Scaling limits of permutation classes with a finite specification

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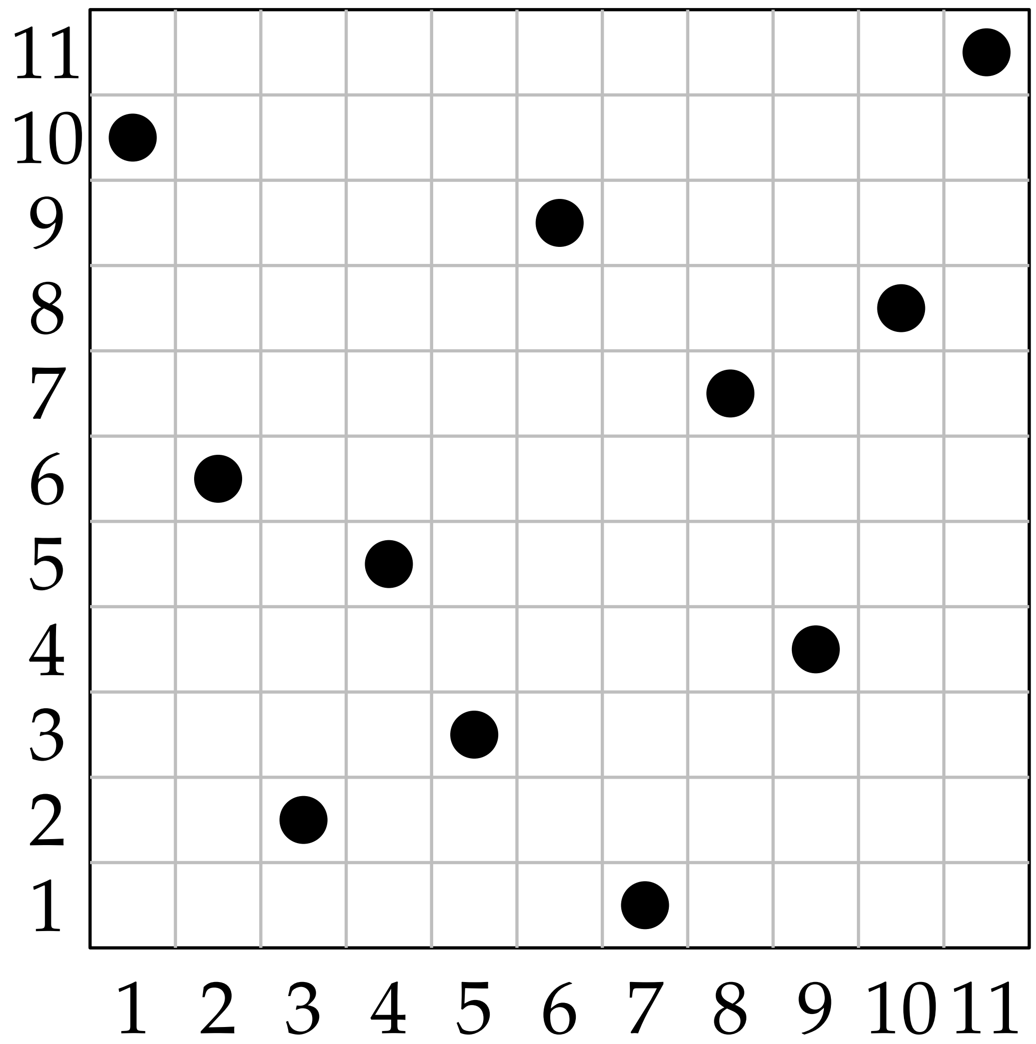
Joint work with F. Bassino, M. Bouvel, V. Féray, L. Gerin and A. Pierrot (LIPN-P13, Zürich^2, CMAP-Polytechnique, LMO-Orsay)

Oxford probability seminar
17 february 2020

Part 0 : Introduction

Permutation patterns

$$\sigma = (10, 6, 2, 5, 3, 9, 1, 7, 4, 8, 11) \in \mathfrak{S}_{11}$$



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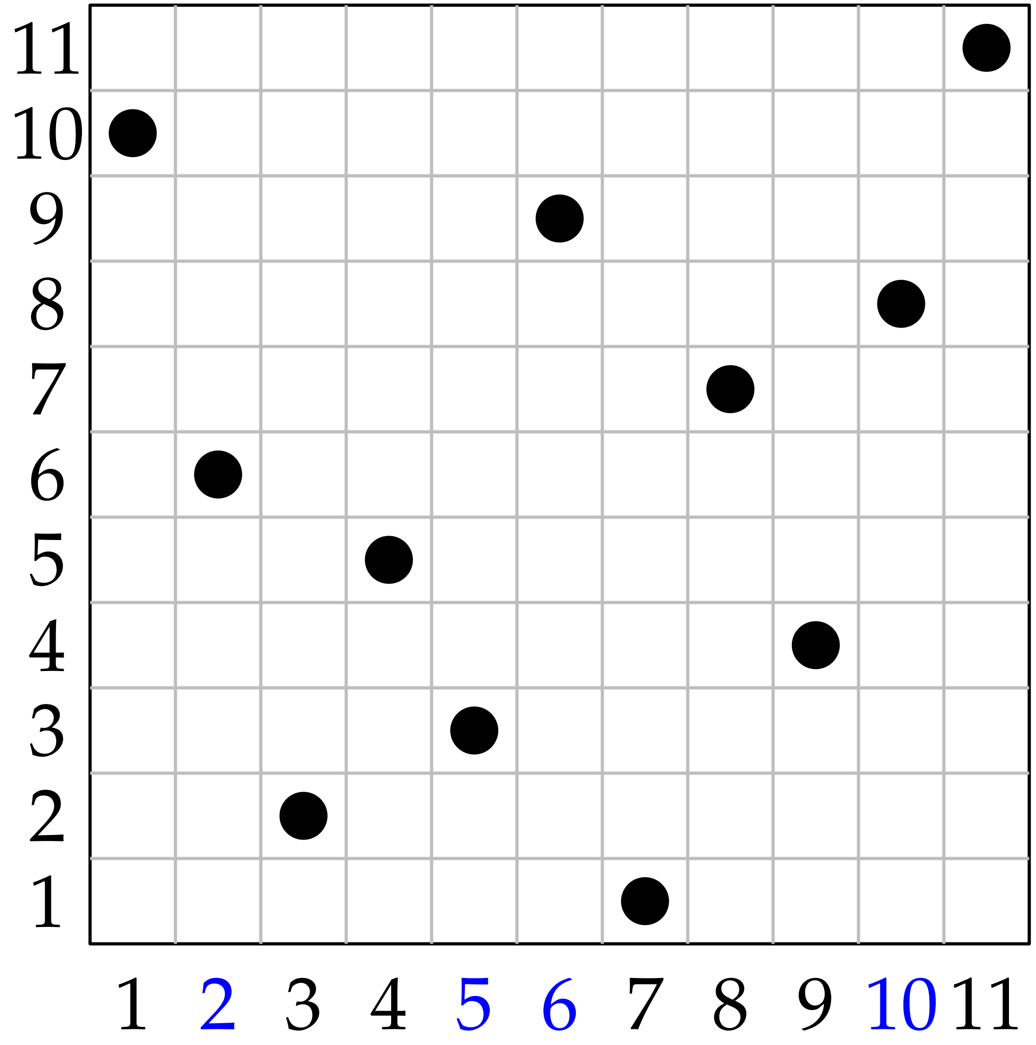

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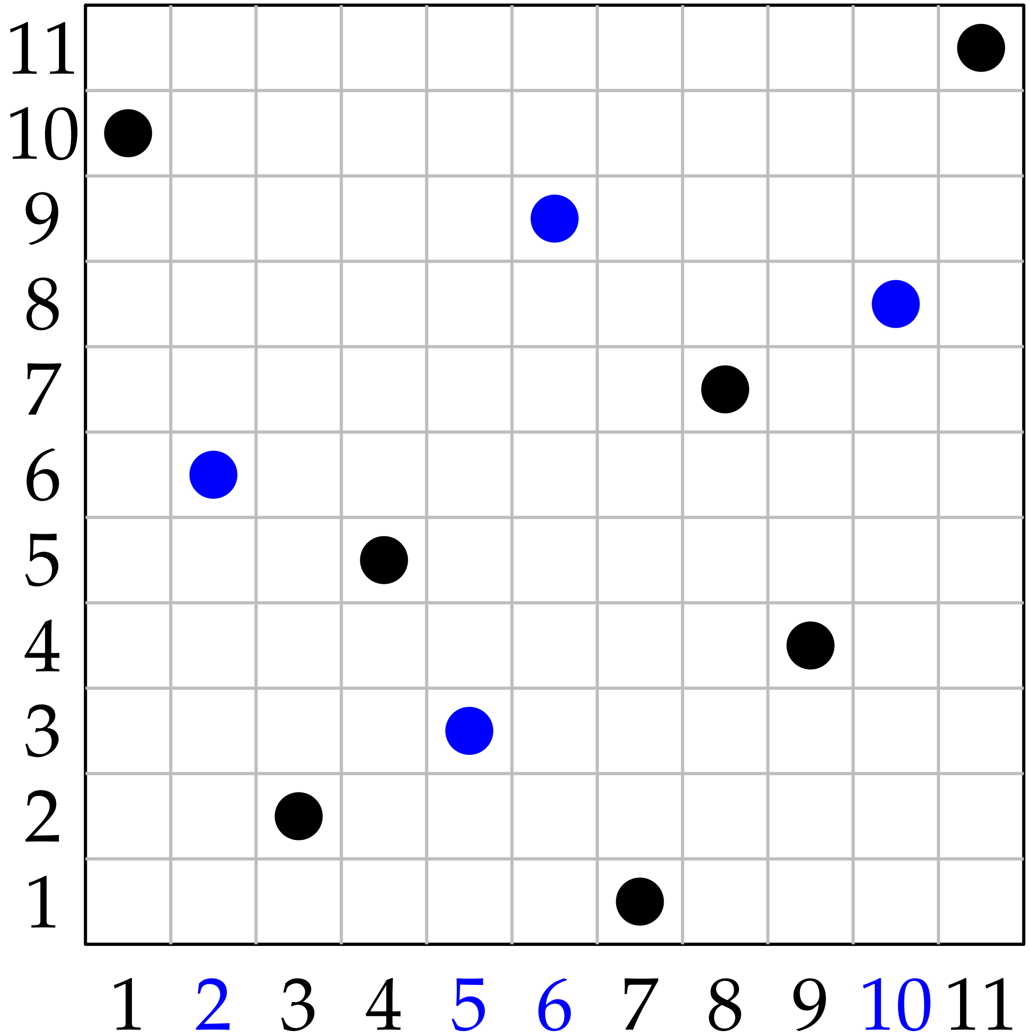

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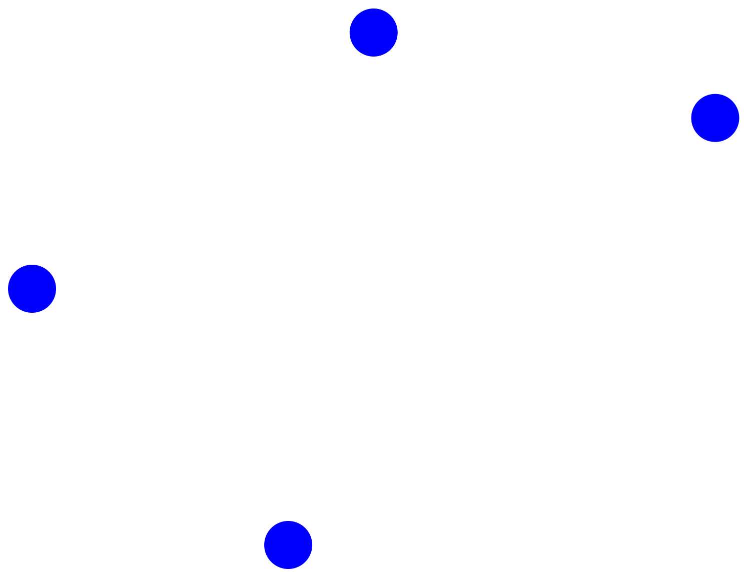

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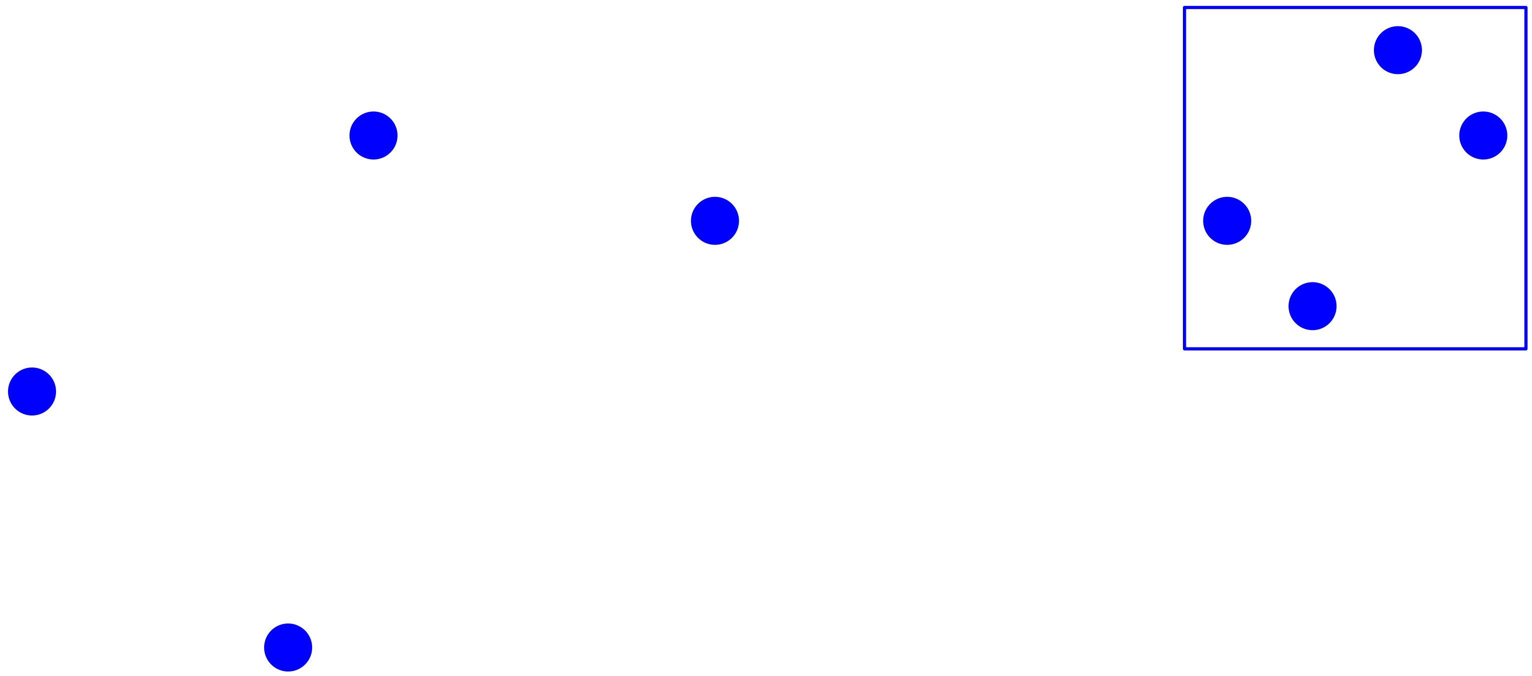

