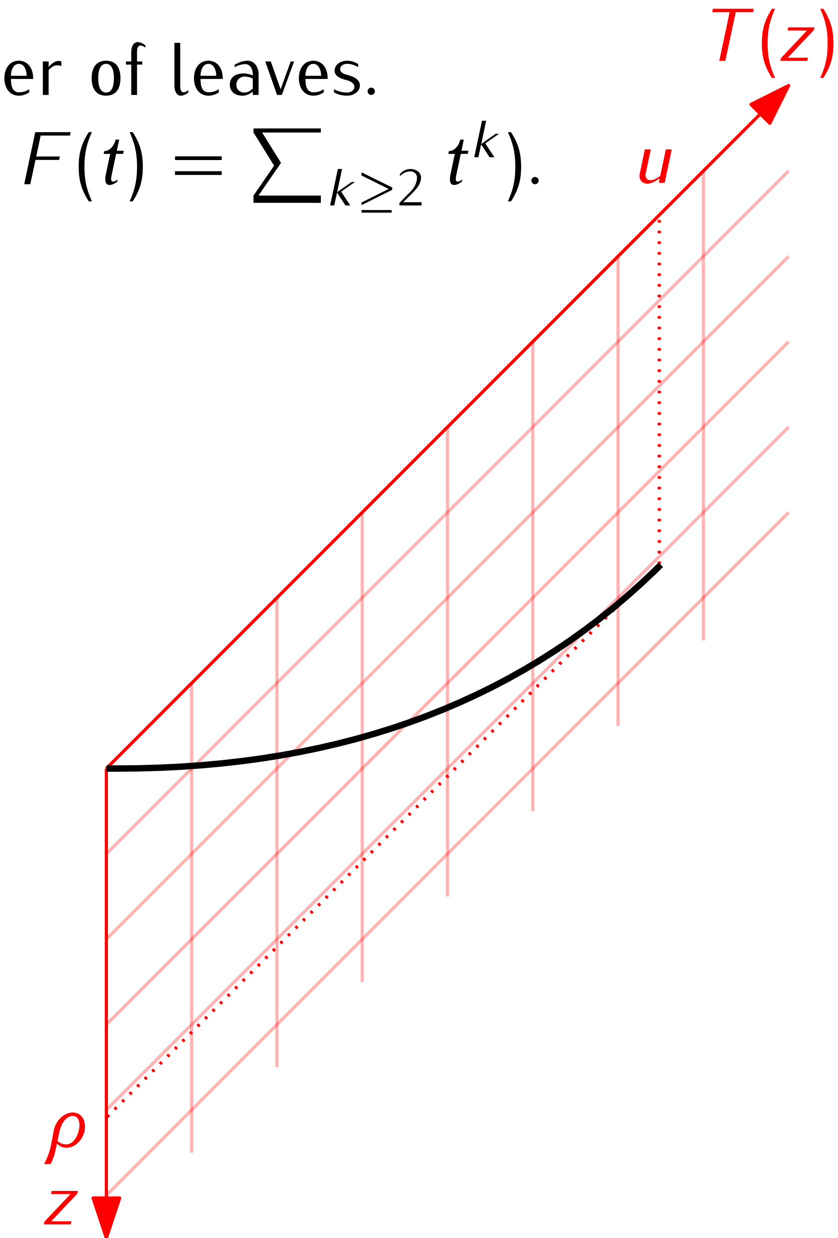
Analytic combinatorics for leaf-counted trees

For "nice" varieties of trees, the uniform k-leaf-subtree in a large tree converges to the uniform binary tree with k leaves.

Recursive trees counted by number of leaves.
$T(z) = z + F(T(z))$ (Schröder: $F(t) = \sum_{k \geq 2} t^k$).

In this case, "nice" $\overset{def}{\iff}$
$\exists \quad 0 < u < R_F, F'(u) = 1.$
Then T is Δ-analytic at ρ with $T(\rho) = u$ and a square-root singularity (smooth implicit function schema).

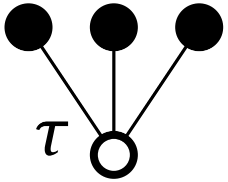
This is the case for Schröder (F rational)



Uniform k-subtree in large unsigned trees

T has square-root singularity at ρ and F analytic at $T(\rho)$. Then, the $g.f$ of trees with k marked leaves that induce the k-tree τ is

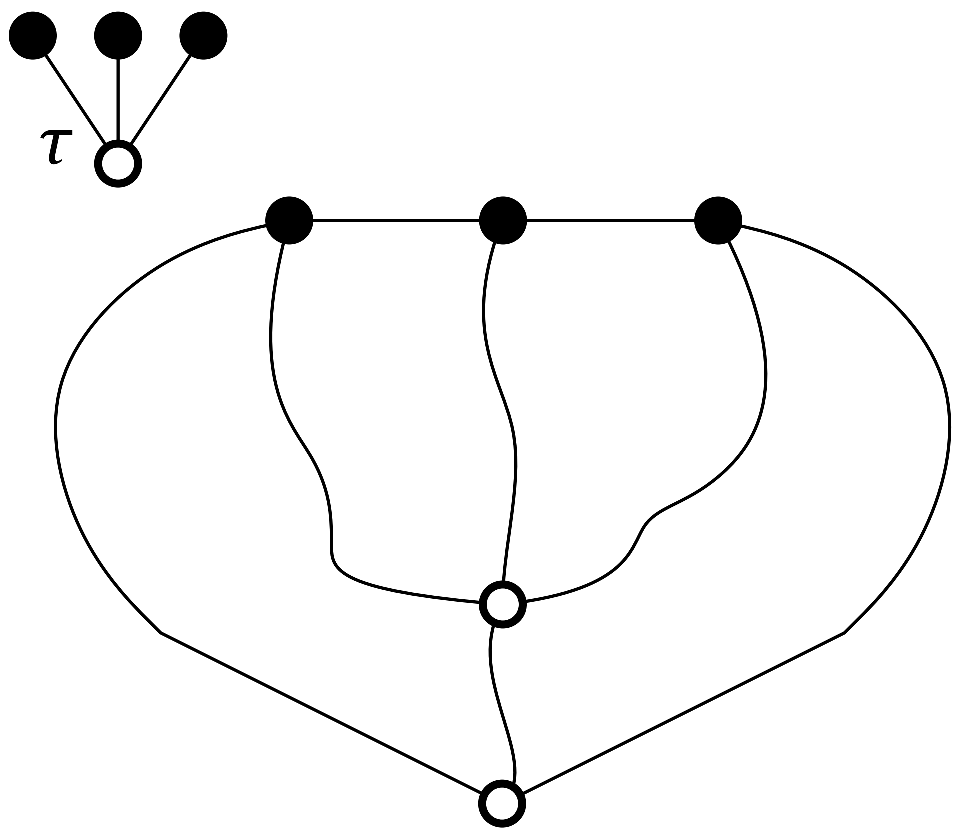
$$z^k T'(z) \prod_{v \text{ internal node of } \tau} T'(z)^{\deg(v)} \frac{1}{\deg(v)!} F^{(\deg(v))}(T(z))$$



Uniform k-subtree in large unsigned trees

T has square-root singularity at ρ and F analytic at $T(\rho)$. Then, the $g.f$ of trees with k marked leaves that induce the k-tree τ is

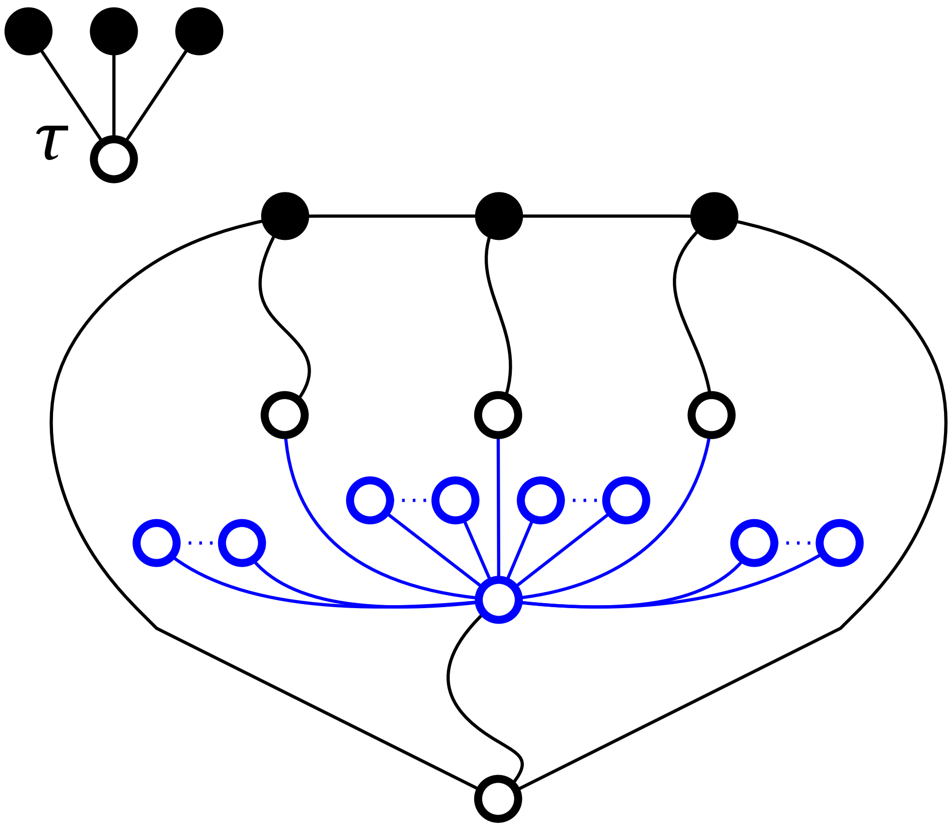
$$z^k T'(z) \prod_{v \text{ internal node of } \tau} T'(z)^{\deg(v)} \frac{1}{\deg(v)!} F^{(\deg(v))}(T(z))$$



