Scaling limits of permutation classes with a finite specification

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Part 0: Introduction
Permutation patterns

\[ \sigma = (10, 6, 2, 5, 3, 9, 1, 7, 4, 8, 11) \in \mathcal{S}_{11} \]
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Classes of permutation and pattern-avoidance

*Permutation class*: set of permutations closed under pattern extraction. Can always be written as $\text{Av}(B)$, the set of permutations that avoid patterns in some *basis* $B$. 
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Example: $\text{Av}(321)$ can be drawn on (MacMahon 1915), $\text{Av}(231)$ stack-sortable permutations (Knuth 1968), $\text{Av}(2413, 3142)$: separable permutations, $\text{Av}(321, 2143, 2413)$ are riffle shuffle permutations, ...
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*Permutation class*: set of permutations closed under pattern extraction. Can always be written as $\text{Av}(B)$, the set of permutations that avoid patterns in some *basis* $B$.

Example: $\text{Av}(321)$ can be drawn on \[\begin{array}{c}
1
2
3
\end{array}\] (MacMahon 1915), $\text{Av}(231)$ stack-sortable permutations (Knuth 1968), $\text{Av}(2413, 3142)$: separable permutations, $\text{Av}(321, 2143, 2413)$ are riffle shuffle permutations, ... 

What does a large permutation in a class *look like*?
$\mathfrak{S}_n$

Av(231)

Av(4321)

Av(2413, 3142, 2143, 34512)

Av(2413, 3142) = \{separables\}

(E. Slivken)

(Madras-Yildirim)
A large uniform separable permutation
A large uniform separable permutation
Permutons

A permuton is a probability measure on $[0, 1]^2$ with both marginals uniform.

$\implies$ compact metric space (with weak convergence).

Permutations of all sizes are densely embedded in permutons.
The Brownian limit of separable permutations

$\sigma_n$ uniform of size $n$ in $C = \text{Av}(2413, 3142) = \{\text{separables}\}$:

**Theorem** (Bassino, Bouvel, Féray, Gerin, Pierrot 2016)

$\sigma_n$ converges in distribution to some random permuton $\mu$, called the Brownian separable permuton.
The main theorem.

**Theorem** (BBFGMP 2019)
Many other classes of permutation converge also to the Brownian permuton, or a 1-parameter deformation. Those behave nicely under the so-called "substitution-decomposition" (precise statement later)
The main theorem.

**Theorem** When $\mathcal{C} = \text{Av}(31452, 41253, 41352, 531642, 25413, 35214, 25314, 246135)$, $\mu_{\sigma_n}$ also converges to the Brownian permuton.
The main theorem.

Theorem: When $C = Av(2413, 1243, 2341, 531642, 41352)$, $\mu_{\sigma_n}$ converges to a deterministic V-shape.
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\[ x \approx 0.818632668576995 \] is the only real root of
\[ 19168x^5 - 86256x^4 + 155880x^3 - 141412x^2 + 64394x - 1177 \]
Part 1 - the proof method

(illustrated on the case of separable permutations)
0 - General idea and limit object

Characterization of separable permutations:

Signed tree $\tau$
0 - General idea and limit object

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Separable permutation

$\text{perm}(\tau) = (1 \, 2 \, 10 \, 7 \, 6 \, 5 \, 8 \, 9 \, 4 \, 3)$
Signed tree $\tau$

Separable permutation

$\text{perm}(\tau) = (12107658943)$
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Signed tree $\tau$

Separable permutation
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Alternating-signs Schröder tree

Separable permutation
perm(τ) = (1 2 10 7 6 5 8 9 4 3)
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Alternating-signs Schröder tree

Separable permutation

perm(τ) = (1 2 10 7 6 5 8 9 4 3)

Counted by large Schröder numbers

1, 2, 6, 22, 90, 394, 1806, 8558, ... \approx (3 + \sqrt{8})^n n^{-3/2}
0 - General idea and limit object
Many "nice" models of random trees \((t_n)_n\) where \(n\) is the size, converge to (a multiple of) the Brownian CRT when distances are rescaled by \(\sqrt{n}\). More precisely, if \(C_n\) is the contour function of \(t_n\), for some constant \(c > 0\), \(cn^{-1/2}C_n\) converges in distribution to the normalized Brownian excursion.
Many "nice" models of random trees \((t_n)_n\) where \(n\) is the size, converge to (a multiple of) the Brownian CRT when distances are rescaled by \(\sqrt{n}\). More precisely, if \(C_n\) is the contour function of \(t_n\), for some constant \(c > 0\), \(cn^{-1/2}C_n\) converges in distribution to the normalized Brownian excursion.
Leaf-counted Schröder trees are (critical, finite-variance) BGW trees conditioned on the number of leaves and fall in this category (Kortchemski ’12, Pitman-Rizzolo ’12)

\[ cn^{-1/2}C_n(t) \xrightarrow{d} e(t) \]
0 - General idea and limit object

The main point: signs at macroscopic branching points become independent as the tree gets larger. This tells us how the corresponding permutation looks like in the large scale.