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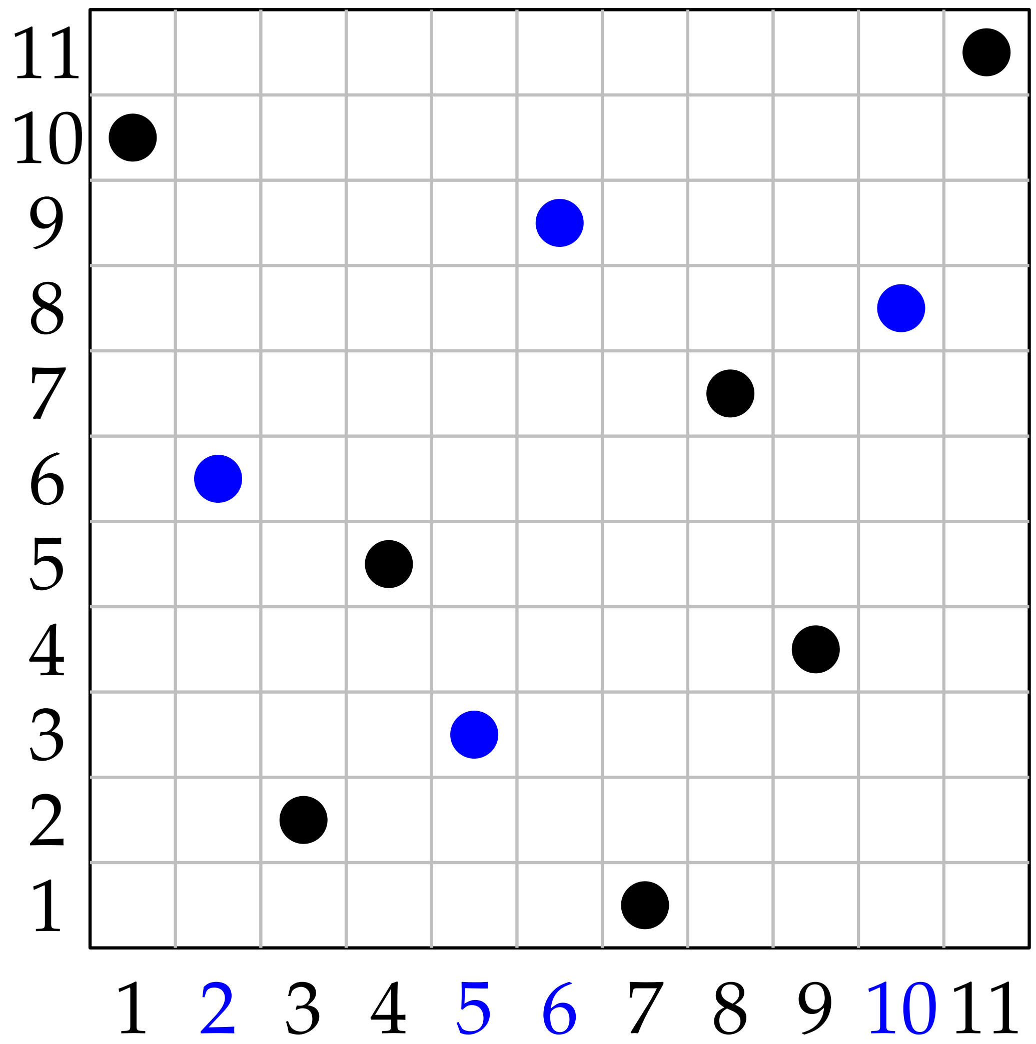

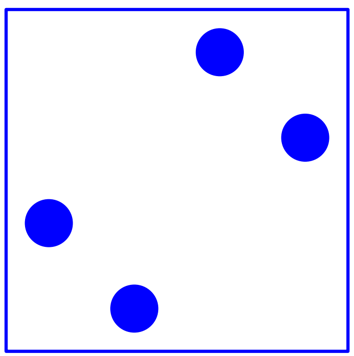

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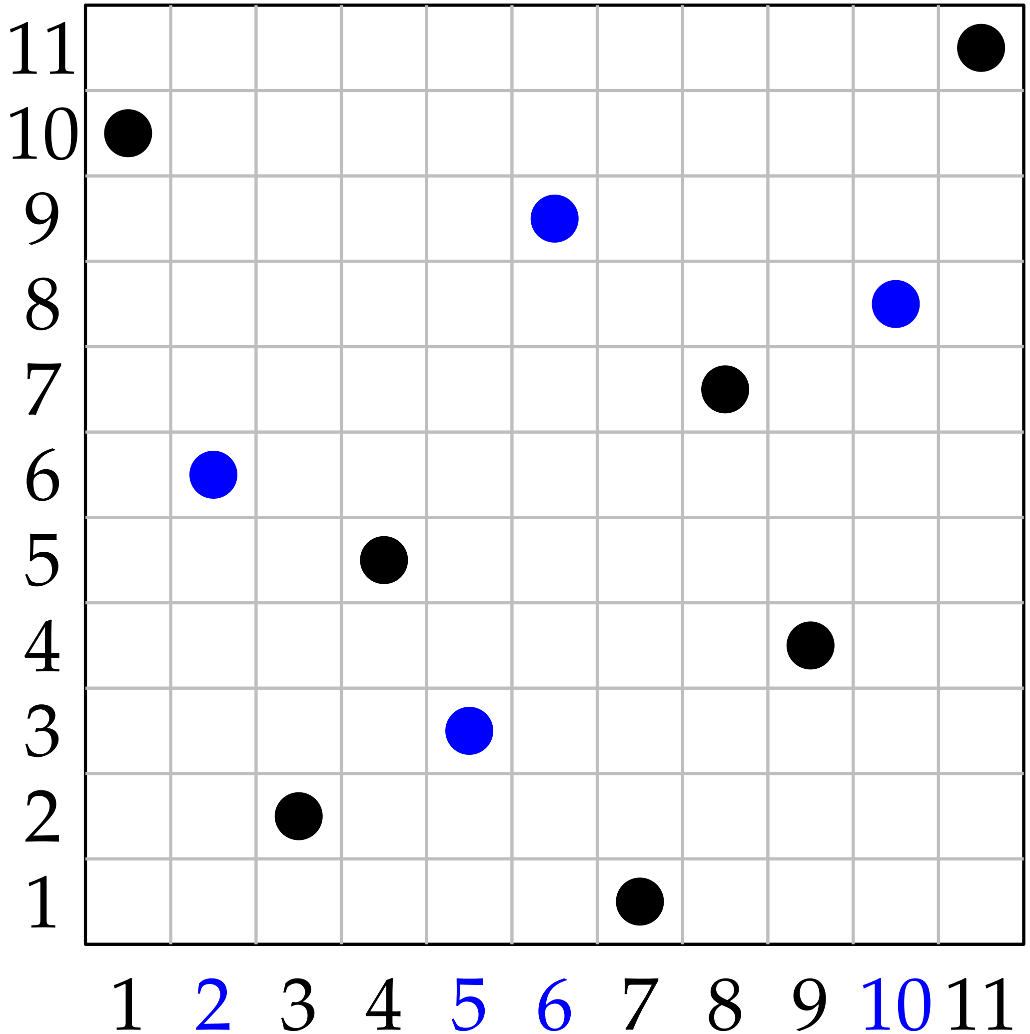

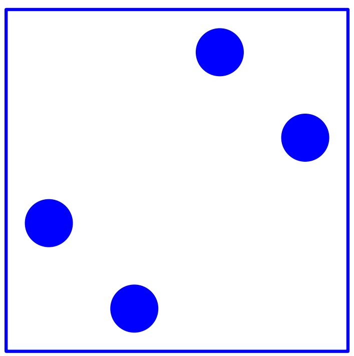

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Classes of permutation and pattern-avoidance

Permutation class: set of permutations closed under pattern extraction. Can always be written as $\mathrm{Av}(B)$, the set of permutations that avoid patterns in some *basis* B.

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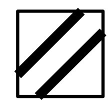
Example: $\mathrm{Av}(321)$ can be drawn on  (MacMahon 1915), $\mathrm{Av}(231)$ stack-sortable permutations (Knuth 1968), $\mathrm{Av}(2413, 3142)$: separable permutations, $\mathrm{Av}(321, 2143, 2413)$ are riffle shuffle permutations, ...

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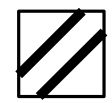
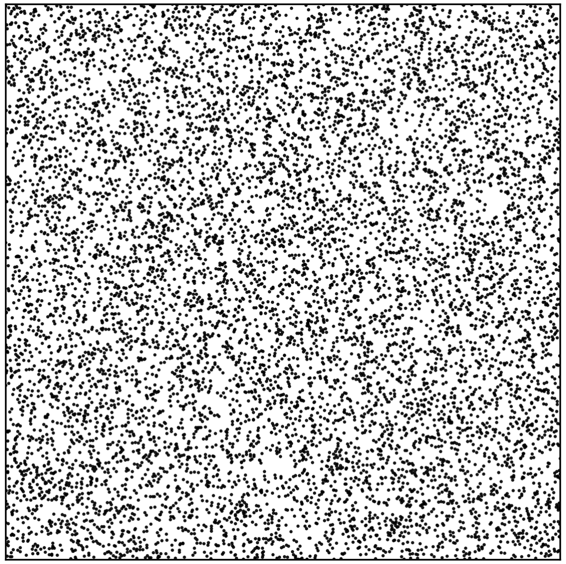
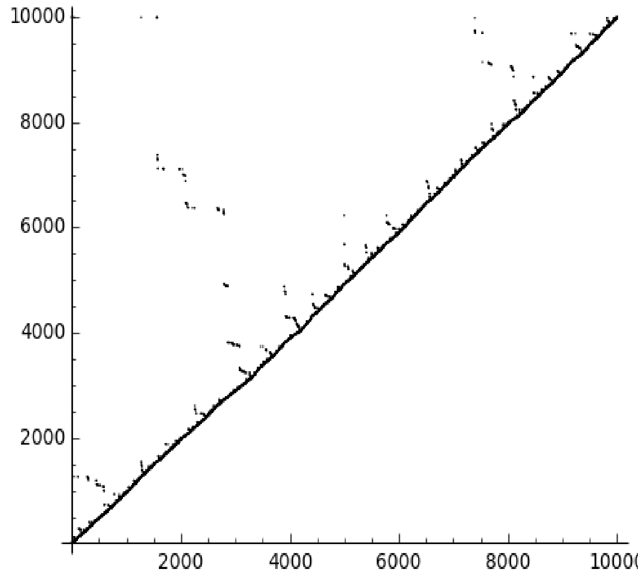
Example: $\mathrm{Av}(321)$ can be drawn on  (MacMahon 1915), $\mathrm{Av}(231)$ stack-sortable permutations (Knuth 1968), $\mathrm{Av}(2413, 3142)$: separable permutations, $\mathrm{Av}(321, 2143, 2413)$ are riffle shuffle permutations, ...

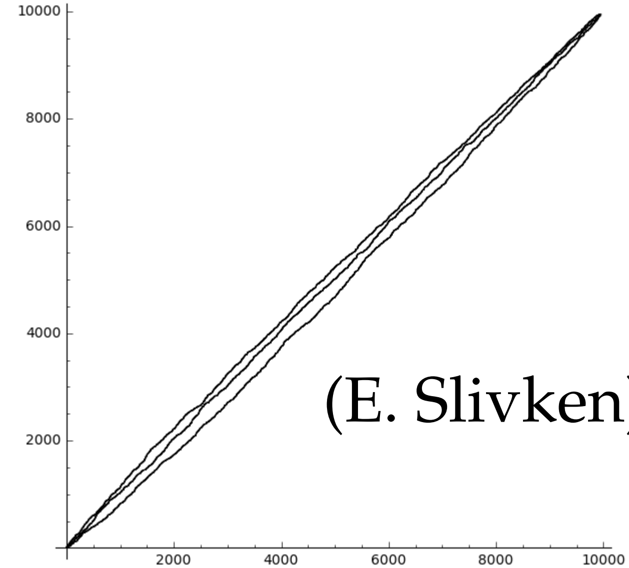
What does a large permutation in a class *look like*?



$\mathfrak{S}_n$

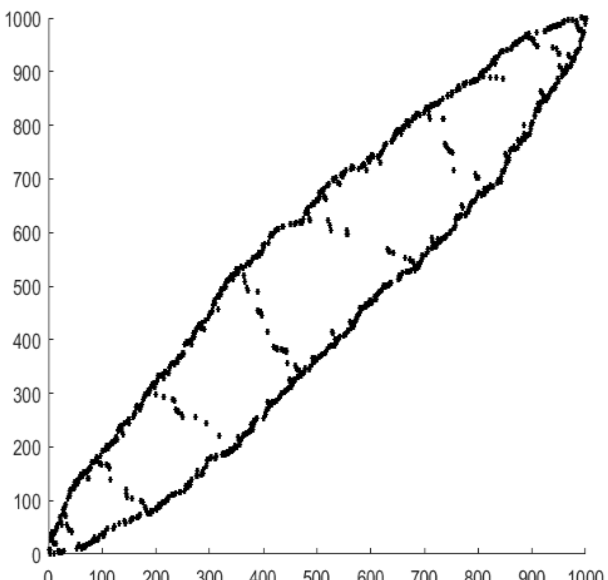


Av(231)



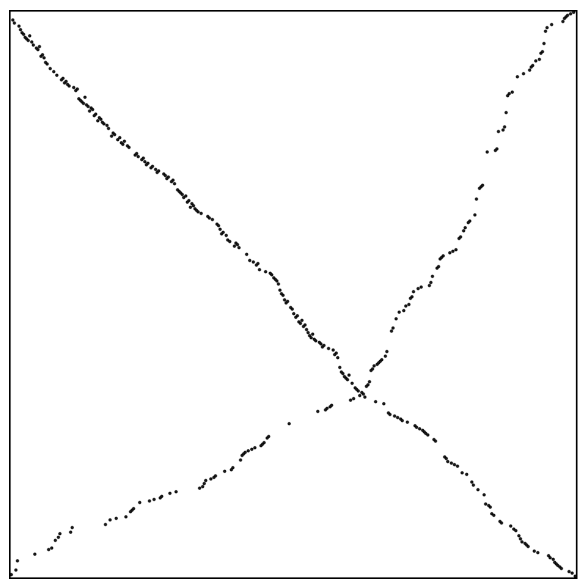
(E. Slivken)

Av(4321)