Uniform k-subtree in large unsigned trees

T has square-root singularity at ρ and F analytic at $T(\rho)$. Then, the $g.f$ of trees with k marked leaves that induce the k-tree τ is

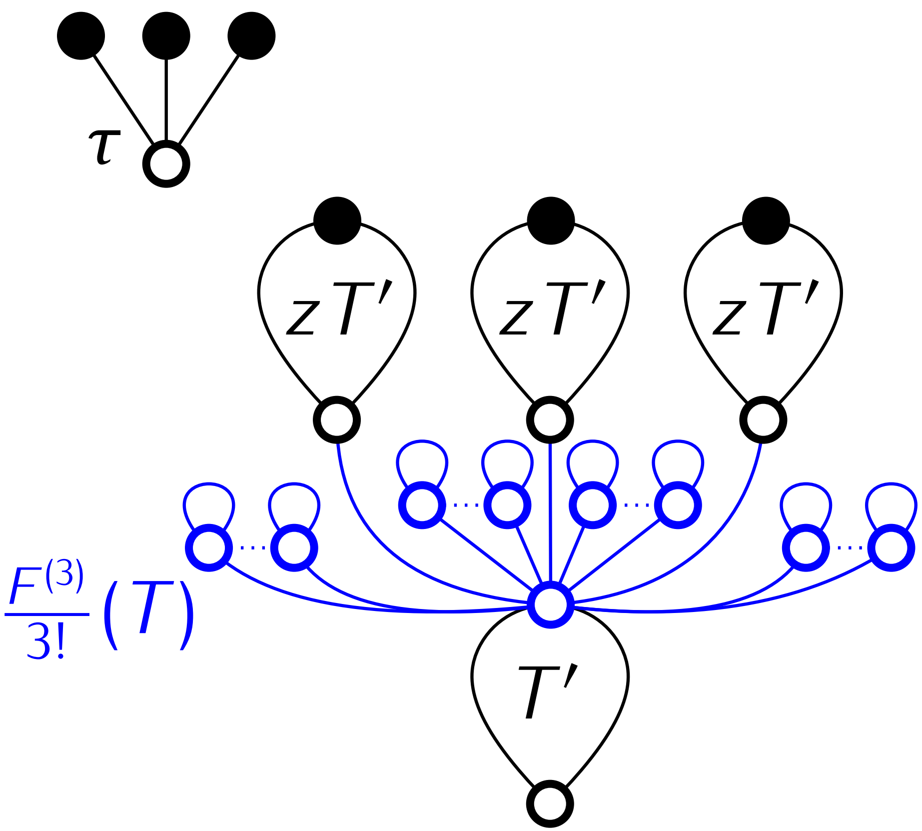
$$z^k T'(z) \prod_{v \text{ internal node of } \tau} T'(z)^{\deg(v)} \frac{1}{\deg(v)!} F^{(\deg(v))}(T(z))$$



Uniform k-subtree in large unsigned trees

T has square-root singularity at ρ and F analytic at $T(\rho)$. Then, the $g.f$ of trees with k marked leaves that induce the k-tree τ is

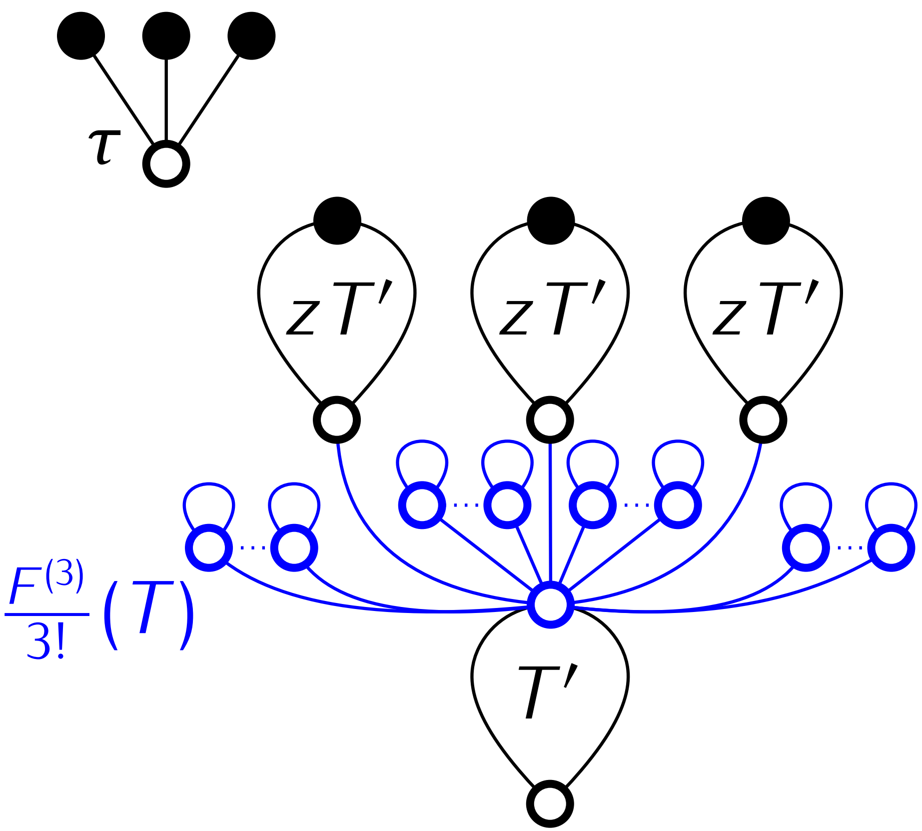
$$z^k T'(z) \prod_{v \text{ internal node of } \tau} T'(z)^{\deg(v)} \frac{1}{\deg(v)!} F^{(\deg(v))}(T(z))$$



Uniform k-subtree in large unsigned trees

T has square-root singularity at ρ and F analytic at $T(\rho)$. Then, the $g.f$ of trees with k marked leaves that induce the k-tree τ is

$$z^k T'(z) \prod_{v \text{ internal node of } \tau} T'(z)^{\deg(v)} \frac{1}{\deg(v)!} F^{(\deg(v))}(T(z))$$



$\sim_\rho C_\tau(\rho - z)^{-\#\{\text{nodes in } \tau\}/2}$.
Dominates when τ binary.
(Then C_τ doesn't depend on τ).
Transfer: $t_{n|I_n^k}$ converges in distribution to a uniform binary tree.

Uniform k-subtree in large signed trees

Counting signed trees that induce a given signed tree τ: adding parity constraints on the height of the marked leaf in the marked trees.

Uniform k-subtree in large signed trees

Counting signed trees that induce a given signed tree τ: adding parity constraints on the height of the marked leaf in the marked trees.
Replace instances of T' by T'_0 (even height) or T'_1 (odd height). $T'_0 + T'_1 = T'$ and $T'_1 = F'(T)T'_0$, so $T'_0 \sim T'_1 \sim \frac{1}{2}T'$.

Uniform k-subtree in large signed trees

Counting signed trees that induce a given signed tree τ: adding parity constraints on the height of the marked leaf in the marked trees.
Replace instances of T' by T'_0 (even height) or T'_1 (odd height). $T'_0 + T'_1 = T'$ and $T'_1 = F'(T)T'_0$, so $T'_0 \sim T'_1 \sim \frac{1}{2}T'$.

g.f. of Trees with k marked leaves that induce the signed k-tree τ :

$$z^k (T'_0 + T'_1) {T'_0}^b {T'_1}^a {T'}^k \prod_{v \text{ internal node of } \tau} \frac{1}{\deg(v)!} F^{(\deg(v))}(T(z))$$

where a (resp. b) is the number of edges of τ incident to two nodes of the same (resp. different) signs

Uniform k-subtree in large signed trees

Counting signed trees that induce a given signed tree τ: adding parity constraints on the height of the marked leaf in the marked trees.
Replace instances of T' by T'_0 (even height) or T'_1 (odd height). $T'_0 + T'_1 = T'$ and $T'_1 = F'(T)T'_0$, so $T'_0 \sim T'_1 \sim \frac{1}{2}T'$.

g.f. of Trees with k marked leaves that induce the signed k-tree τ :

$$z^k (T'_0 + T'_1) {T'_0}^b {T'_1}^a {T'}^k \prod_{v \text{ internal node of } \tau} \frac{1}{\deg(v)!} F^{(\deg(v))}(T(z))$$

where a (resp. b) is the number of edges of τ incident to two nodes of the same (resp. different) signs

Hence all signed binary trees have the same asymptotic probability, what whe needed for permuton convergence.

Idea of proof (general substitution-closed families)

Substitution-closed classes

Here the trees are described by a context-free grammar with three types:

$$T = z + \frac{T_{\text{not}\oplus}^2}{1 - T_{\text{not}\oplus}} + \frac{T_{\text{not}\ominus}^2}{1 - T_{\text{not}\ominus}} + S(T)$$

$$T_{\text{not}\oplus} = z + \frac{T_{\text{not}\oplus}^2}{1 - T_{\text{not}\oplus}} + S(T)$$

$$T_{\text{not}\ominus} = z + \frac{T_{\text{not}\ominus}^2}{1 - T_{\text{not}\ominus}} + S(T)$$

Substitution-closed classes

Here the trees are described by a context-free grammar with three types:

$$T = z + \frac{T_{\text{not}\oplus}^2}{1 - T_{\text{not}\oplus}} + \frac{T_{\text{not}\ominus}^2}{1 - T_{\text{not}\ominus}} + S(T)$$

$$T_{\text{not}\oplus} = z + \frac{T_{\text{not}\oplus}^2}{1 - T_{\text{not}\oplus}} + S(T)$$

$$T_{\text{not}\ominus} = z + \frac{T_{\text{not}\ominus}^2}{1 - T_{\text{not}\ominus}} + S(T)$$

Which reduces to

$$T_{\text{not}\ominus} = T_{\text{not}\oplus} = z + \frac{T_{\text{not}\oplus}^2}{1 - T_{\text{not}\oplus}} + S\left(\frac{T_{\text{not}\oplus}}{1 - T_{\text{not}\oplus}}\right) = z + \Lambda(T_{\text{not}\oplus}).$$

$$T = \frac{T_{\text{not}\oplus}}{1 - T_{\text{not}\oplus}}.$$

Then

- If $a > 0$, then Λ' reaches 1 before its singularity and we end up in the smooth implicit function schema (hence the Brownian behavior)
- If $a = 0$ then $\Lambda'(R_\Lambda) = 1$. If $\delta > 1$ is the singularity exponent of S and Λ then the one of $T_{\text{not}\oplus}$ is $(\delta \wedge 2)^{-1}$.
- If $a < 0$ then $\Lambda'(R_\Lambda) < 1$. Then S, Λ and $T_{\text{not}\oplus}$ have the same singularity exponent $\delta > 1$.