\[ cn^{-1/2}C_n(t) \]
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$e$ Brownian excursion, $S$ i.i.d. balanced signs indexed by the local minima of $e$. 
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![Diagram of Brownian excursion with signs indexed by local minima]
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balanced signs indexed by the
local minima of $e$.

Define a shuffled pseudo-order
on $[0, 1]$: $x \prec_S y$ if and only if

$$
\begin{align*}
x & \bigoplus y \\
y & \bigominus x
\end{align*}
$$
Brownian excursion, $S$ i.i.d. balanced signs indexed by the local minima of $e$.

Define a shuffled pseudo-order on $[0, 1]$: $x \triangleleft^S_e y$ if and only if

or

$\varphi(t) = \text{Leb}\left(\{u \in [0, 1], u \triangleleft^S_e t\}\right)$ is the only (up to a.e. equality) Lebesgue-preserving function sending $\leq$ to $\triangleleft^S_e$.
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Define a shuffled pseudo-order on $[0, 1]$: $x \triangleleft^S_e y$ if and only if

\[
\begin{align*}
x & \oplus y \\
y & \ominus x
\end{align*}
\]

or

\[
\varphi(t) = \operatorname{Leb}(\{u \in [0, 1], u \triangleleft^S_e t\})
\]

is the only (up to a.e. equality) Lebesgue-preserving function sending $\leq$ to $\triangleleft^S_e$

Then $\mu = (\operatorname{id}, \varphi)_* \operatorname{Leb}$ is the Brownian separable permuton (M. 2017)
For $\sigma \in \mathcal{S}_n$ and $k \leq n$, $\text{perm}_k(\sigma)$ is a uniform subpermutation of length $k$ in $\sigma$. 
For \( \sigma \in S_n \) and \( k \leq n \), \( \text{perm}_k(\sigma) \) is a uniform subpermutation of length \( k \) in \( \sigma \).

This notion is extended to permutons: \( \text{perm}_k(\mu) \) is the random permutation that is order-isomorphic to an i.i.d. pick according to \( \mu \).
I - Permuton convergence and patterns

For $\sigma \in \mathcal{S}_n$ and $k \leq n$, $\text{perm}_k(\sigma)$ is a uniform subpermutation of length $k$ in $\sigma$.

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**Theorem** (Hoppen et. al. ’2013, BBFGMP ’2017)

The random permutons $(\mu_{\sigma_n})$ converge in distribution to $\mu$ iff for every $k$, $\text{perm}_k(\sigma_n) \xrightarrow{d} \text{perm}_k(\mu)$. 
II - Patterns and the tree encoding

A subpermutation of $\sigma_n$ can be read on a reduced tree of $t_n$
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Consider a uniform $k$-reduced tree of a Schröder tree of size $n$. Here $k = 3$. 

\[
\begin{align*}
I_n^k &\quad t_n | I_n^k &\quad \text{pat}_{I_n^k}(\sigma_n)
\end{align*}
\]
Consider a uniform $k$-reduced tree of a Schröder tree of size $n$. Here $k = 3$. What does it look like as $n \to \infty$?
A subpermutation of $\sigma_n$ can be read on a reduced tree of $t_n$.

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What does it look like as $n \to \infty$?
III - Patterns in the Brownian permuton

$\varphi(x)$

$e(x)$
Reduced trees of the Brownian excursion are uniform binary trees (Aldous ’93, Le Gall ’93)
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Hence $\text{perm}_k(\mu)$ has the distribution of $\text{perm}(b_k)$ where $b_k$ is a uniform signed signed binary tree with $k$ leaves.
Fix a signed binary tree $\tau$ with $k$ leaves. We need only show that

$$\frac{\#\{\text{Schröder trees of size } n \text{ with } k \text{ marked leaves inducing } \tau\}}{\#\{\text{Schröder trees of size } n \text{ with } k \text{ marked leaves}\}}$$

converges to

$$\mathbb{P}(b_k = \tau) = \frac{1}{2^{k-1}\text{Cat}_{k-1}}.$$
Let \((a_n)_n\) be a nonnegative sequence and \(A(z) = \sum_n a_n z^n\) its generating function of radius \(\rho\).