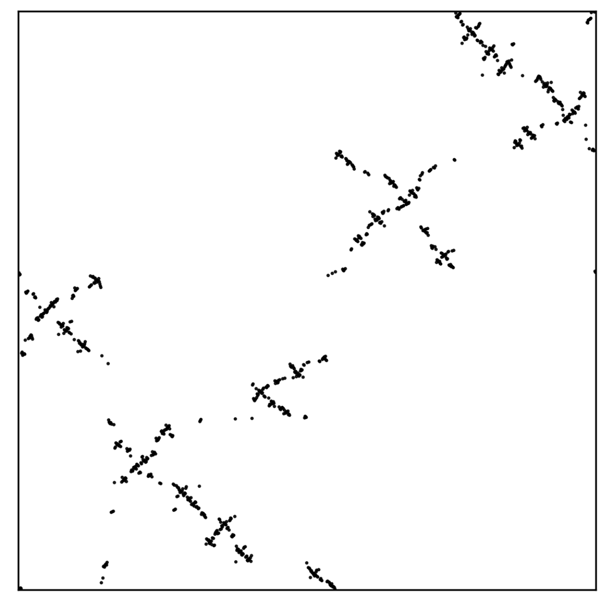


Av(4231)

(Madras-Yildrim)



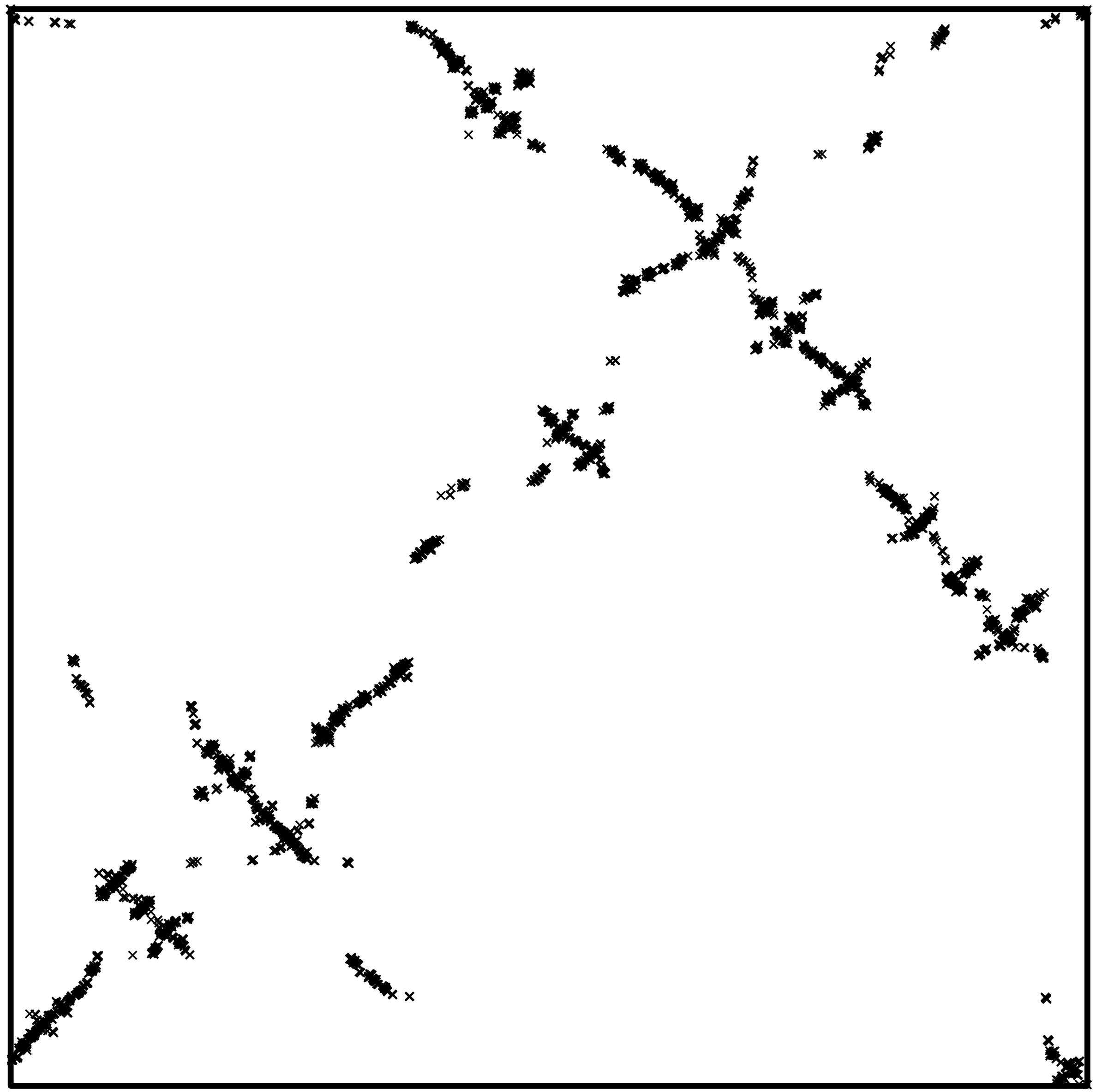
Av(2413, 3142, 2143, 34512)



Av(2413,3142)

={separables}

A large uniform separable permutation



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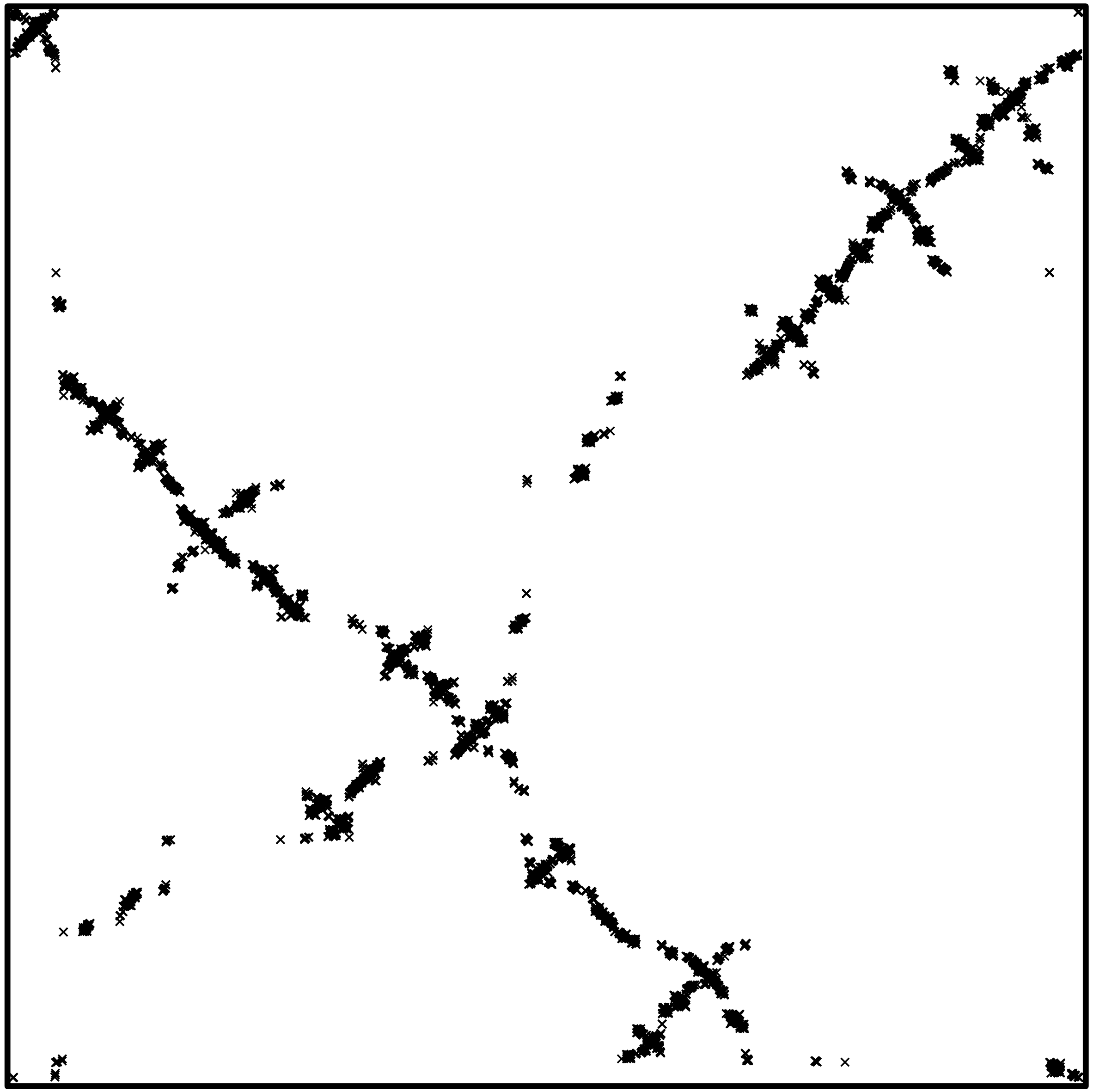
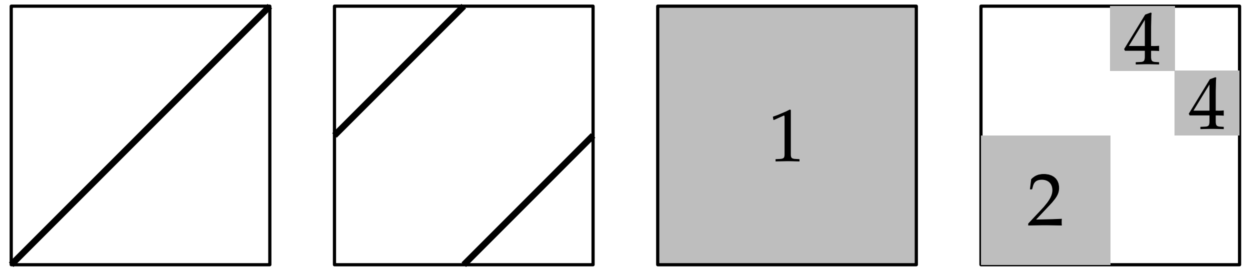


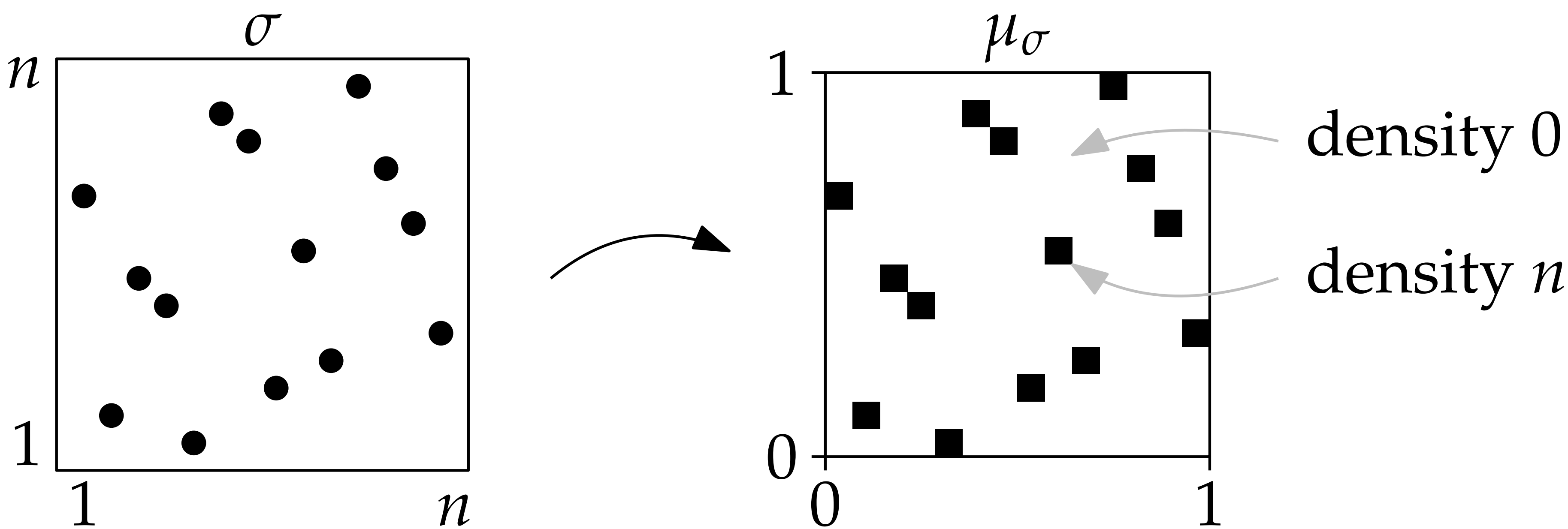
Permutons

A permuton is a probability measure on $[0,1]^2$ with both marginals uniform.



$\implies$ compact metric space (with weak convergence).

Permutations of all sizes are densely embedded in permutons.

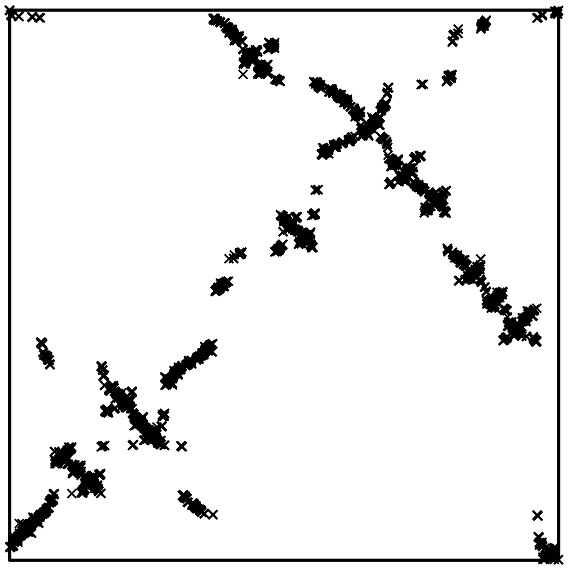


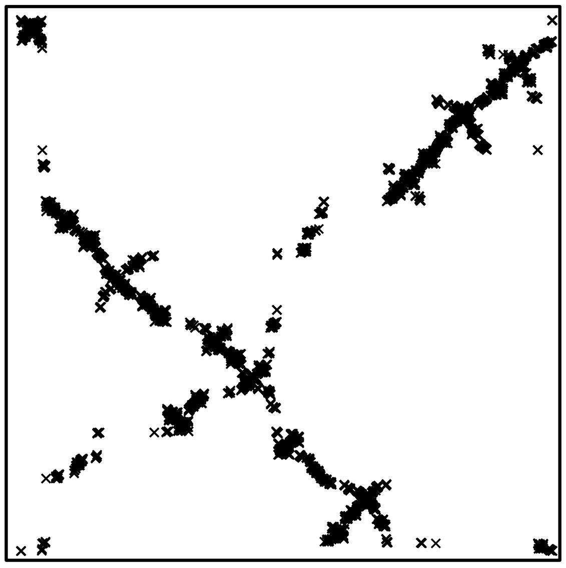
The Brownian limit of separable permutations

σ_n uniform of size n in $\mathcal{C} = \mathrm{Av}(2413, 3142) = \{\text{separables}\}$:

Theorem (Bassino, Bouvel, Féray, Gerin, Pierrot 2016)

σ_n converges in distribution to some random permuton μ, called the Brownian separable permuton.