Then

- If $a > 0$, then Λ' reaches 1 before its singularity and we end up in the smooth implicit function schema (hence the Brownian behavior)
- If $a = 0$ then $\Lambda'(R_\Lambda) = 1$. If $\delta > 1$ is the singularity exponent of S and Λ then the one of $T_{\text{not}\oplus}$ is $(\delta \wedge 2)^{-1}$.
- If $a < 0$ then $\Lambda'(R_\Lambda) < 1$. Then S, Λ and $T_{\text{not}\oplus}$ have the same singularity exponent $\delta > 1$.

In the first two cases, the 3×3 matrix

(g.f. of trees of type i with a marked leaf of type $j)_{i,j \in \{\emptyset, \text{not}\oplus, \text{not}\ominus\}}$

is asymptotic to $T'_{\text{not}\oplus}$ times a constant matrix of rank 1.

Then

- If $a > 0$, then Λ' reaches 1 before its singularity and we end up in the smooth implicit function schema (hence the Brownian behavior)
- If $a = 0$ then $\Lambda'(R_\Lambda) = 1$. If $\delta > 1$ is the singularity exponent of S and Λ then the one of $T_{\text{not}\oplus}$ is $(\delta \wedge 2)^{-1}$.
- If $a < 0$ then $\Lambda'(R_\Lambda) < 1$. Then S, Λ and $T_{\text{not}\oplus}$ have the same singularity exponent $\delta > 1$.

In the first two cases, the 3×3 matrix

(g.f. of trees of type i with a marked leaf of type $j)_{i,j \in \{\emptyset, \text{not}\oplus, \text{not}\ominus\}}$

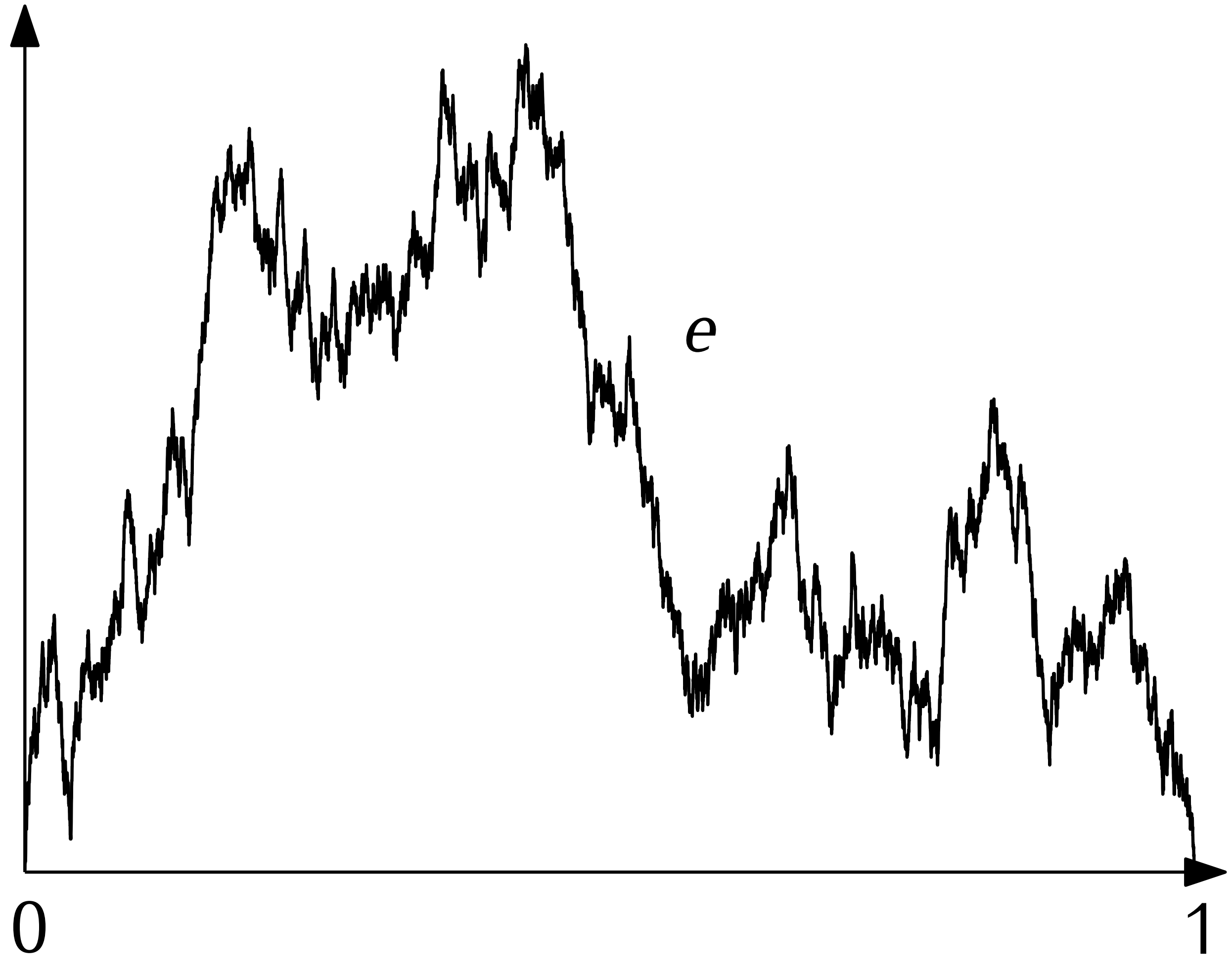
is asymptotic to $T'_{\text{not}\oplus}$ times a constant matrix of rank 1.

This is enough to analyze the probability of uniform subtrees in a large substitution tree and prove the theorem.

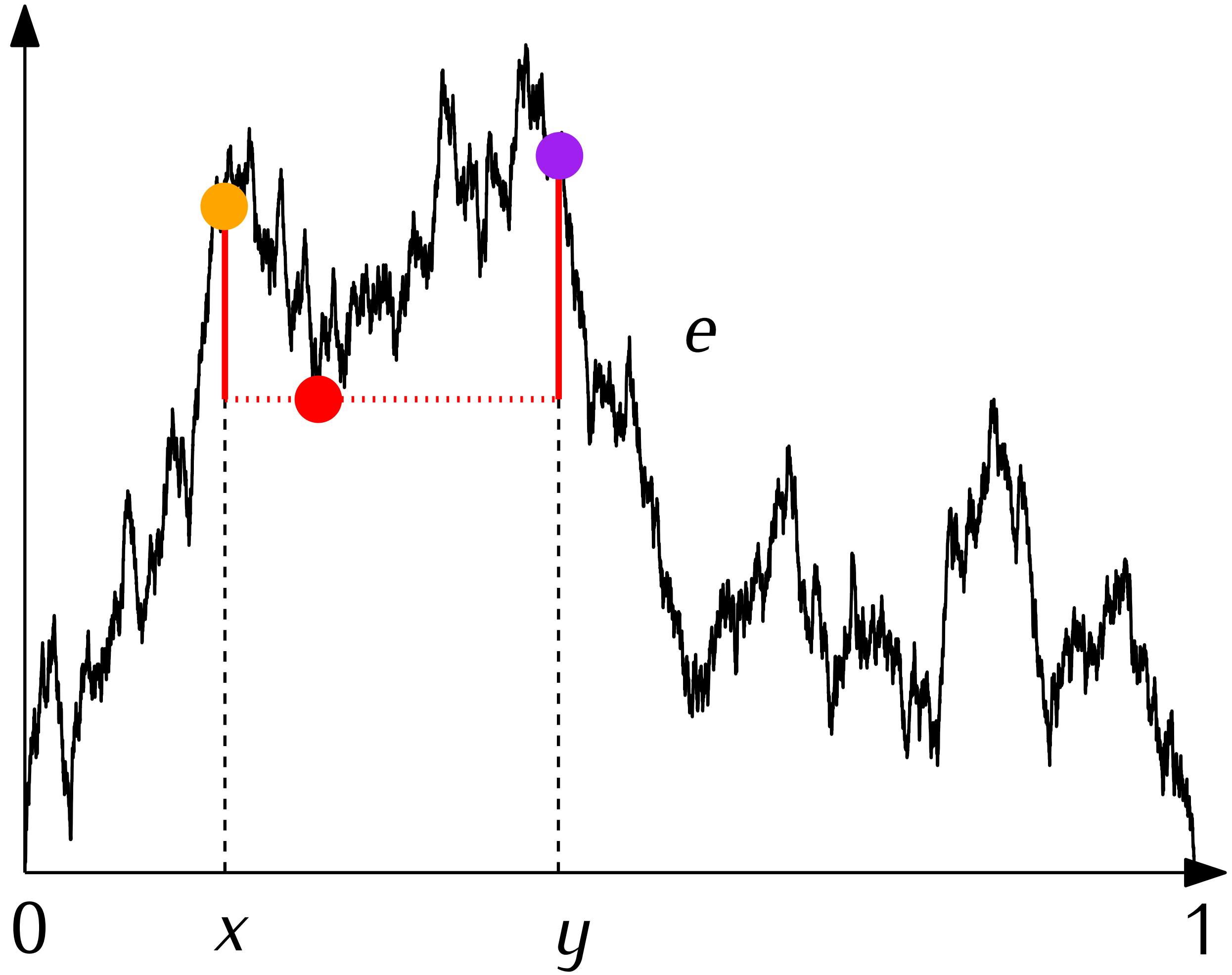
2 – Construction of the Brownian Permuton