**Transfer Theorem (Flajolet & Odlyzko)** If

- \(A\) is defined on a \(\Delta\)-domain at \(\rho > 0\) (e.g. is algebraic)
- \(A(z) \xrightarrow{z \to \rho} g(z) + (C + o(1))(\rho - z)^\delta\) with \(g\) analytic,
  \[\delta \notin \mathbb{N},\]
then \(a_n \xrightarrow{n \to \infty} \left(\frac{C}{\Gamma(-\delta)} + o(1)\right)\rho^{-n} n^{-1-\delta}\)

**Proposition (Singular differentiation)** Under the same hypotheses, \(A'(z) \xrightarrow{z \to \rho} g'(z) + \delta(C + o(1))(\rho - z)^{\delta-1}\)
Analytic combinatorics for leaf-counted trees

Recall: nice trees converge to the Brownian CRT.
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Recursive trees counted by number of leaves.

\[ T(z) = z + F(T(z)) \quad \text{(Schröder: } F(t) = \sum_{k \geq 2} t^k \text{)}. \]
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In this case, "very nice" if
\[ \exists \ 0 < u < R_F, F'(u) = 1. \]
Then \( T \) is \( \Delta \)-analytic at \( \rho \) with \( T(\rho) = u \) and a square-root singularity (smooth implicit function schema).
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This is the case for Schröder (\( F \) rational)
Uniform \( k \)-subtree in large unsigned trees

\( T \) has square-root singularity at \( \rho \) and \( F \) analytic at \( T(\rho) \). Then, the g.f of trees with \( k \) marked leaves that induce the \( k \)-tree \( \tau \) is

\[
z^{k} T'(z) \prod_{v \text{ internal node of } \tau} T'(z)^{\text{deg}(v)} \frac{1}{\text{deg}(v)!} F^{(\text{deg}(v))}(T(z))
\]
Uniform $k$-subtree in large unsigned trees

$T$ has square-root singularity at $\rho$ and $F$ analytic at $T(\rho)$. Then, the g.f of trees with $k$ marked leaves that induce the $k$-tree $\tau$ is

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$$\sim_{\rho} C_\tau (\rho - z)^{-\#\{\text{nodes in } \tau\}/2}.$$ Dominates when $\tau$ binary. (Then $C_\tau$ doesn’t depend on $\tau$).

Transfer: $t_n |_{I_n^k}$ converges in distribution to a uniform binary tree.
Uniform $k$-subtree in large signed trees

Counting signed trees that induce a given signed tree $\tau$: adding parity constraints on the height of the marked leaf in the marked trees.
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Counting signed trees that induce a given signed tree $\tau$: adding parity constraints on the height of the marked leaf in the marked trees.

Replace instances of $T'$ by $T'_0$ (even height) or $T'_1$ (odd height). $T'_0 + T'_1 = T'$ and $T'_1 = F'(T)T'_0$, so $T'_0 \sim T'_1 \sim \frac{1}{2}T'$. 
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g.f. of Trees with \( k \) marked leaves that induce the signed \( k \)-tree \( \tau \):

\[
z^k(T'_0 + T'_1)T'_0^b T'_1^a T'^k \prod_{v \text{ internal node of } \tau} \frac{1}{\deg(v)!} F^{(\deg(v))}(T(z))
\]

where \( a \) (resp. \( b \)) is the number of edges of \( \tau \) incident to two nodes of the same (resp. different) signs.
Uniform $k$-subtree in large signed trees

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g.f. of Trees with $k$ marked leaves that induce the signed $k$-tree $\tau$:

$$z^k(T'_0 + T'_1)T'_0^b T'_1^a T'^k \prod_{\text{internal node of } \tau} \frac{1}{\text{deg}(v)!} F^{(\text{deg}(v))}(T(z))$$

where $a$ (resp. $b$) is the number of edges of $\tau$ incident to two nodes of the same (resp. different) signs.