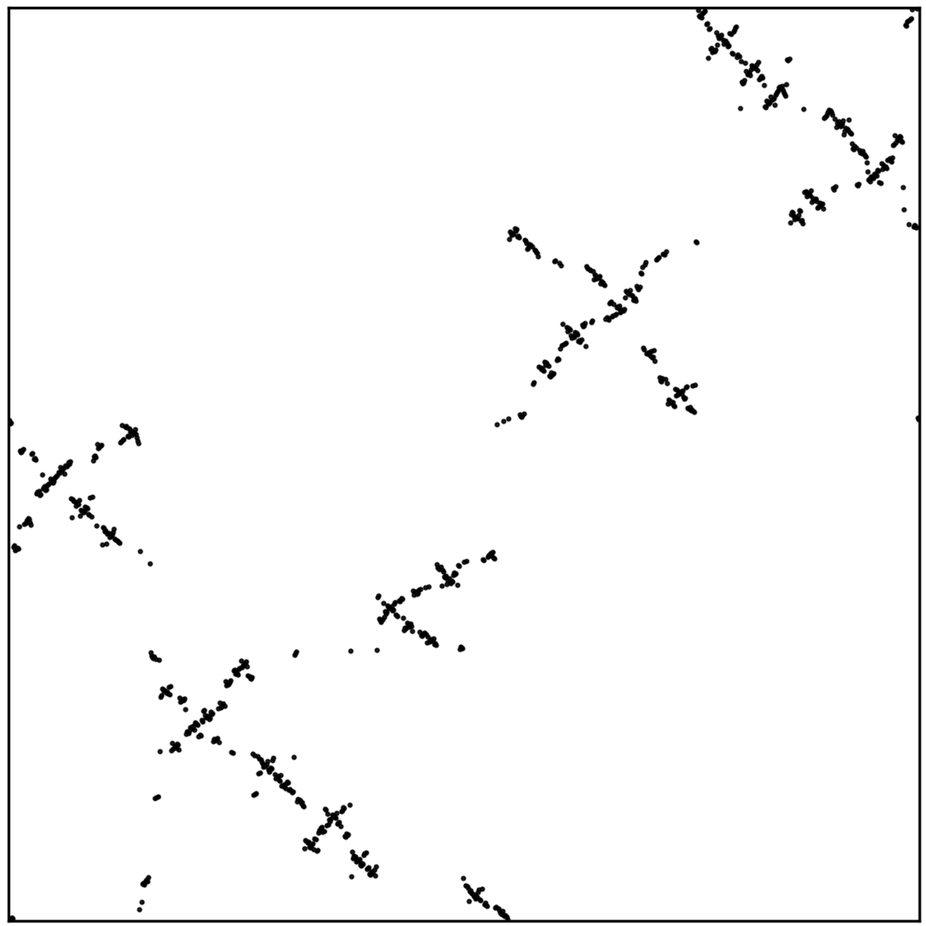


The main theorem.

Theorem (BBFGMP 2019)
Many other classes of permutation converge also to the Brownian permuton, or a 1-parameter deformation. Those behave nicely under the so-called "substitution-decomposition" (precise statement later)

The main theorem.

Theorem When $\mathcal{C} =$ Av$(31452, 41253, 41352, 531642, 25413, 35214, 25314, 246135)$, μ_{σ_n} also converges to the Brownian permuton.



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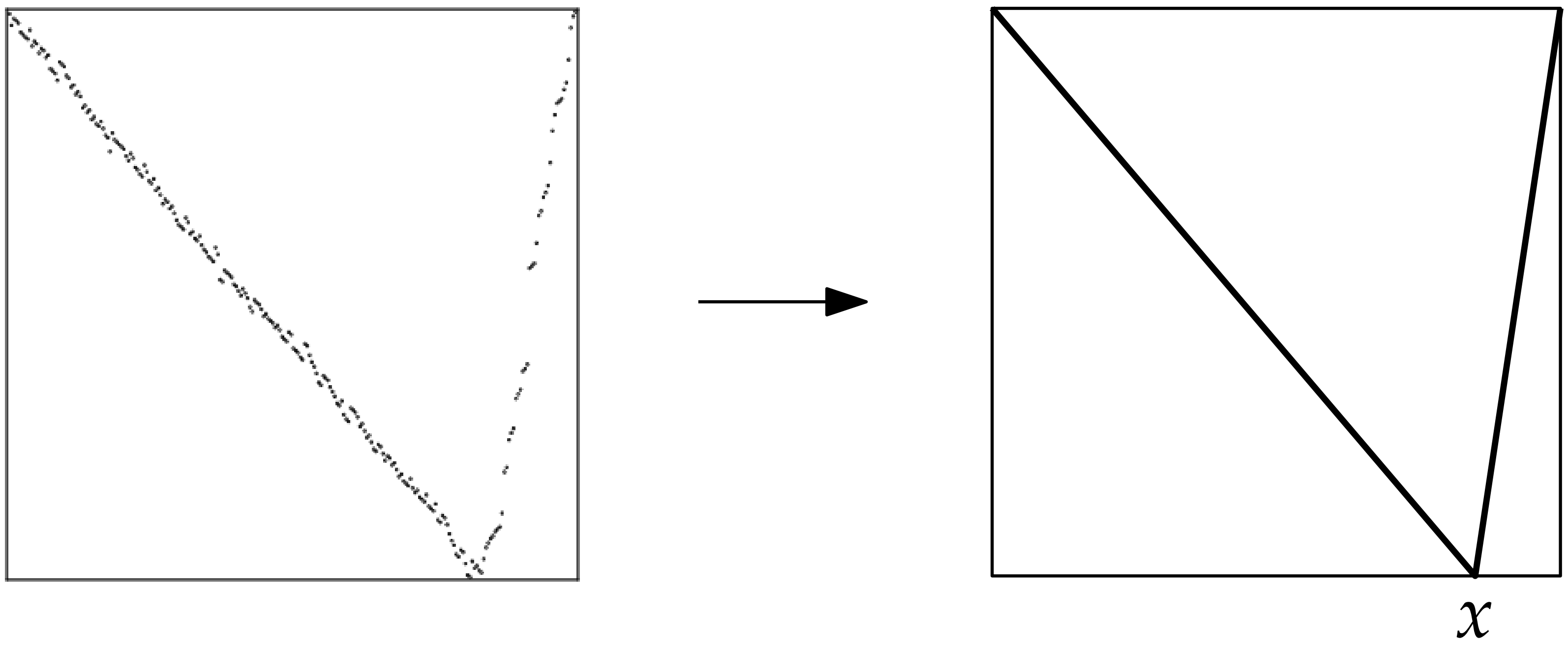

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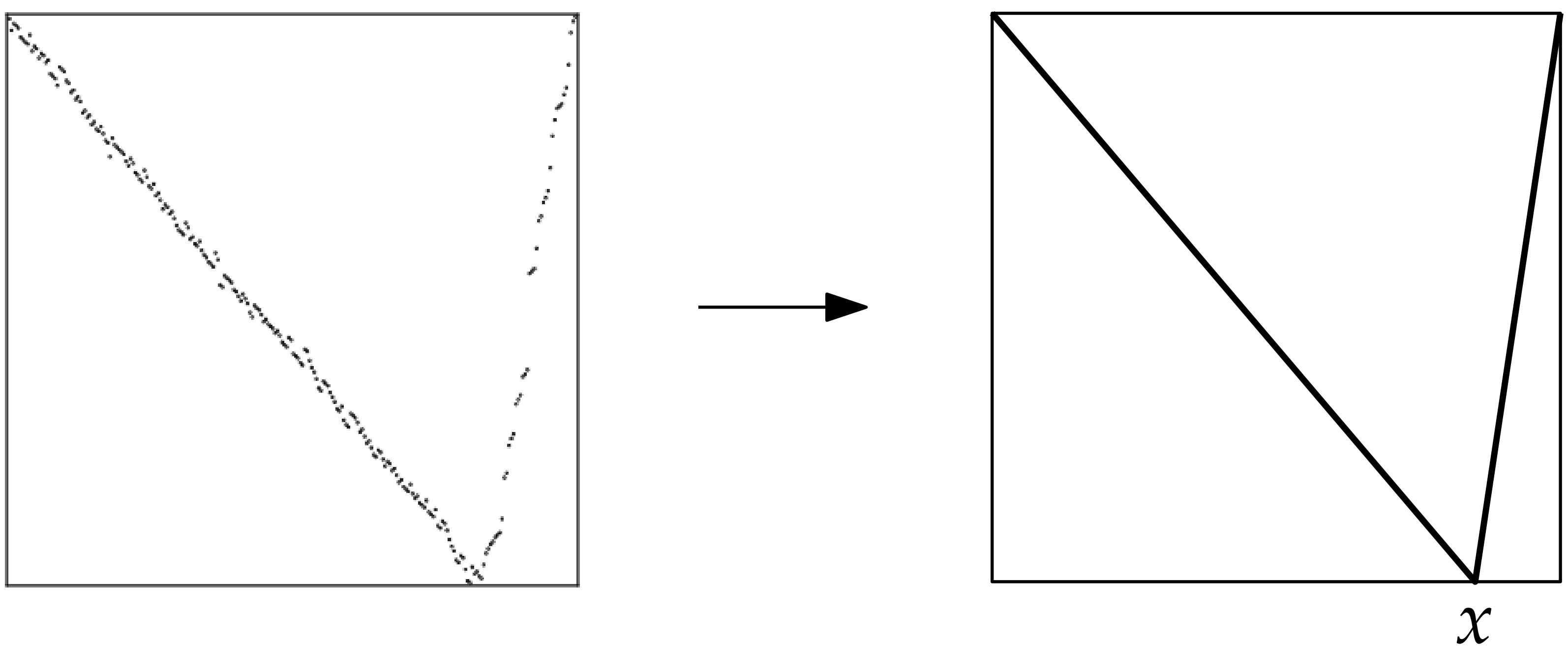


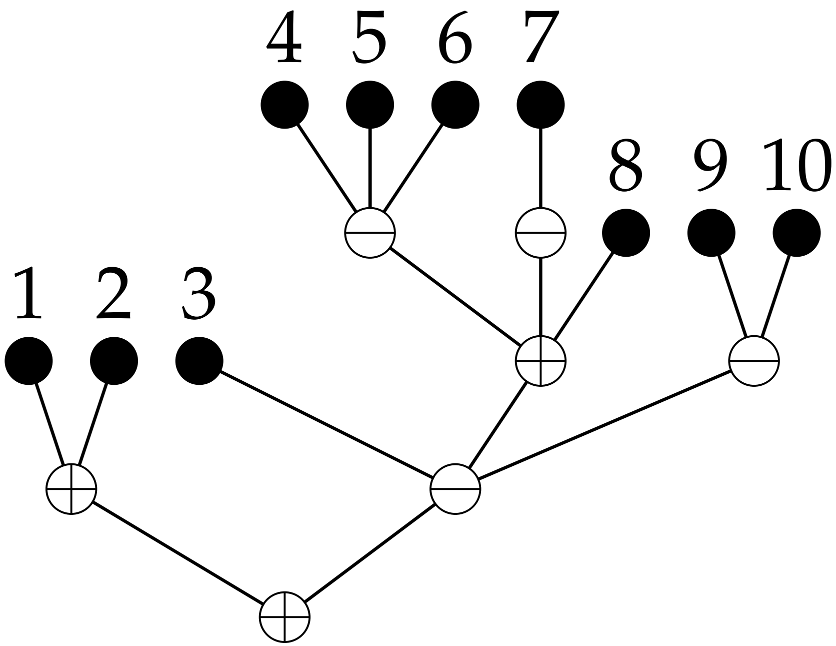
$x \approx 0.818632668576995$ is the only real root of
$19168x^5 - 86256x^{+}155880x^3 - 141412x^2 + 64394x - 1177$

Part 1 - the proof method

(illustrated on the case of separable permutations)

0 - General idea and limit object

Characterization of separable permutations:



Signed tree τ

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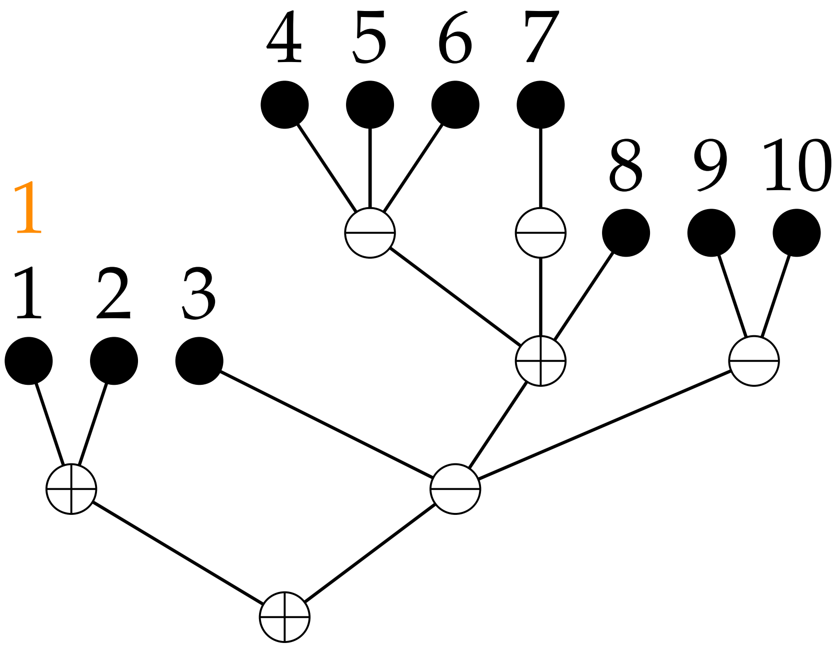