[arXiv:1711.08986]

The Brownian excursion and CRT

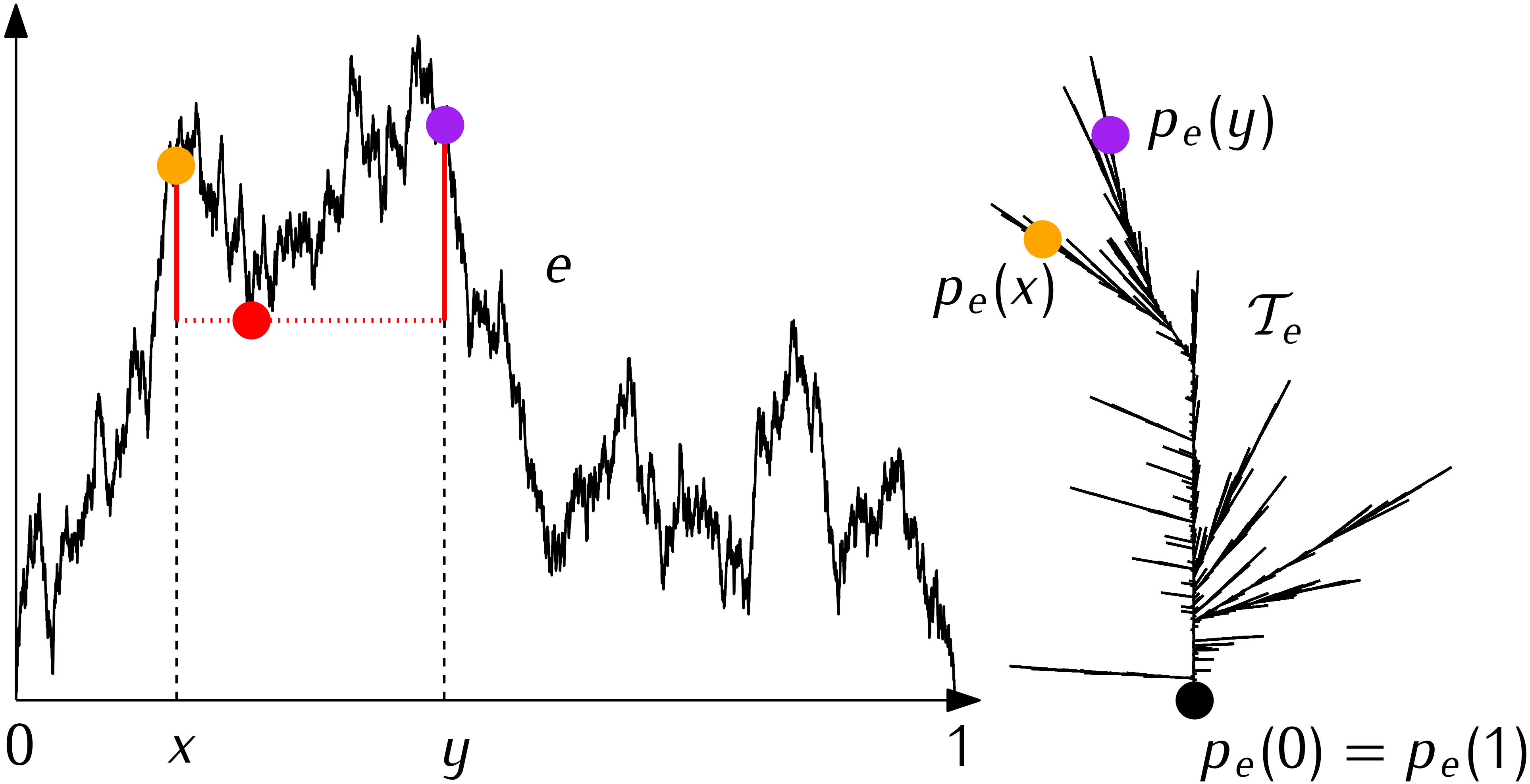


The Brownian excursion and CRT



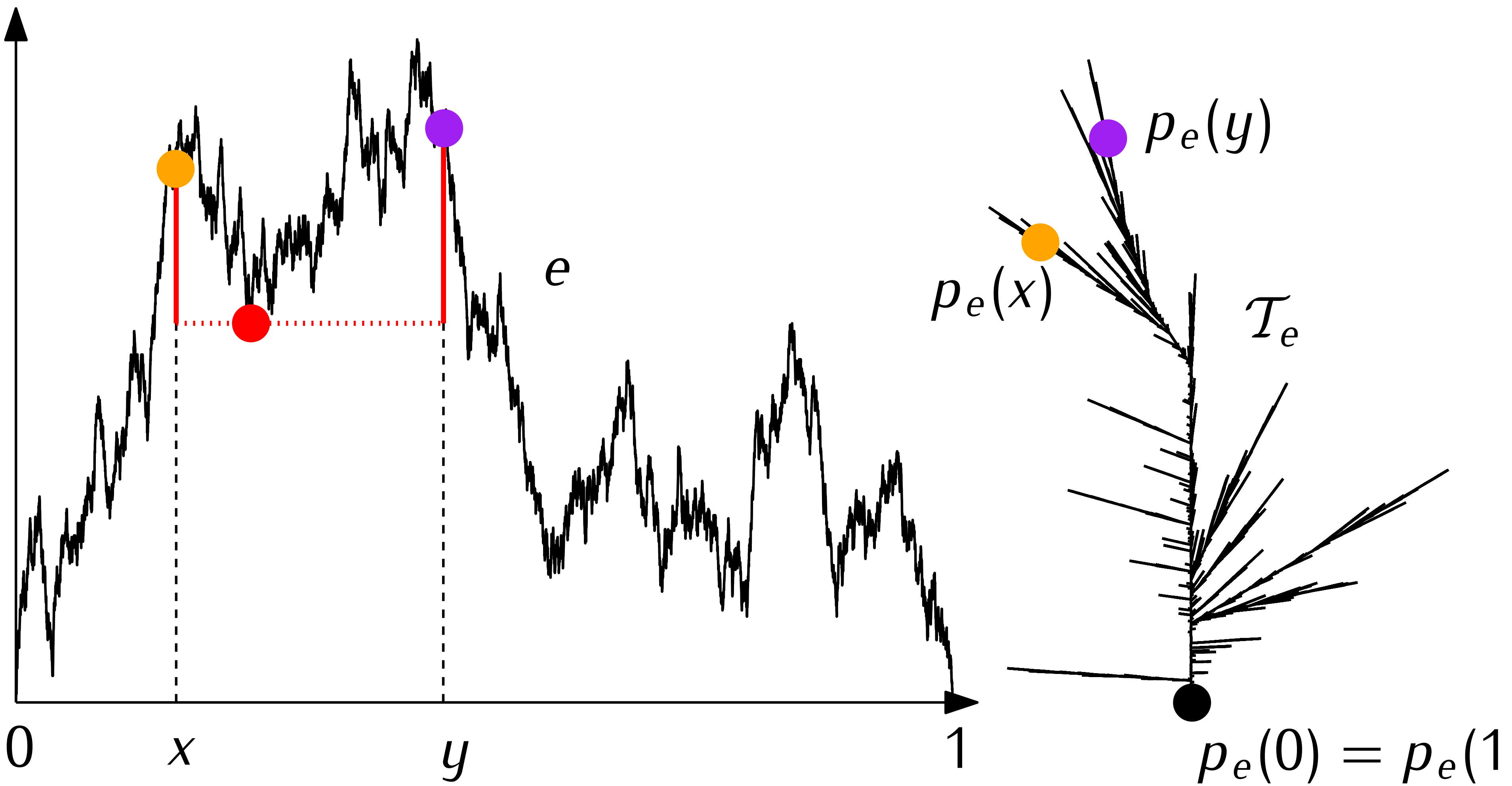
$$d_e(x, y) = e(x) + e(y) - 2\min_{[x,y]} e$$

The Brownian excursion and CRT



$d_e(x, y) = e(x) + e(y) - 2 \min_{[x,y]} e$ $\quad$ $\mathcal{T}_e = ([0, 1]/\{d_e = 0\}, d_e)$