Hence all signed binary trees have the same asymptotic probability, what whe needed for permuton convergence.
Part 2 - statement
Substitution decomposition

For $\sigma \in \mathfrak{S}_k$, $\rho_1, \ldots, \rho_k \in \mathfrak{S}$, define $\sigma[\rho_1, \ldots, \rho_k]$ by replacing the $i$-th dot in $\sigma$ by $\pi_i$.

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Example: \( 132[21, 312, 2413] = 219784635 \).
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Example: $132[21, 312, 2413] = 219784635$. $\oplus$ and $\ominus$ are just substitutions into increasing and decreasing permutations.
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Given $\sigma$, either:
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Given \( \sigma \), either:

- We can find a proper interval mapped to an interval, and then \( \sigma \) can be written as a substitution of smaller permutations.
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$\oplus$ and $\ominus$ are just substitutions into increasing and decreasing permutations.

Given $\sigma$, either:

- We can find a proper interval mapped to an interval, and then $\sigma$ can be written as a substitution of smaller permutations.
- Or $\sigma$ can’t be decomposed by a nontrivial substitution: $\sigma$ is a **simple permutation**. Ex: $1, 12, 21, 2413, 3142, 31524, \ldots \sim \frac{n!}{e^2}$. 


Substitution decomposition

(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)
Substitution decomposition

\( (8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5) \)
Substitution decomposition

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(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)

42513

2413

⊕

⊖
Substitution decomposition

(8, 10, 9, 2, 11, 1, 4, 7, 3, 6, 5)

\[42513 \oplus 2413 \ominus \]
Theorem (Albert, Atkinson 2005): Any permutation can be decomposed into a substitution tree with nodes labeled by simple permutations, unique as long as no $\oplus$ is the left child of a $\oplus$ (same for $\ominus$)
Study classes using substitution

$\mathcal{S} \subset \{\text{simple permutations} \}.$

$\tilde{\mathcal{S}} = \{\text{permutations built by substituting simples of } \mathcal{S}\}.$
Study classes using substitution

\( S \subset \{ \text{simple permutations} \} \).

\( \tilde{S} = \{ \text{permutations built by substituting simples of } S \} \).

**Proposition:** Let \( C = \text{Av}(B) \) be a class. Then \( C \subset \tilde{S}_C \) where \( S_C \) is the set of simple permutations in \( C \).

When \( B \) has only simples, then \( C = \tilde{S}_C \). We say that \( C \) is substitution-closed.
Study classes using substitution

$S \subset \{ \text{simple permutations } \}$. 
\[ \tilde{S} = \{ \text{permutations built by substituting simples of } S \}. \]

**Proposition:** Let $C = \text{Av}(B)$ be a class. Then $C \subset \tilde{S}_C$ where $S_C$ is the set of simple permutations in $C$. 
When $B$ has only simples, then $C = \tilde{S}_C$. We say that $C$ is substitution-closed.

This is the case of the separable permutations 
$\text{Av}(2413, 3142) = \{ \oplus, \ominus \}$. 
Specifications

A substitution-closed-class $\mathcal{T}$ has the following specification:
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$$\mathcal{T} = \{\bullet\} \uplus \oplus [\mathcal{T}^{\mathrm{not} \oplus}, \mathcal{T}] \uplus \ominus [\mathcal{T}^{\mathrm{not} \ominus}, \mathcal{T}] \uplus \left( \biguplus_{\pi \in S_{\mathcal{T}}, |\pi| \geq 4} \pi[\mathcal{T}, \ldots, \mathcal{T}] \right)$$

$$\mathcal{T}^{\mathrm{not} \oplus} = \{\bullet\} \uplus \ominus [\mathcal{T}^{\mathrm{not} \ominus}, \mathcal{T}] \uplus \left( \biguplus_{\pi \in S_{\mathcal{T}}, |\pi| \geq 4} \pi[\mathcal{T}, \ldots, \mathcal{T}] \right)$$

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Specifications

A substitution-closed-class \( \mathcal{T} \) has the following specification:

\[
\mathcal{T} = \{ \bullet \} \uplus \ominus [\mathcal{T}^{\text{not} \oplus}, \mathcal{T}] \uplus \ominus [\mathcal{T}^{\text{not} \ominus}, \mathcal{T}] \uplus \left( \bigcup_{\pi \in S_{\mathcal{T}}, |\pi| \geq 4} \pi[\mathcal{T}, \ldots, \mathcal{T}] \right)
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→ system of equations on the generating functions of the specified families, made of analytic functions with nonnegative coefficients.
→ a Boltzmann sampler for the class.
Specifications