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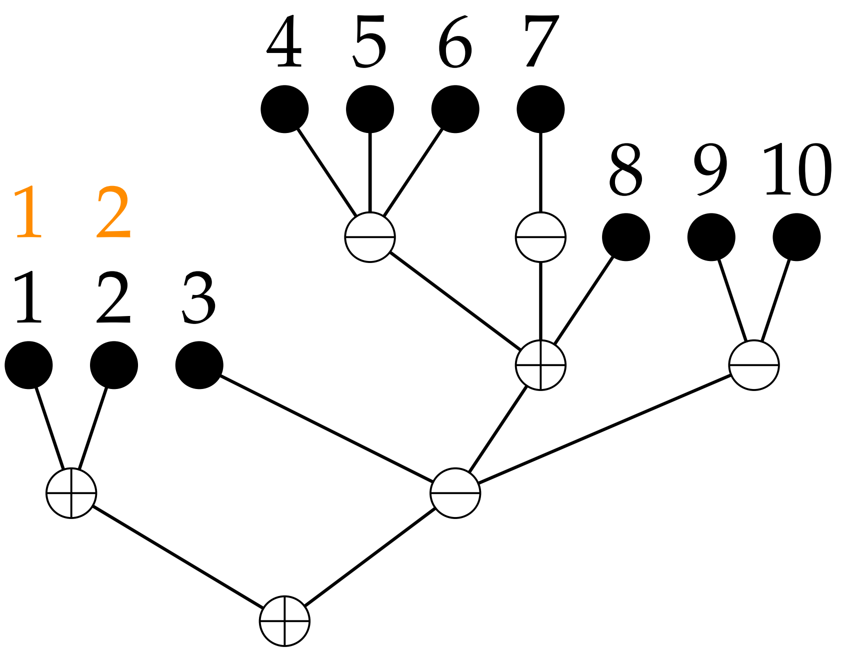

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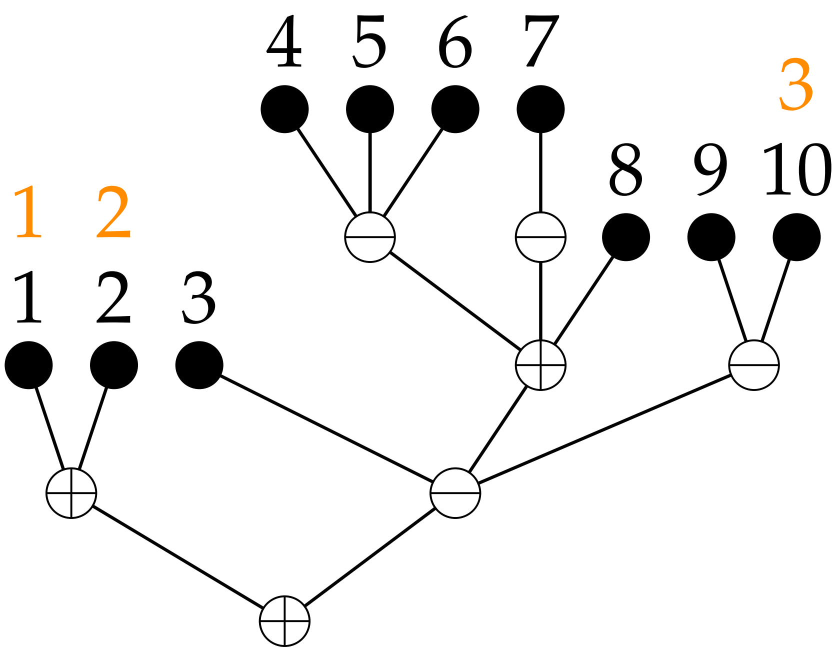

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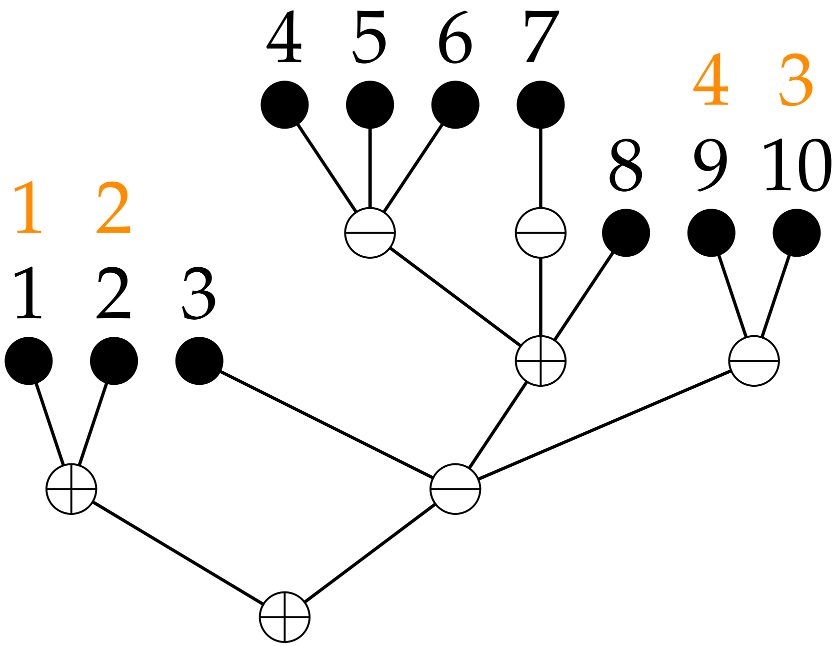

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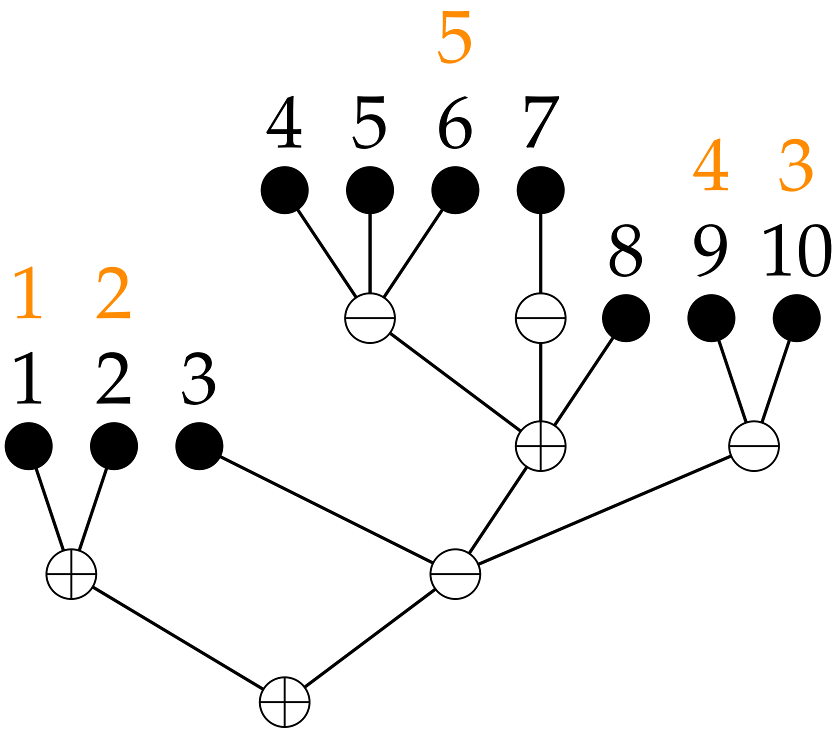

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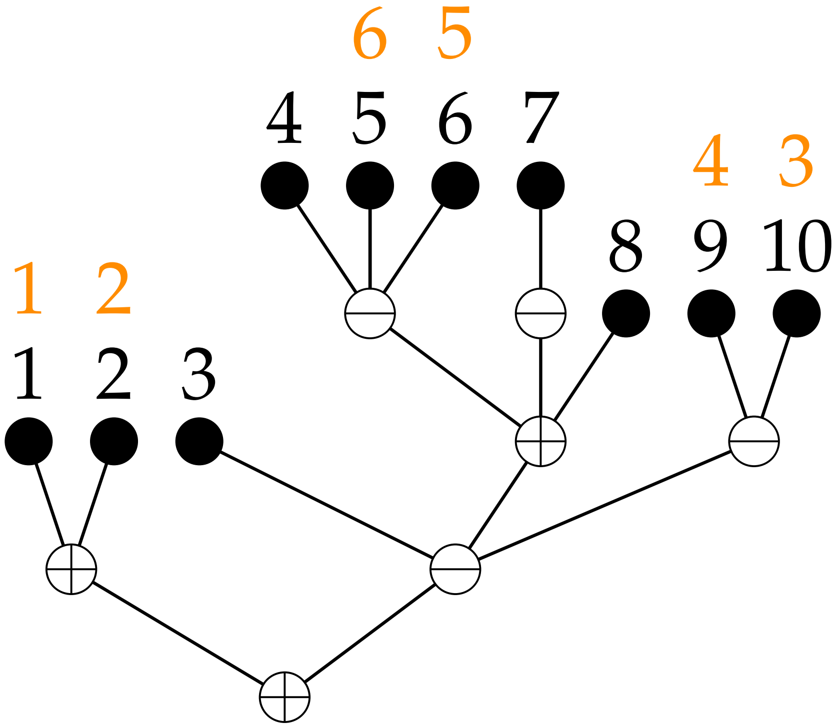

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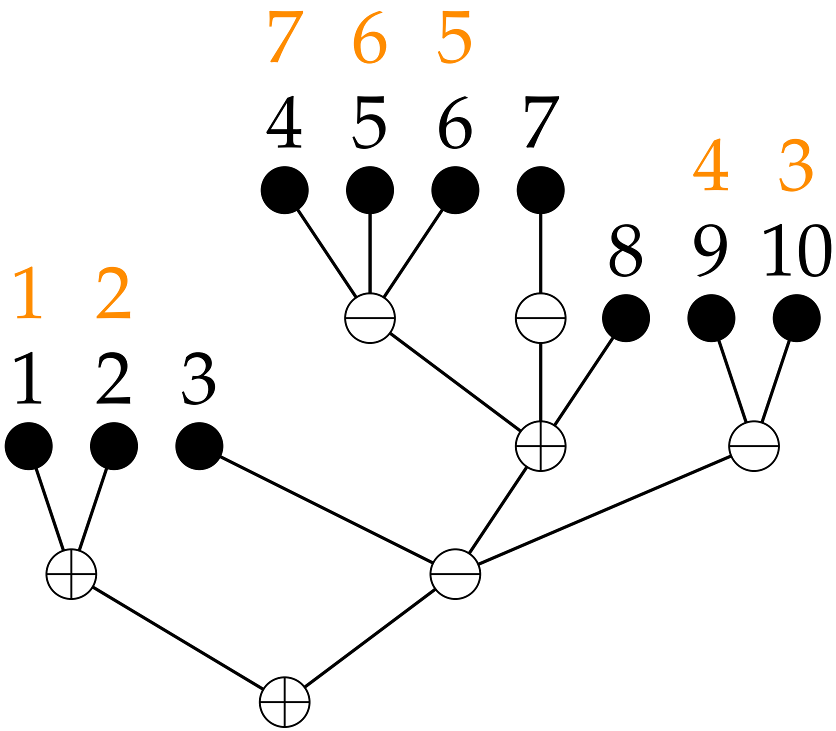

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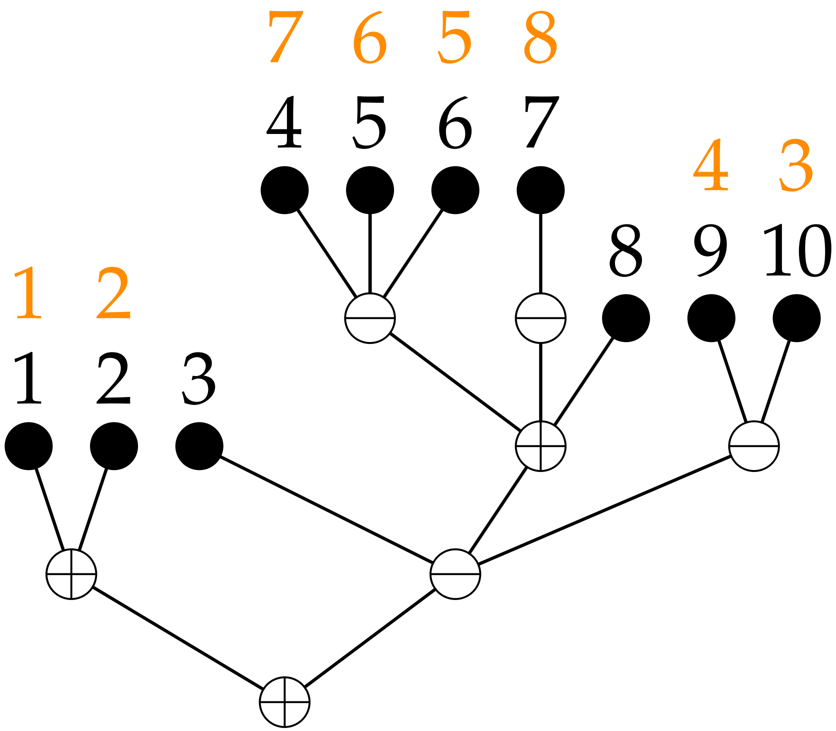

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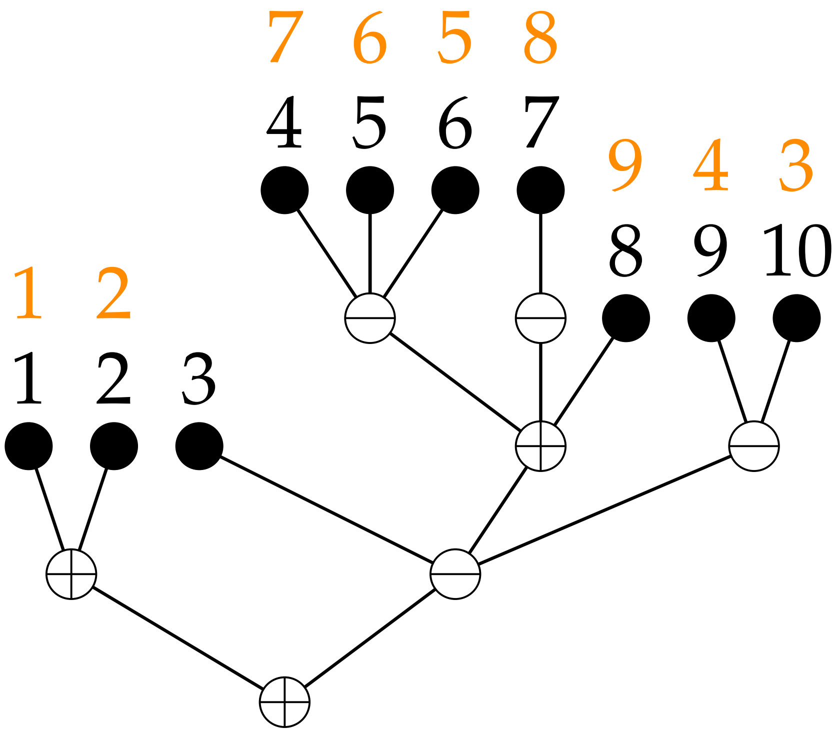