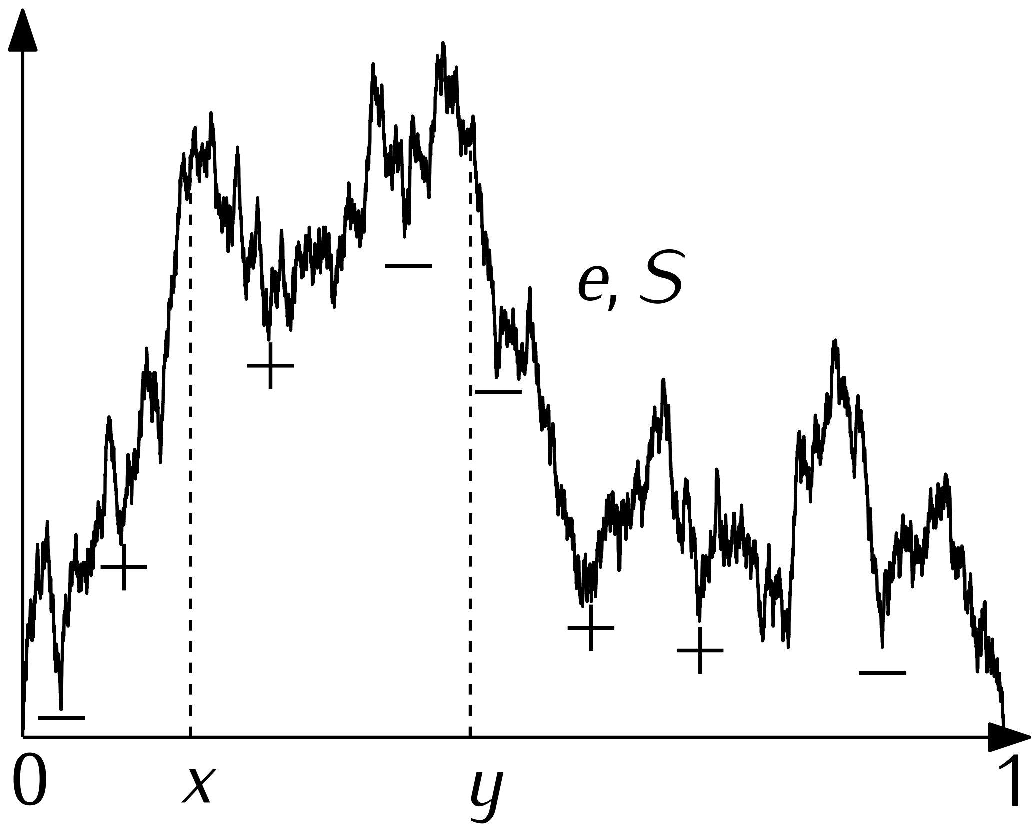
The Brownian excursion and CRT



$d_e(x,y) = e(x) + e(y) - 2\min_{[x,y]} e \qquad \mathcal{T}_e = ([0,1]/\{d_e = 0\}, d_e)$

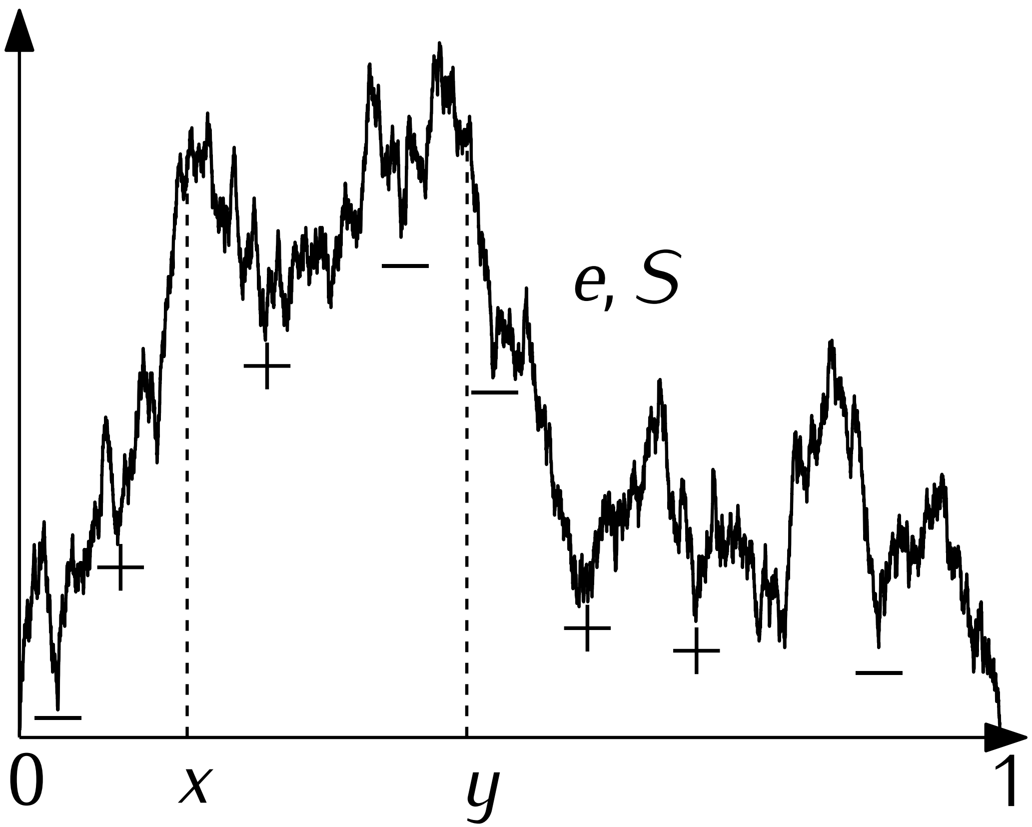
e contains more information than the metric space $\mathcal{T}_e$: 1) a mass measure 2) a DFS ordering of the vertices, $\iff$ an ordering of the two subtrees at each branching point.

The signed Brownian excursion



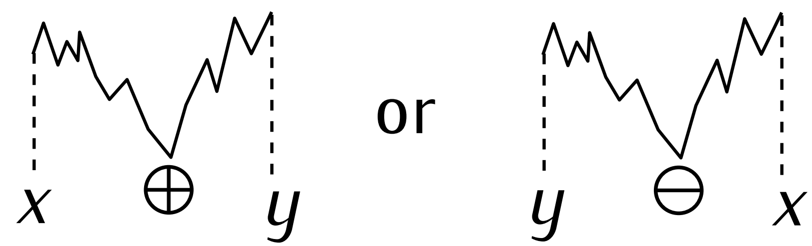
e Brownian excursion, S i.i.d. signs indexed by the local minima of e.

The signed Brownian excursion

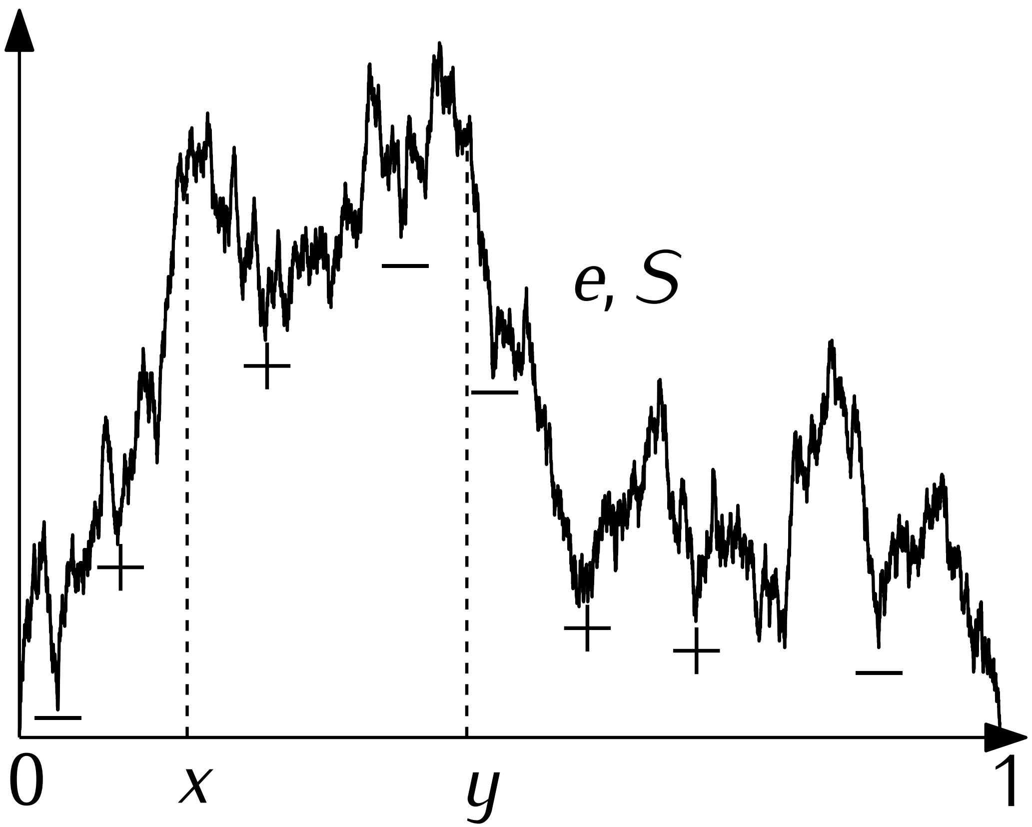


e Brownian excursion, S i.i.d. signs indexed by the local minima of e.