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→ system of equations on the generating functions of the specified families, made of analytic functions with nonnegative coefficients.
→ a Boltzmann sampler for the class.
→ trees coding specification-closed classes are 3-type Galton-Watson trees conditioned on their number of leaves. In BBFGMP 2017 we treat substitution-closed classes in wider generality.
Theorem (Bassino, Bouvel, Pivoteau, Pierrot, Rossin 2017) If $S_T$ is finite, then there is a finite specification

$$T_i = \varepsilon_i \{\bullet\} \cup \bigcup_{\pi \in S_T} \bigcup_{(k_1, \ldots, k_{|\pi|}) \in K_{\pi}} \pi[T_{k_1}, \ldots, T_{k_{|\pi|}}]$$

where $T = T_0 \supset T_1, \ldots, T_d$ and $\varepsilon_i \in \{0, 1\}$. Moreover, there is an algorithm (implemented!) to find it.

→ system of equations on the generating functions of the specified families, made of analytic functions with nonnegative coefficients.
→ a Boltzmann sampler for the class.
The case of $\text{Av}(132)$

\[
\mathcal{T} = \{ \bullet \} \bigcup \oplus [\mathcal{T}^\text{not} \oplus, \mathcal{T}_{\langle 21 \rangle}] \bigcup \ominus [\mathcal{T}^\text{not} \ominus, \mathcal{T}]
\]

$\mathcal{T}^\text{not} \oplus = \{ \bullet \} \bigcup \ominus [\mathcal{T}^\text{not} \ominus, \mathcal{T}]$

$\mathcal{T}^\text{not} \ominus = \{ \bullet \} \bigcup \ominus [\mathcal{T}^\text{not} \ominus, \mathcal{T}_{\langle 21 \rangle}]$

$\mathcal{T}_{\langle 21 \rangle} = \{ \bullet \} \bigcup \ominus [\mathcal{T}^\text{not} \ominus, \mathcal{T}_{\langle 21 \rangle}]
$

$\mathcal{T}^\text{not} \ominus_{\langle 21 \rangle} = \{ \bullet \}$.
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\]

\[
\mathcal{T}_{\langle 21 \rangle} = \{\bullet\} \uplus \ominus[\mathcal{T}_{\langle 21 \rangle}^{\text{not} \ominus}, \mathcal{T}_{\langle 21 \rangle}]
\]

\[
\mathcal{T}_{\langle 21 \rangle}^{\text{not} \ominus} = \{\bullet\}.
\]

We plot the dependency graph of the system. In gray, critical families, of maximal growth rate (minimal radius of convergence)
The main theorem

Theorem (BBFGMP 2019) Consider the specification of a class $C$ with a finite number of simples. Assume that there is only one strongly connected critical component.
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**Theorem** (BBFGMP 2019) Consider the specification of a class $C$ with a finite number of simples. Assume that there is only one strongly connected critical component.

If the specification is *linear* in the critical families, then $\sigma_n$ converges to a $X$-permutoon with explicit parameters.
The main theorem

**Theorem** (BBFGMP 2019) Consider the specification of a class $C$ with a finite number of simples. Assume that there is only one strongly connected critical component.

If the specification is *linear* in the critical families, then $\sigma_n$ converges to a $X$-permuton with explicit parameters.

Otherwise, $\sigma_n$ converges to a biased Brownian permuton of explicit parameter.
Examples: linear case

The $V$-shape class from earlier:

\[
\begin{align*}
\mathcal{T}_0 &= \{\bullet\} \uplus \oplus [\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus [\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus [\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus [\mathcal{T}_5, \mathcal{T}_0] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\
\mathcal{T}_1 &= \{\bullet\} \uplus \ominus [\mathcal{T}_7, \mathcal{T}_1] \\
\mathcal{T}_2 &= \{\bullet\} \uplus \oplus [\mathcal{T}_7, \mathcal{T}_2] \\
\mathcal{T}_3 &= \ominus [\mathcal{T}_8, \mathcal{T}_2] \uplus \ominus [\mathcal{T}_9, \mathcal{T}_6] \\
\mathcal{T}_4 &= \ominus [\mathcal{T}_{10}, \mathcal{T}_{11}] \uplus \ominus [\mathcal{T}_{10}, \mathcal{T}_1] \uplus \ominus [\mathcal{T}_7, \mathcal{T}_{11}] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\
\mathcal{T}_5 &= \{\bullet\} \uplus \ominus [\mathcal{T}_1, \mathcal{T}_1] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\
\mathcal{T}_6 &= \{\bullet\} \uplus \ominus [\mathcal{T}_{12}, \mathcal{T}_2] \uplus \ominus [\mathcal{T}_9, \mathcal{T}_6] \\
\mathcal{T}_7 &= \{\bullet\} \\
\mathcal{T}_8 &= \ominus [\mathcal{T}_9, \mathcal{T}_6] \\
\mathcal{T}_9 &= \{\bullet\} \uplus \ominus [\mathcal{T}_1, \mathcal{T}_7] \\
\mathcal{T}_{10} &= \ominus [\mathcal{T}_1, \mathcal{T}_1] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1] \\
\mathcal{T}_{11} &= \ominus [\mathcal{T}_1, \mathcal{T}_2] \uplus \ominus [\mathcal{T}_1, \mathcal{T}_3] \uplus \ominus [\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus [\mathcal{T}_{10}, \mathcal{T}_{11}] \uplus \ominus [\mathcal{T}_{10}, \mathcal{T}_1] \uplus \ominus [\mathcal{T}_7, \mathcal{T}_{11}] \\
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\mathcal{T}_{12} &= \{\bullet\} \uplus \ominus[\mathcal{T}_9, \mathcal{T}_6]
\end{align*}
\]

Critical series are \( \mathcal{T}_0, \mathcal{T}_4, \mathcal{T}_{11} \). The critical system is not strongly connected, but aae permutation of \( \mathcal{T}_0 \) is in \( \mathcal{T}_{11} \). Removing \( \mathcal{T}_0 \) we can apply the theorem.
Examples: linear case

$Av(2413, 1243, 2341, 41352, 531642)$

$Av(2413, 3142, 2143, 34512)$
Examples: linear case

\[ \text{Av}(2413, 1243, 2341, 41352, 531642) \quad \text{Av}(2413, 3142, 2143, 34512) \]
Examples: nonlinear case.

$A v(132)$
$p = 1$

$A v(2413, 31452, 41253, 531642, 41352)$
$p \approx 0.47$ is algebraic of degree 9.
Examples: nonlinear case.

$Av(132)$

$p = 1$

$Av(2413, 31452, 41253, 531642, 41352)$

$p \approx 0.47$ is algebraic of degree 9.
Part 3 - proof of the main theorem

(in the nonlinear case)
Substitution decomposition and patterns

\[ \sigma = 24387156 \]

\[ \text{pat}_I(\sigma) = 4123 \]
Our goal

Fix a signed binary tree $\tau$ with $k$ leaves. We need only show that

$$\frac{\#\{\text{trees in } T \text{ of size } n \text{ with } k \text{ marked leaves inducing } \tau\}}{\#\{\text{trees in } T \text{ of size } n \text{ with } k \text{ marked leaves}\}}$$

converges to

$$\mathbb{P}(b^p_k = \tau) = \frac{p^\# \oplus (1 - p)^\# \ominus}{\text{Cat}_{k-1}}.$$
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Fix a signed binary tree $\tau$ with $k$ leaves. We need only show that

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converges to

$$P(b_k^p = \tau) = \frac{p^{#\oplus}(1 - p)^{#\ominus}}{\text{Cat}_{k-1}}.$$

The denominator is $[z^{n-k}] T_0^{(k)}$. 

G.F. of the numerator
$\sum_{a,b,c,d,e,f,g,h,i}$

G.F. of the numerator
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\[ \sum_{a,b,c,d,e,f,g,h,i} \]
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G.F. of the numerator

\[ G.F. \text{ of the numerator} \]

\[ \sum_{a,b,c,d,e,f,g,h,i} \partial_{e,f}^{-} F_{d}(z, T) \]

\[ \partial_{b,c}^{+} F_{a}(z, T) \]

\[ \partial_{h,i}^{-} F_{g}(z, T) \]

\[ T_{a} \]

\[ T_{b} \]

\[ T_{c} \]

\[ T_{d} \]

\[ T_{e} \]

\[ T_{f} \]

\[ T_{g} \]

\[ T_{h} \]

\[ T_{i} \]
DLW Theorem

We can apply the following theorem to our system of equations, partially applied in the subcritical series.
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**Theorem** (Drmota 2009) Let \( T = \Phi(z, T) \) be a system of equations, \( \Phi = \Phi(z, t) \) with nonnegative coefficients and no constant term or \( t_i \) term. Assume that \( \Phi \) is analytic in \( z \) with radius \( > \rho \), polynomial and nonlinear in \( T \). Assume the graph of dependence is strongly connected. Then