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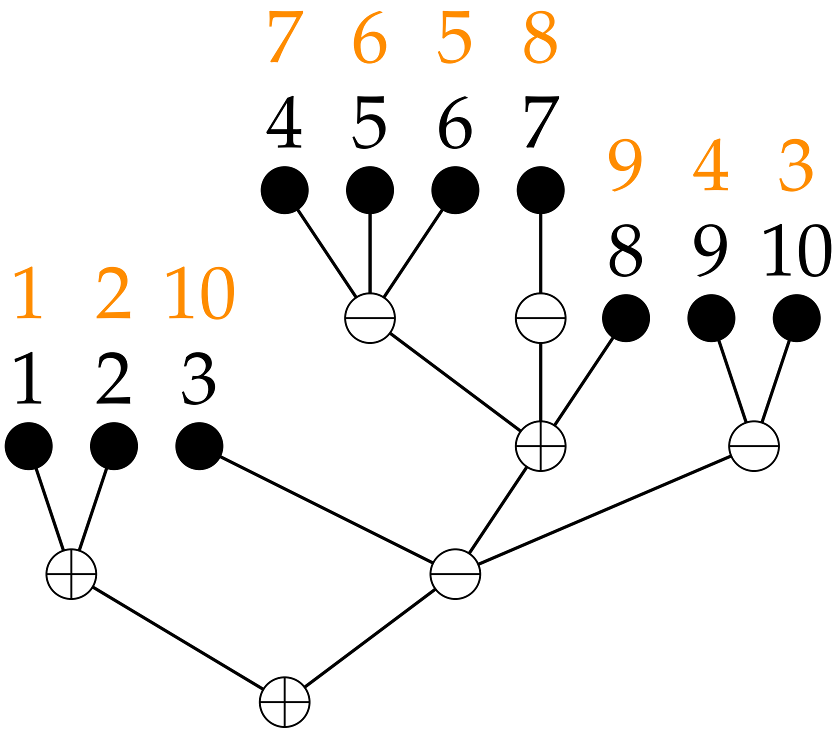

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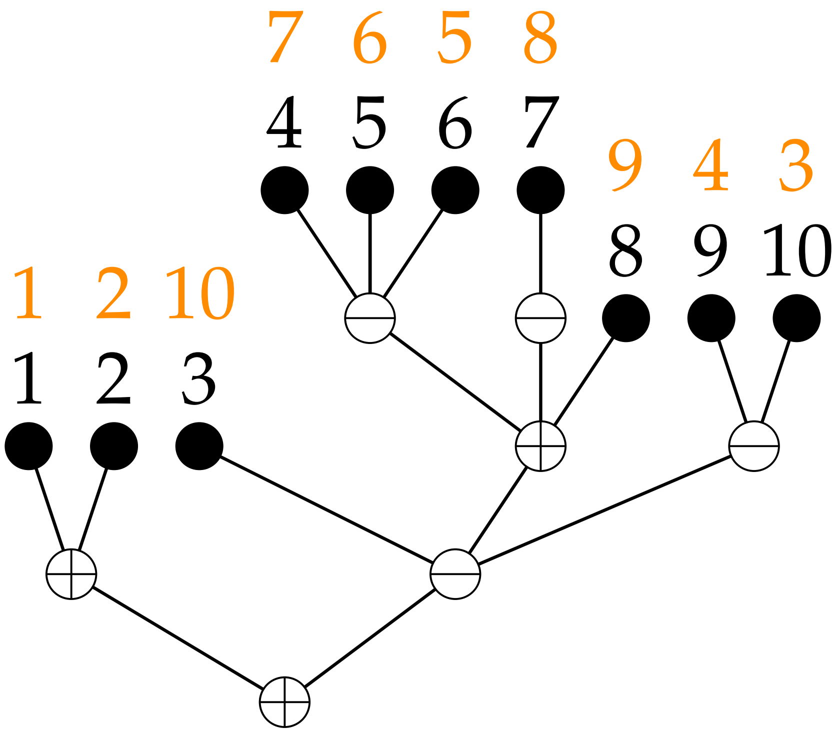

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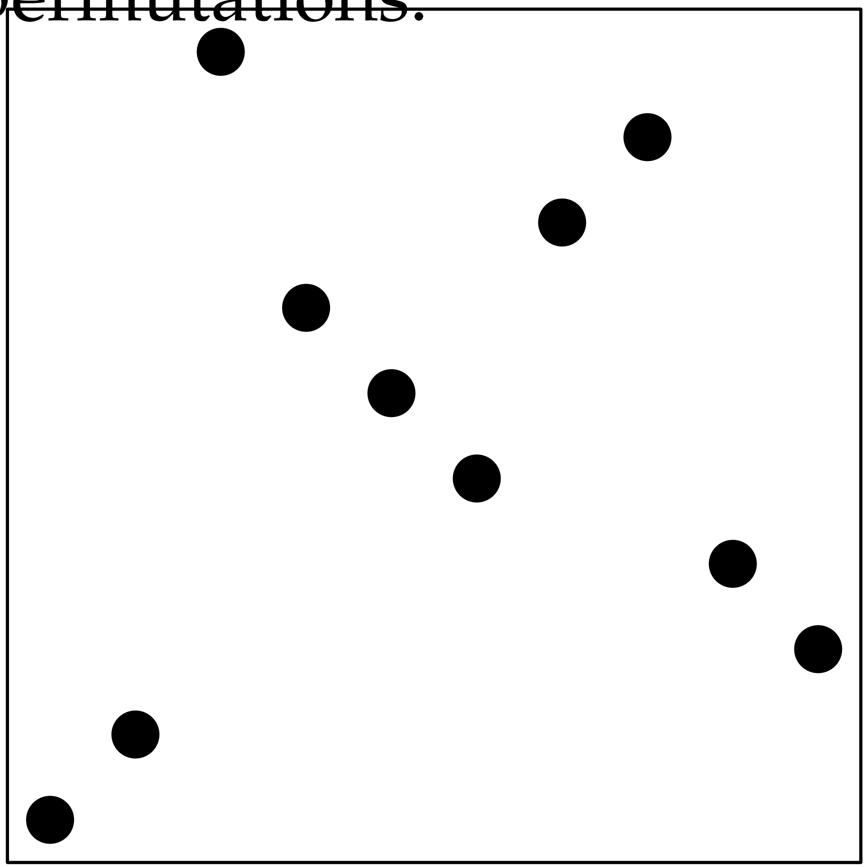

Separable permutation
$\mathrm{perm}(\tau) = (1\,2\,10\,7\,6\,5\,8\,9\,4\,3)$

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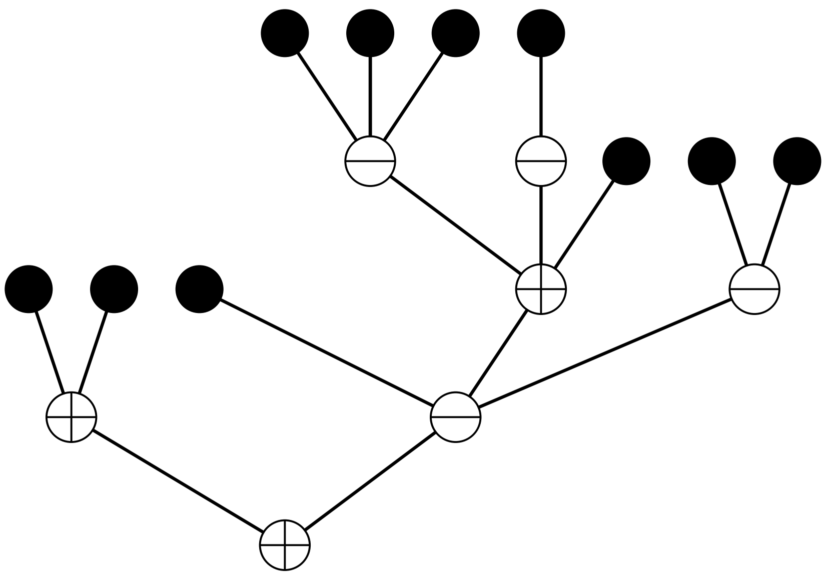

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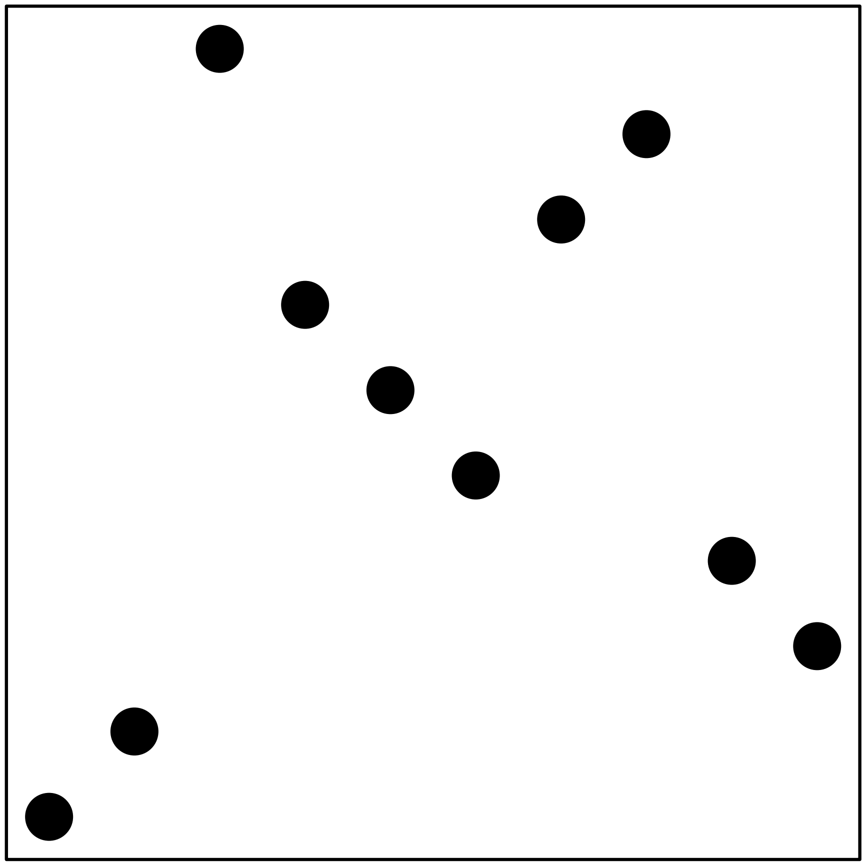

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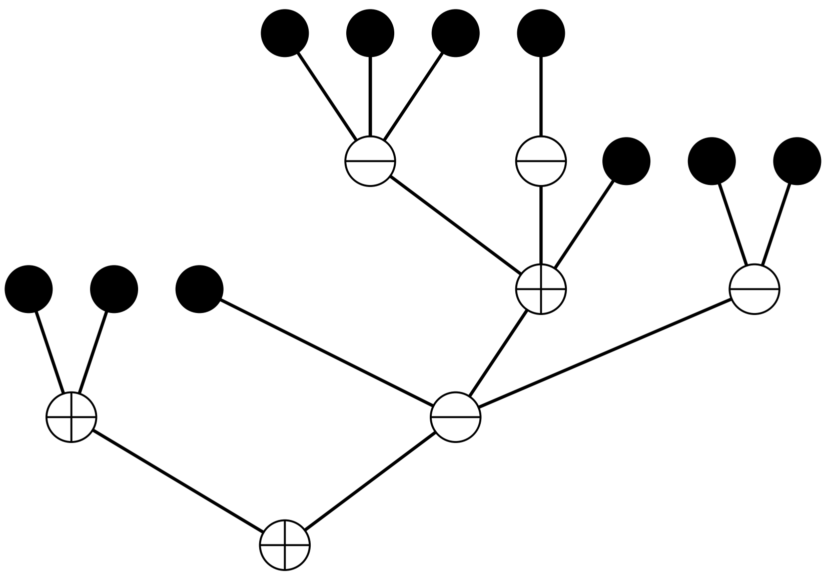

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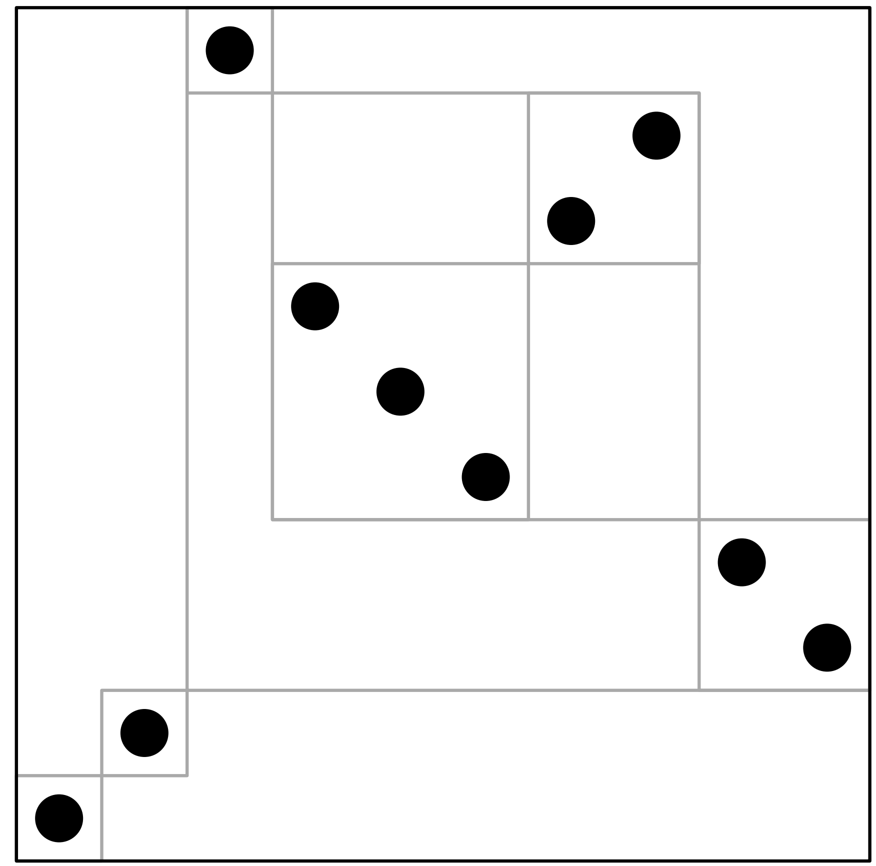