Define a shuffled pseudo-order on $[0, 1]$: $x \lhd_e^S y$ if and only if

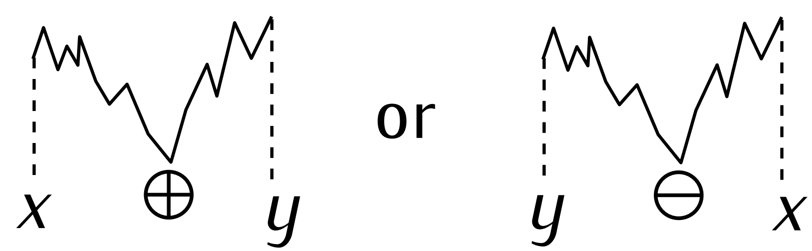


The signed Brownian excursion



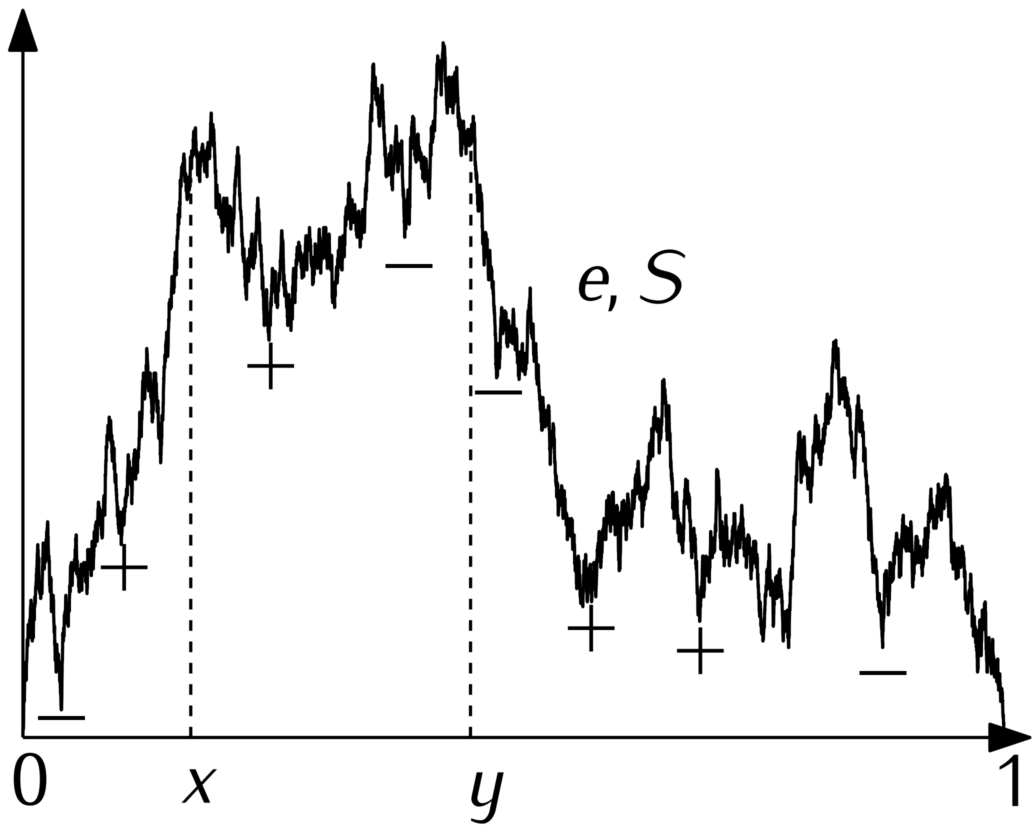
e Brownian excursion, S i.i.d. signs indexed by the local minima of e.

Define a shuffled pseudo-order on $[0, 1]$: $x \lhd_e^S y$ if and only if



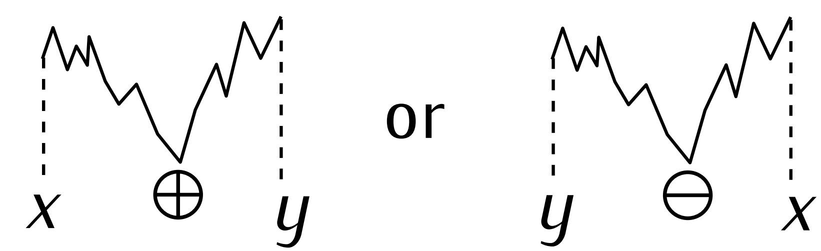
We can interpret $\lhd_e^S$ as a DFS ordering on the tree $\mathcal{T}_e$, different from the one given by e.

The signed Brownian excursion



e Brownian excursion, S i.i.d. signs indexed by the local minima of e.

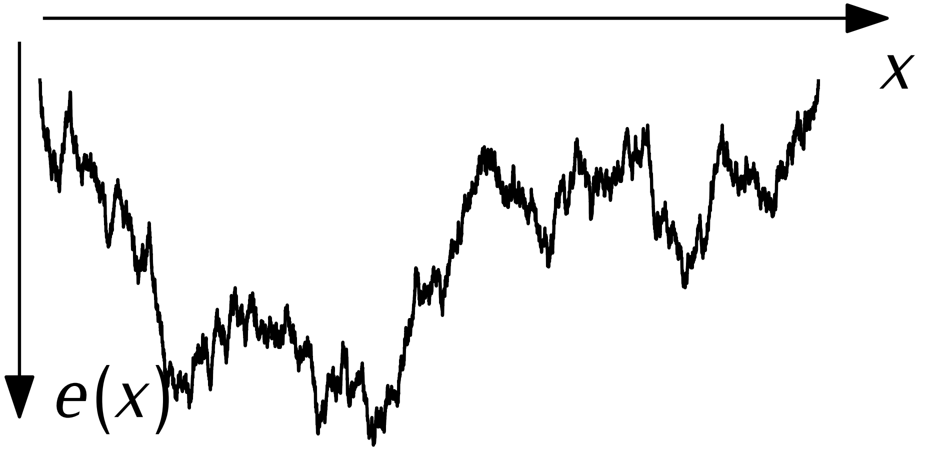
Define a shuffled pseudo-order on $[0,1]$: $x \lhd_e^S y$ if and only if



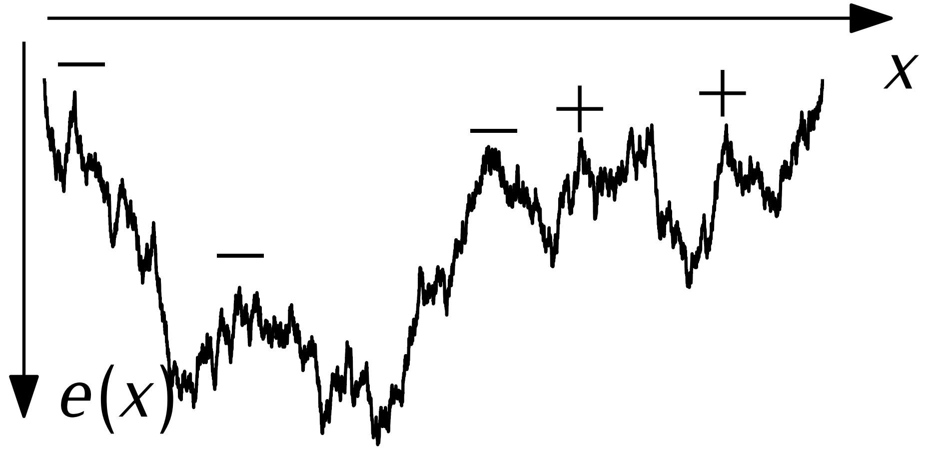
We can interpret $\lhd_e^S$ as a DFS ordering on the tree $\mathcal{T}_e$, different from the one given by e.

We set $\varphi(t) = \mathrm{Leb}(\{u \in [0,1], u \lhd_e^S t\})$.