1. All \( T_i \) have a square root singularity at \( \rho \)
   \[
   T(z) = T(\rho) - c(v + o(1))\sqrt{z - \rho}.
   \]

2. Defining \( (M_{i,j}(z))_{i,j} = \text{Jac}_T \Phi(z, T(z)) \), then \( M(\rho) \) has Perron eigenvalue 1 with left and right eigenvectors \( u \) and \( v \). Moreover
   \[
   (T^j_i)_{i,j} = (\text{Id} - M(z))^{-1} \sim_{z \to \rho} Cv u^T \frac{1}{\sqrt{z-\rho}}.
   \]
Asymptotics of numerator

\[ \partial^{-}_{e,f} F_a(z, T) \]

\[ \partial^{+}_{b,c} F_a(z, T) \]

\[ \sum_{a,b,c,d,e,f,g,h,i} \]

[Diagram with nodes and edges labeled with variables and operations]
Asymptotics of numerator

\[ K \sum_{a,b,c,d,e,f,g,h,i} \]
Asymptotics of numerator

\[ K \sum_{a,b,c,d,e,f,g,h,i} (z - \rho)^{-1/2} \]
Asymptotics of numerator

\[ u_d c_{def} v_e \]

\[ c_{abc} v_b \]

\[ K \sum_{a,b,c,d,e,f,g,h,i} (z - \rho)^{-2/2} \]

\[ K \sum_{a,b,c,d,e,f,g,h,i} \]

\[ T_{i}' \]

\[ T_{h}' \]

\[ T_{f}' \]

\[ c_{ghi} \]

\[ T_{a}^g \]

\[ T_{c}^{g} \]

\[ T_{0}^a \]
Asymptotics of numerator

\[ K \sum_{a,b,c,d,e,f,g,h,i} (z - \rho)^{-7/2} \]
Asymptotics of numerator

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\[ K \sum_{a,b,c,d,e,f,g,h,i} (z - \rho)^{-7/2} \]

\[ \sim KA_+^1 A_-^2 (z - \rho)^{-7/2} \]
Asymptotics of numerator

\[ K \sum_{a,b,c,d,e,f,g,h,i} (z - \rho)^{-7/2} \sim K A_+^1 A_-^2 (z - \rho)^{-7/2} \]

\[ \sim K_k A_+^{\# \oplus} A_-^{\# \ominus} (z - \rho)^{-1/2 - k} \]

same order as the denominator!
Part 4 - what’s the point?
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of scaling-limit results for pattern-avoiding permutations?
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of scaling-limit results for pattern-avoiding permutations?
On a continuous limiting object, we can compute things,
then recover results on the discrete objects!
Some previous work
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- Extremal combinatorics: Presutti-Stromquist (2009) introduced permutons to provide a lower bound for the packing density of (2413) (conjectured tight)
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- Joint convergence of all pattern densities is automatic.
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• Extremal combinatorics: Presutti-Stromquist (2009) introduced permutons to provide a lower bound for the packing density of (2413) (conjectured tight)

• Joint convergence of all pattern densities is automatic.

• Asymptotics of the number of cycles of fixed length (Mukherjee ’16), of the length of the longest increasing subsequence (Mueller, Starr,’13) and of the total displacement (Bevan, Winkler, ’19) in Mallows permutations using the permuton limit + regularity of convergence.
Expectation of the permuton

As $\mu$ is a random measure, it is natural to compute its average $\mathbb{E}_\mu$, which is the limit of the permuton obtained by stacking all separable permutations of a given size.

Theorem (M. 2017) The permuton $\mathbb{E}_\mu$ has density function

$$\frac{1}{\pi}(\beta(x, y) + \beta(x, 1 - y)), \quad 0 \leq x \leq \min(y, 1 - y)$$

$$\beta(x, y) = \frac{3xy - 2x - 2y + 1}{(1-x)(1-y)} \sqrt{\frac{1-x-y}{xy}} + 3 \arctan \sqrt{\frac{xy}{1-x-y}}.$$  

We recover the expected shape of doubly-alternating Baxter permutations. (Dokos-Pak)