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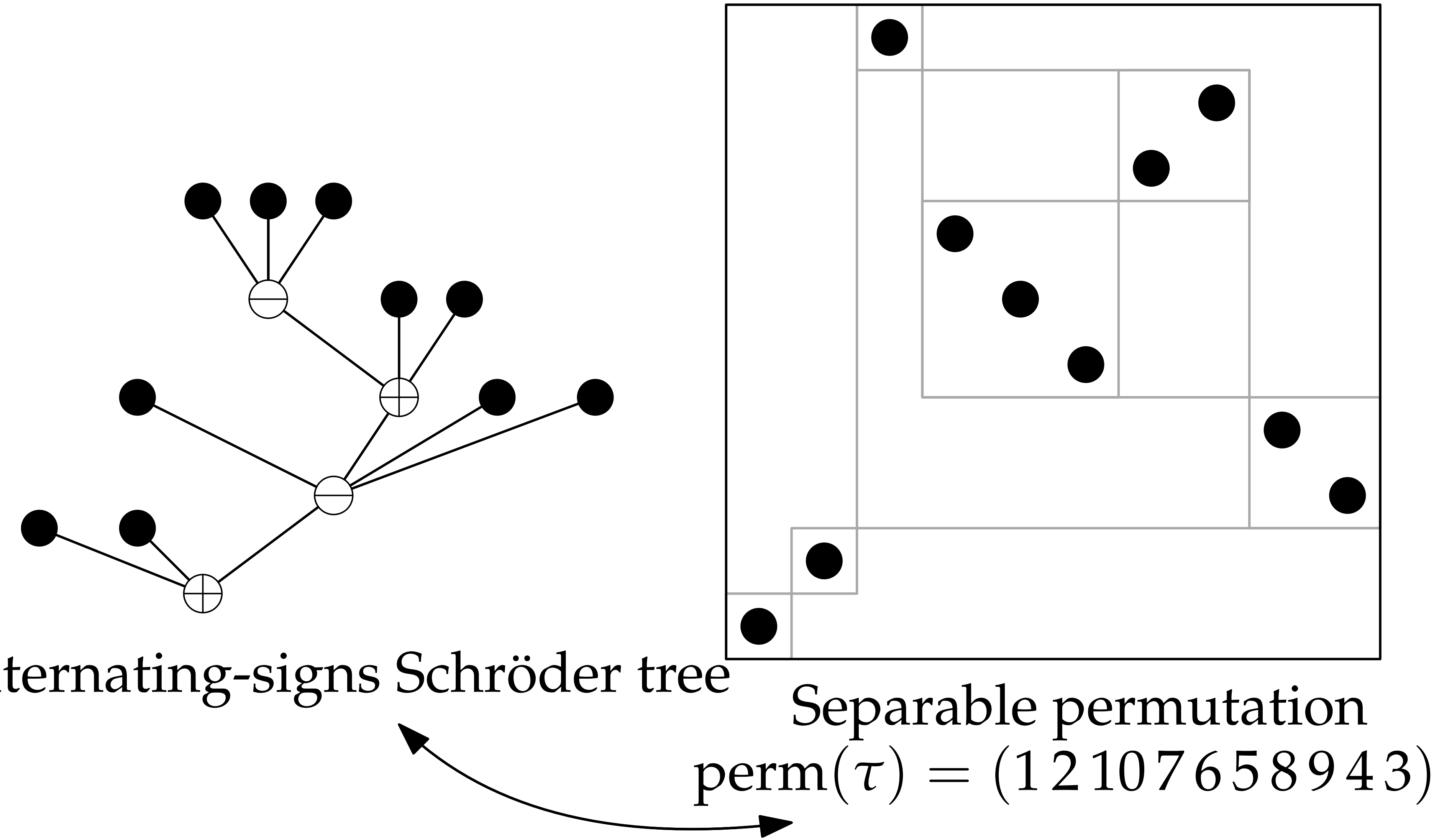

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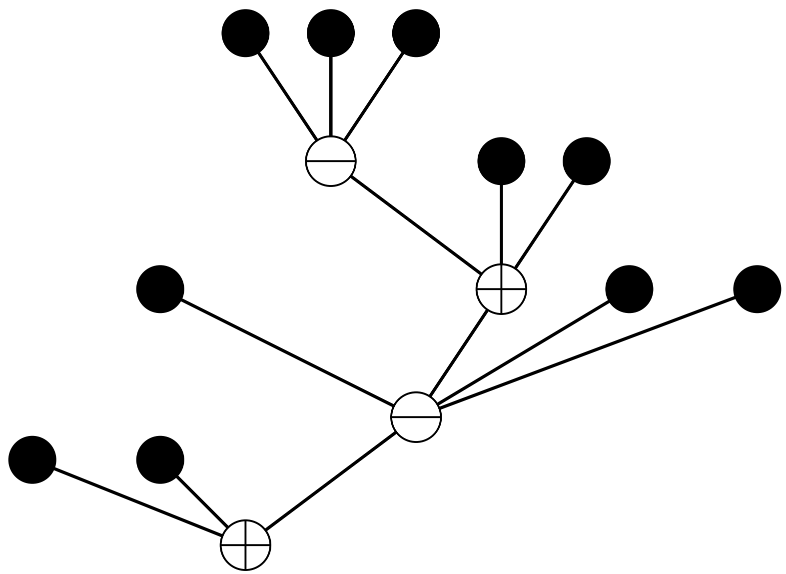
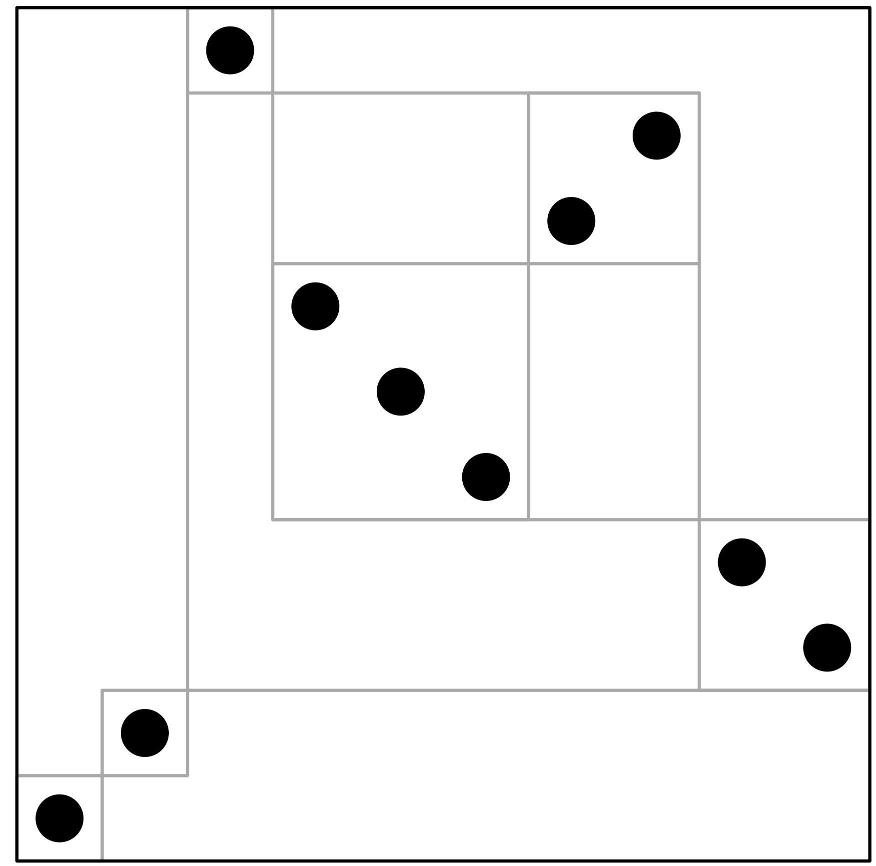


Alternating-signs Schröder tree



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Counted by large Schröder numbers

$1, 2, 6, 22, 90, 394, 1806, 8558, \ldots \asymp (3+\sqrt{8})^n n^{-3/2}$

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Many "nice" models of random trees $(t_n)_n$ where n is the size, converge to (a multiple of) the Brownian CRT when distances are rescaled by $\sqrt{n}$. More precisely, if C_n is the contour function of t_n, for some constant $c > 0$, $cn^{-1/2}C_n$ converges in distribution to the normalized Brownian excursion.

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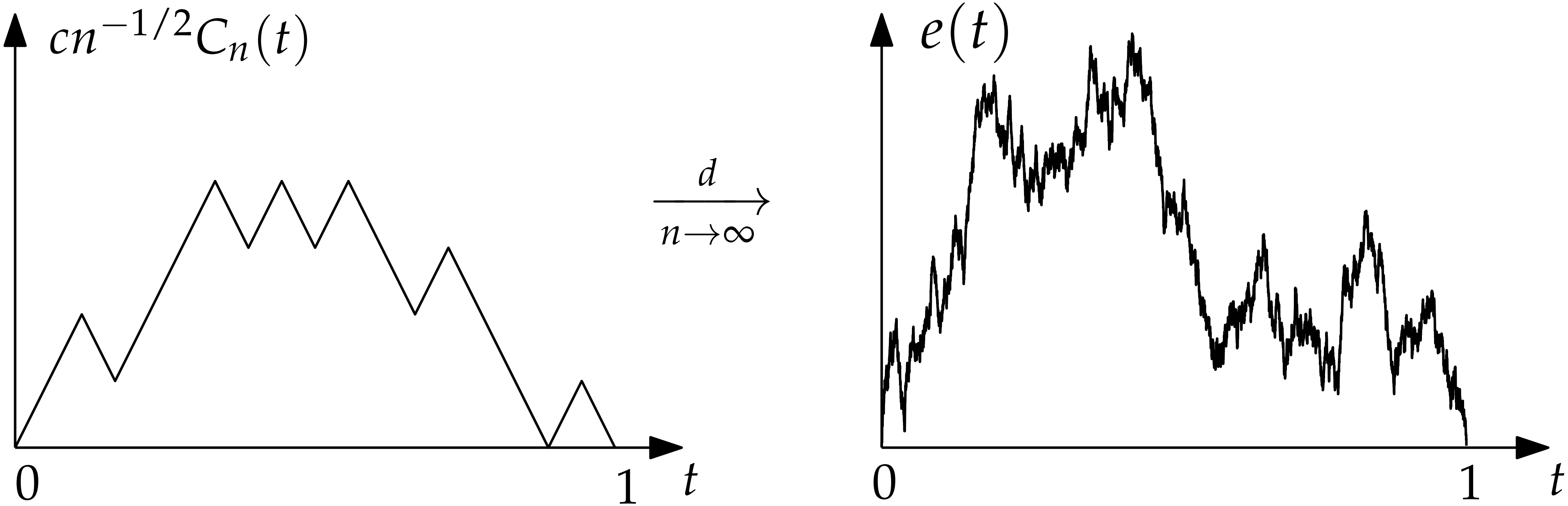

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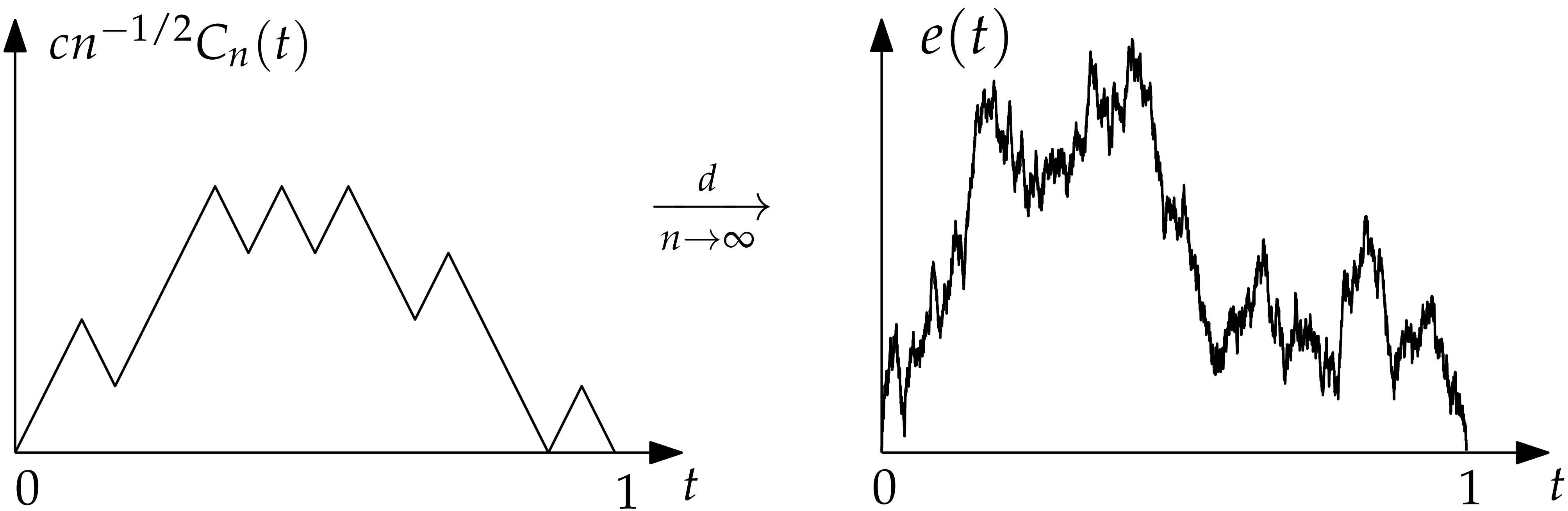
Leaf-counted Schröder trees are (critical, finite-variance) BGW trees conditioned on the number of leaves and fall in this category (Kortchemski '12, Pitman-Rizzolo '12)



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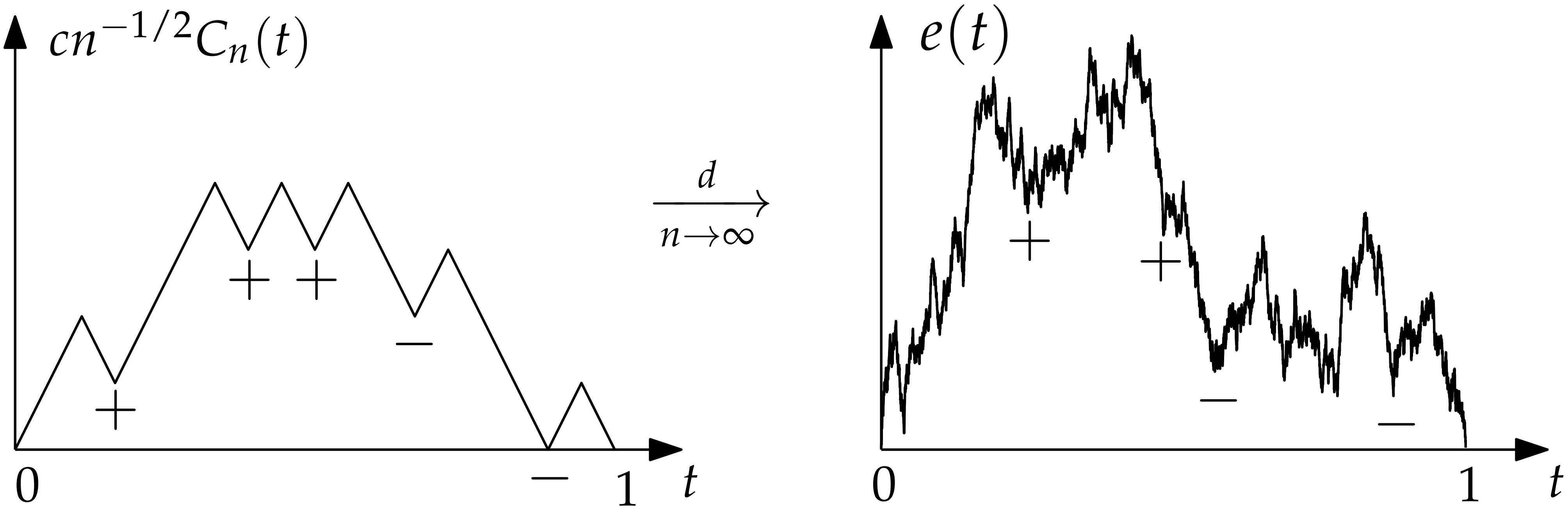
The main point: signs at macroscopic branching points become independent as the tree gets larger. This tells us how the corresponding permutation looks like in the large scale.