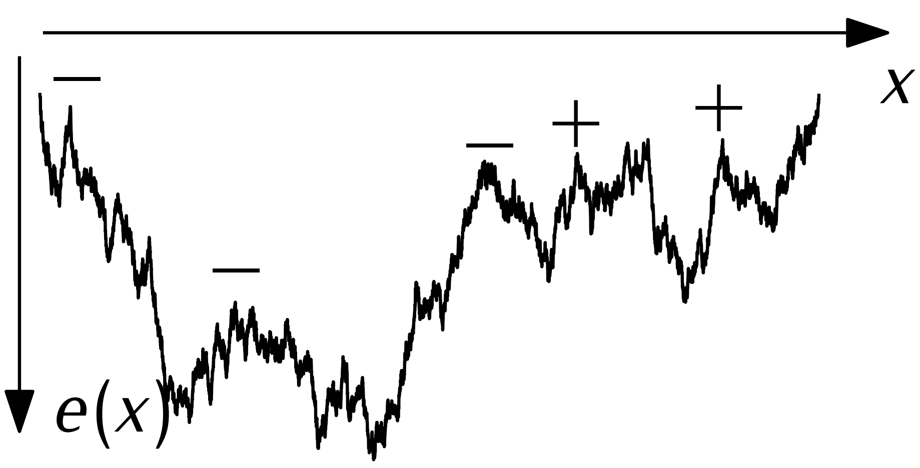
Constructing the Brownian permuton



Constructing the Brownian permuton

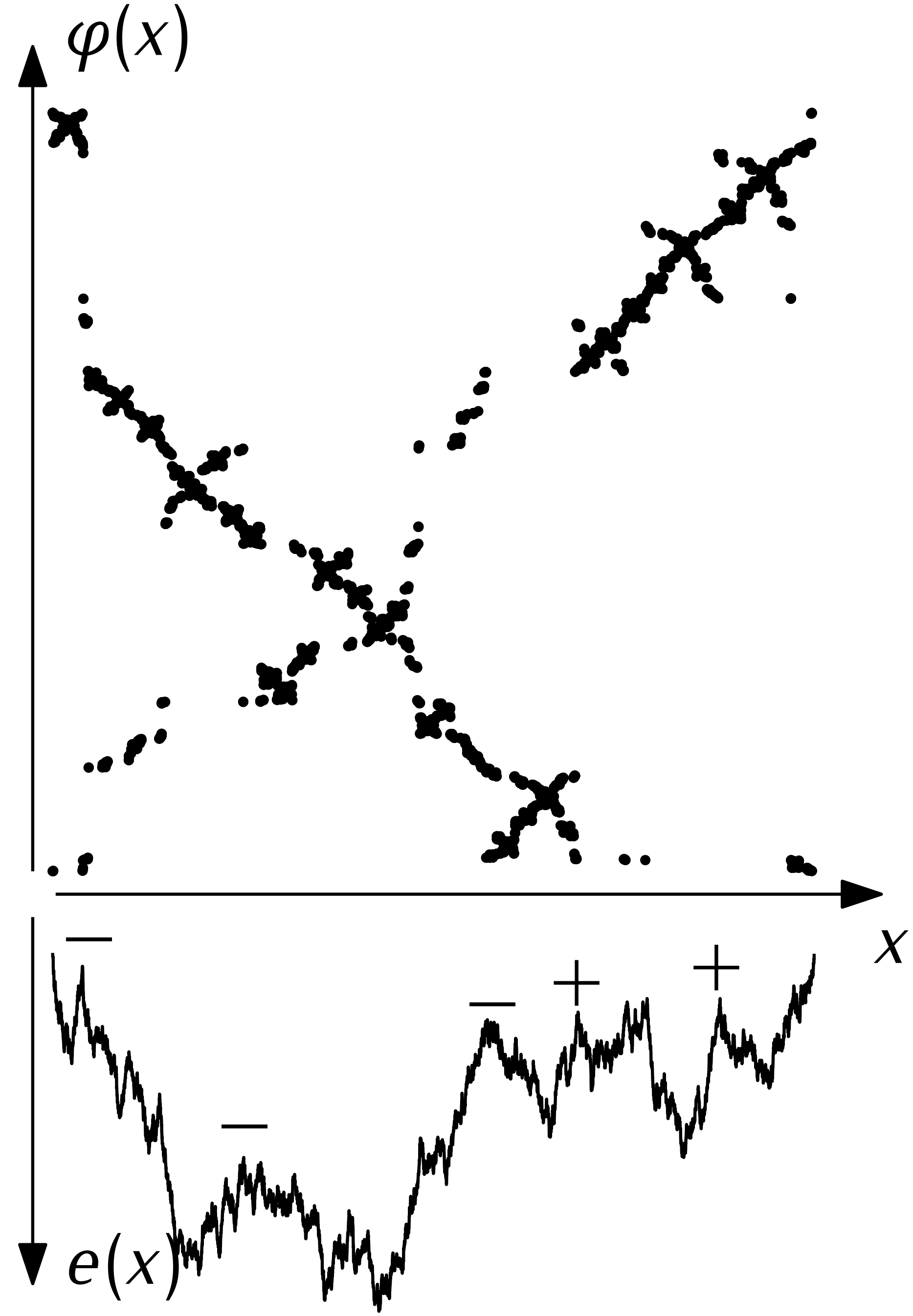


Constructing the Brownian permuton



Constructing the Brownian permuton

Theorem (M. 2017)
A.s. φ is $(\lhd_e^S, \leq)$ increasing and Lebesgue-preserving, uniquely characterized up to a.s. equality by these properties.
The random measure $(\mathrm{id}, \varphi)_\star \mathrm{Leb}$ has the law of the Brownian separable permuton.

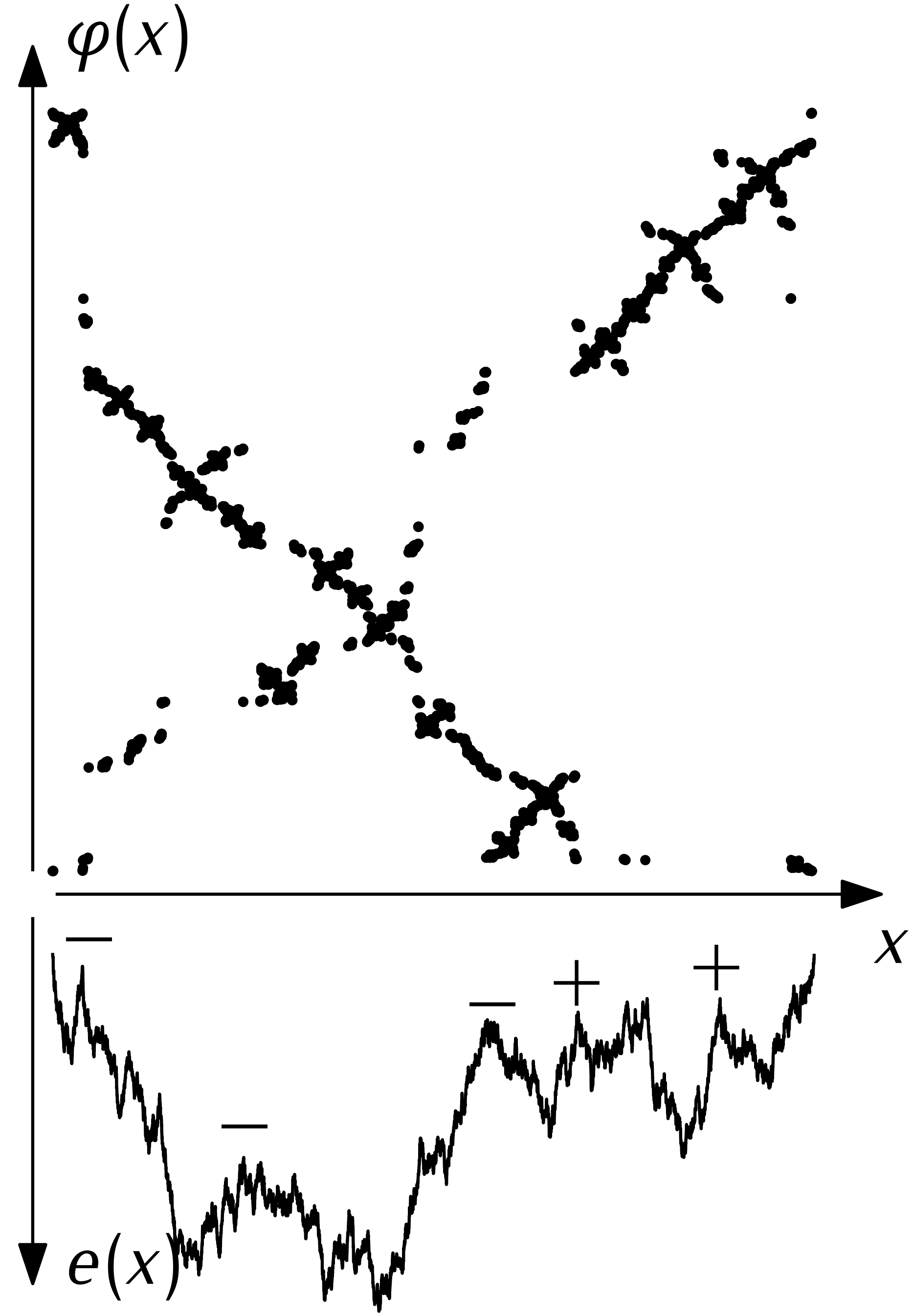


Constructing the Brownian permuton

Theorem (M. 2017)
A.s. φ is $(\lhd_e^S, \leq)$ increasing and Lebesgue-preserving, uniquely characterized up to a.s. equality by these properties.
The random measure $(\mathrm{id}, \varphi)_\star\mathrm{Leb}$ has the law of the Brownian separable permuton.

φ is continuous at every leaf (point which is not a one-sided local minimum) of e (full Lebesgue measure).

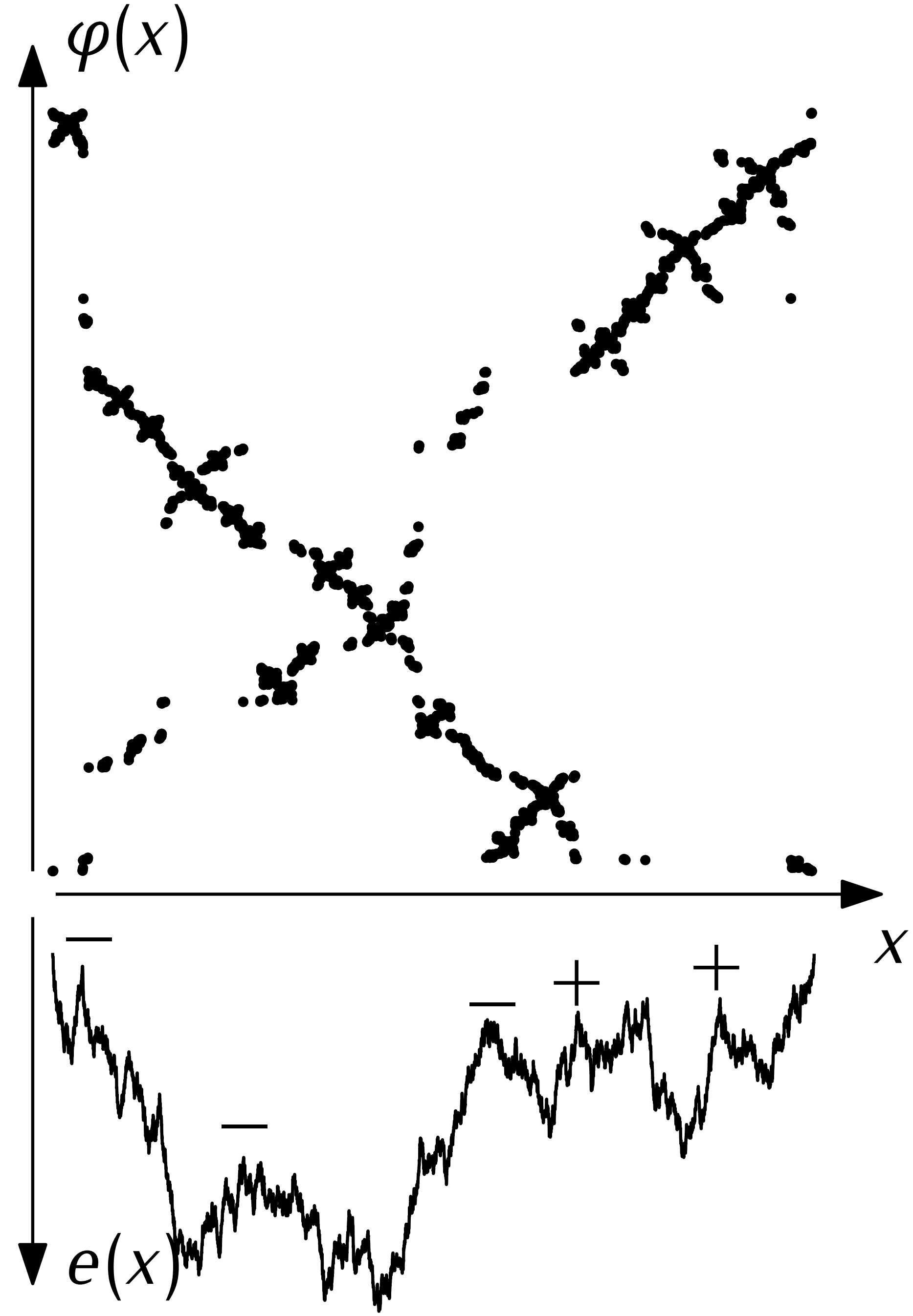
$\rightsquigarrow$ The support of μ is of Hausdorff dimension 1



Constructing the Brownian permuton

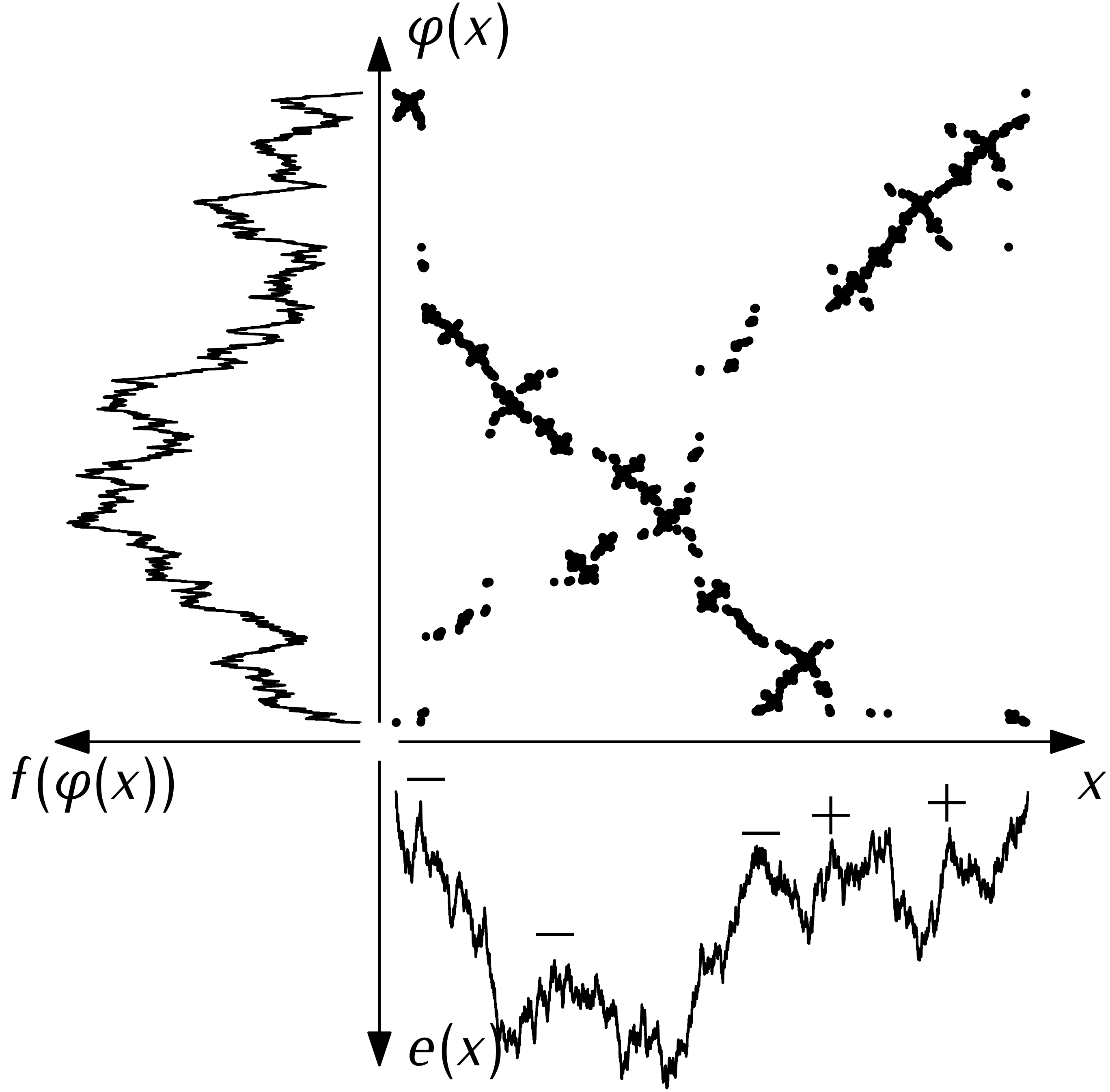
Theorem (M. 2017)
A.s. φ is $(\lhd_e^S, \leq)$ increasing and Lebesgue-preserving, uniquely characterized up to a.s. equality by these properties.
The random measure $(\mathrm{id}, \varphi)_\star\mathrm{Leb}$ has the law of the Brownian separable permuton.

Discontinuities at every strict local minima of e (dense)
$\rightsquigarrow$ The support of μ is totally disconnected.



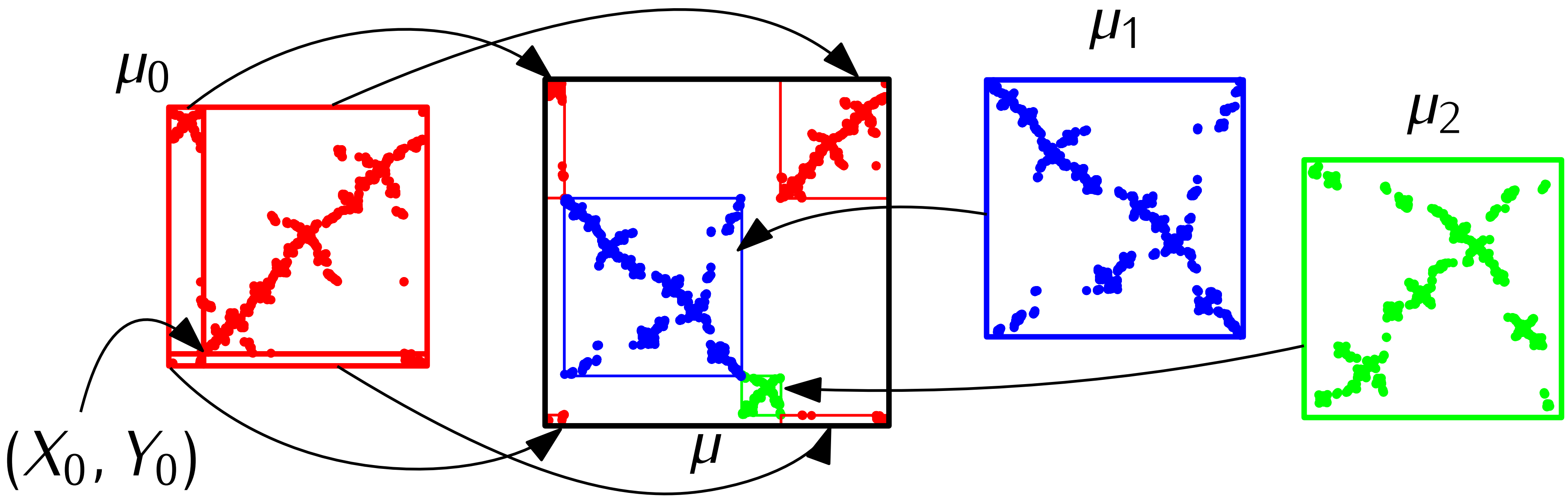
Constructing the Brownian permuton

There exists a Brownian excursion f defined on the same probability space such that $f \circ \varphi = e$.
a.s., $\mathcal{T}_f$ is isometric to $\mathcal{T}_e$.



Self-similarity

The Brownian permuton can be obtained by cut-and-pasting three independent copies in distribution of itself. The first copy μ_0 is cut according to a sample $(X_0, Y_0) \sim \mu_0$.
The scaling is an independent Dirichlet$(1/2, 1/2, 1/2)$ vector.
The relative position of μ_1 and μ_2 is chosen independently and uniformly between $\oplus$ and $\ominus$.

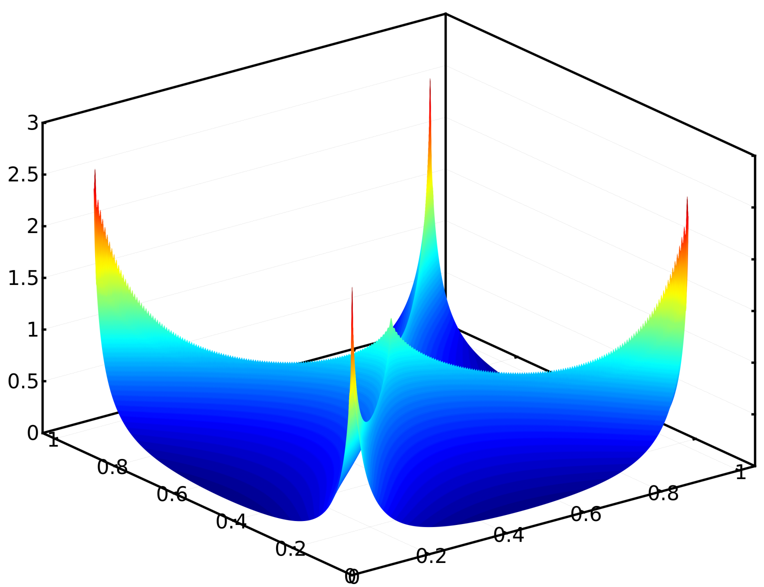


Expectation of the permuton

As μ is a random measure, it is natural to compute its average $\mathbb{E}\mu$, which is the limit of the permuton obtained by stacking all separable permutations of a given size.

Theorem The permuton $\mathbb{E}\mu$ has density function at $(x, y) \in [0,1]^2$

$$\int \frac{3\mathbb{1}_{[max(0,x+y-1),\min(x,y)]}(a)da}{\pi(a(x-a)(1-x-y+a)(y-a))^{\frac{3}{2}}\left(\frac{1}{a}+\frac{1}{(x-a)}+\frac{1}{(1-x-y+a)}+\frac{1}{(y-a)}\right)^{\frac{5}{2}}}.$$



This should be equal to the following formula, computed by Dokos and Pak (picture) for separable Baxter permutations (for $x \leq y \wedge (1-y)$, extended by symmetry)

$$\int_0^x \int_0^{x-u} \frac{dvdu}{4\pi[(u+v)(y-v)(1-y-u)]^{3/2}},$$