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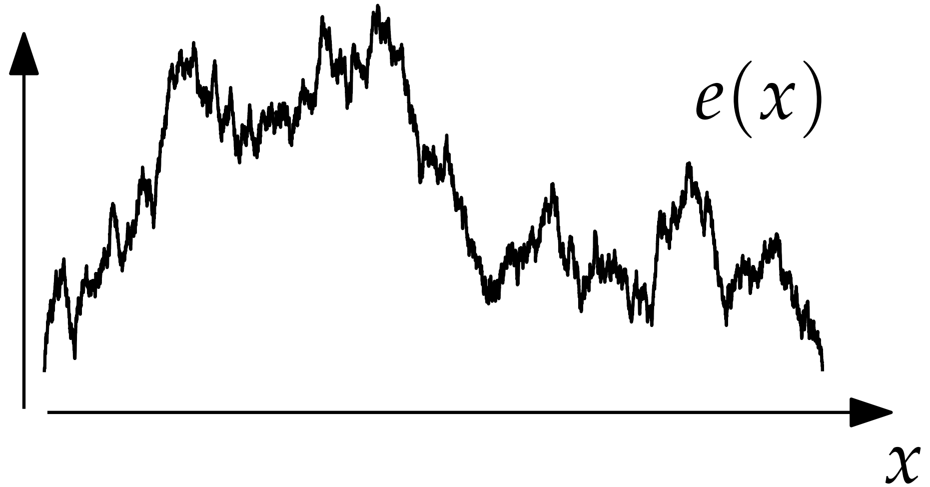
e Brownian excursion, S i.i.d. balanced signs indexed by the local minima of e.



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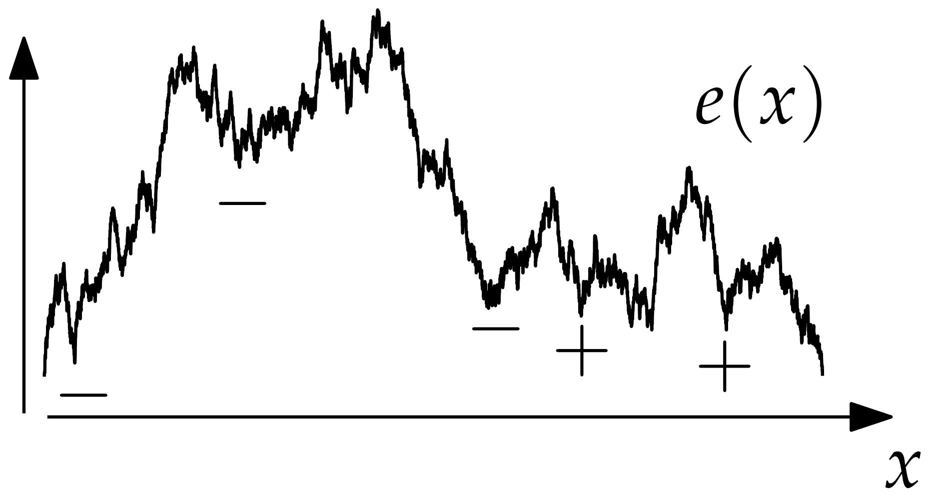

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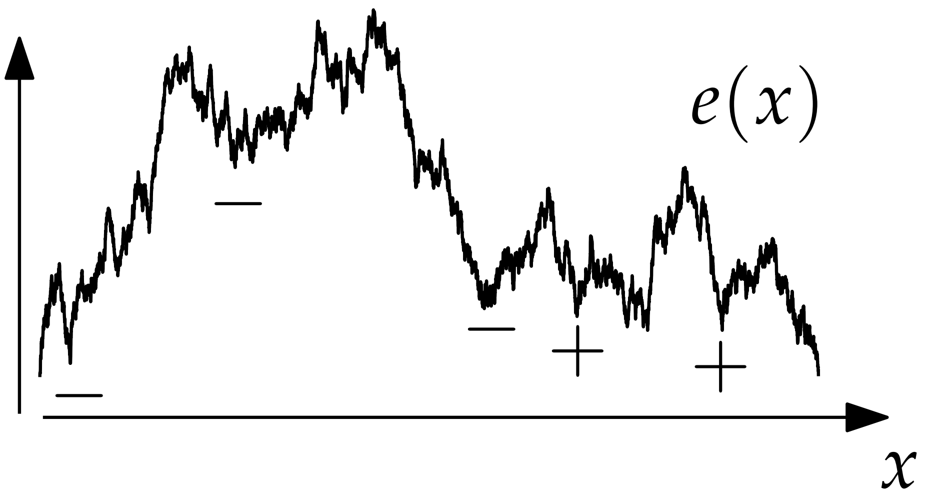
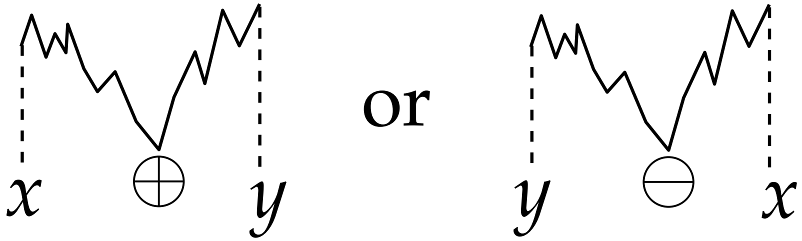


Define a shuffled pseudo-order on $[0,1]$: $x \lhd_e^S y$ if and only if



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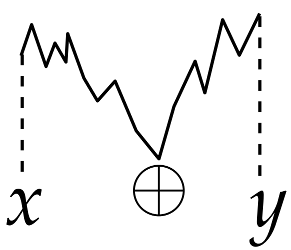
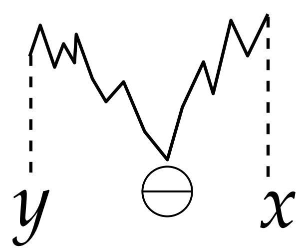
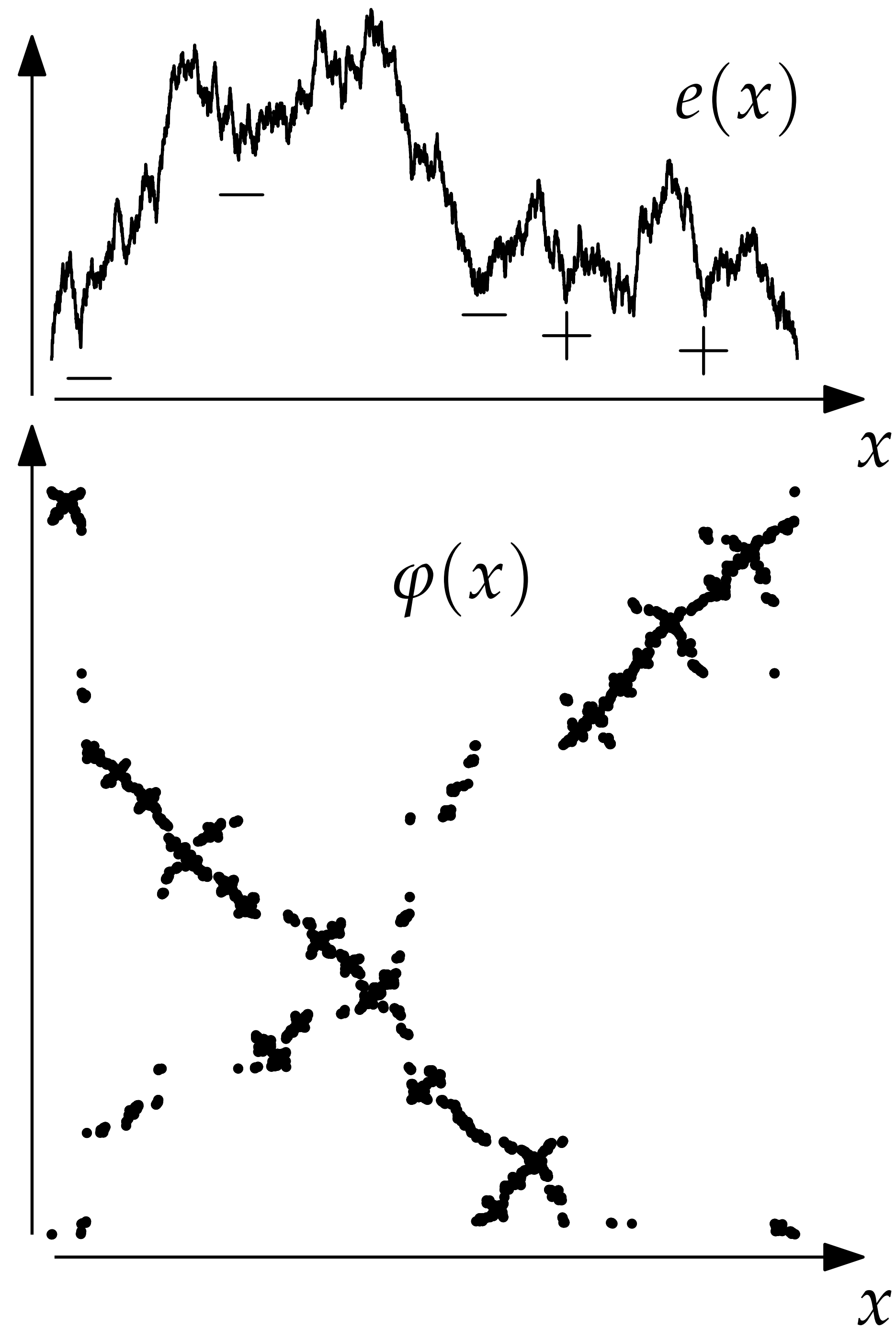


or



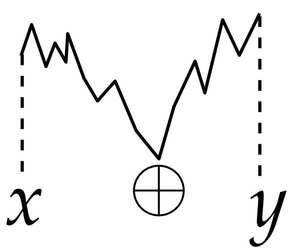
$\varphi(t) = \mathrm{Leb}(\{u \in [0,1], u \lhd_e^S t\})$ is the only (up to a.e. equality) Lebesgue-preserving function sending $\leq$ to $\lhd_e^S$



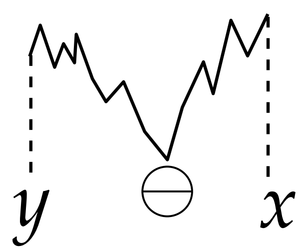
0 - General idea and limit object

e Brownian excursion, S i.i.d. balanced signs indexed by the local minima of e.

Define a shuffled pseudo-order on $[0,1]$: $x \lhd_e^S y$ if and only if



or



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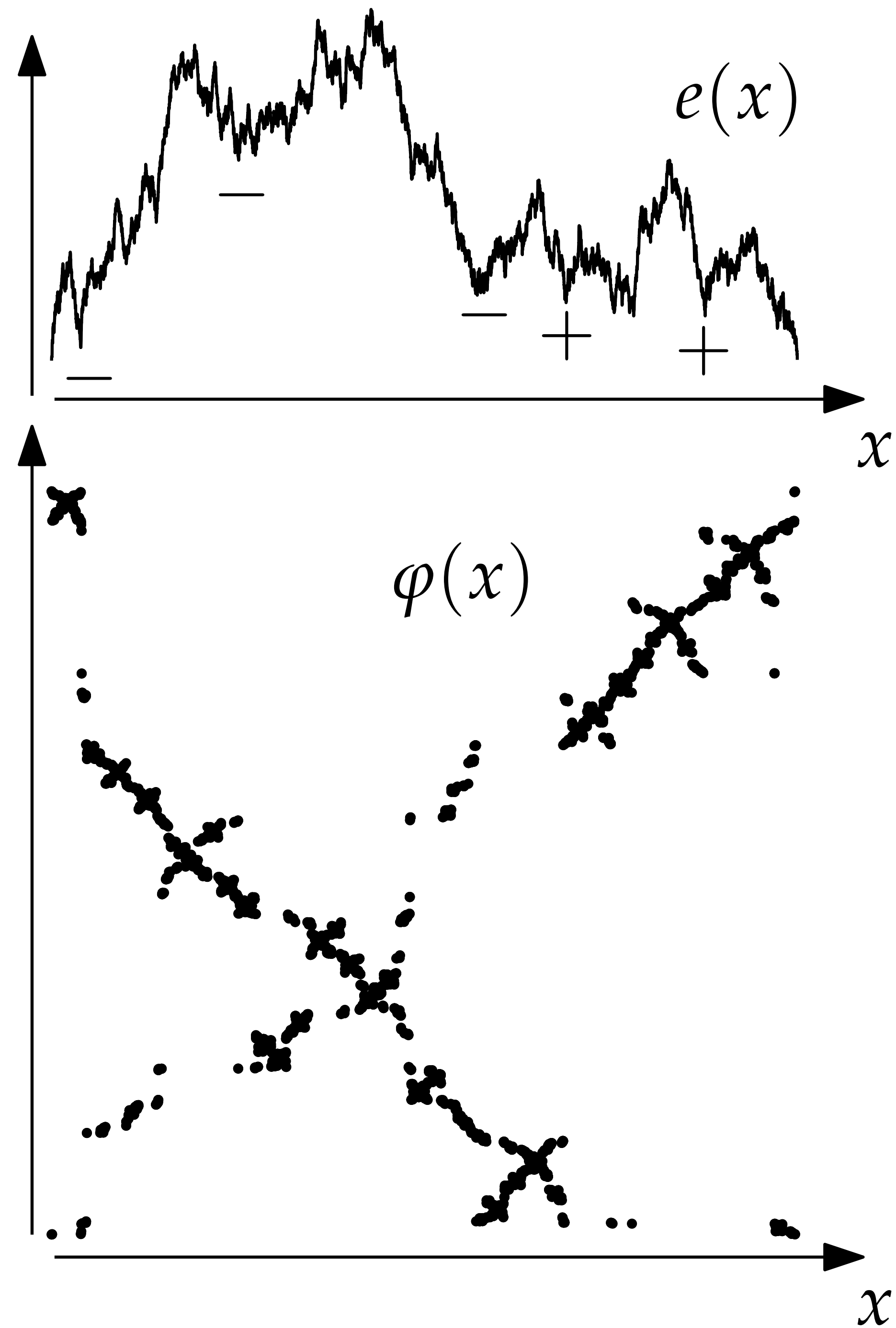
Then $\mu = (\mathrm{id}, \varphi)_\star \mathrm{Leb}$ is the Brownian separable permuton (M. 2017)



I - Permuton convergence and patterns

For $\sigma \in \mathfrak{S}_n$ and $k \leq n$, $\mathrm{perm}_k(\sigma)$ is a uniform subpermutation of length k in σ.

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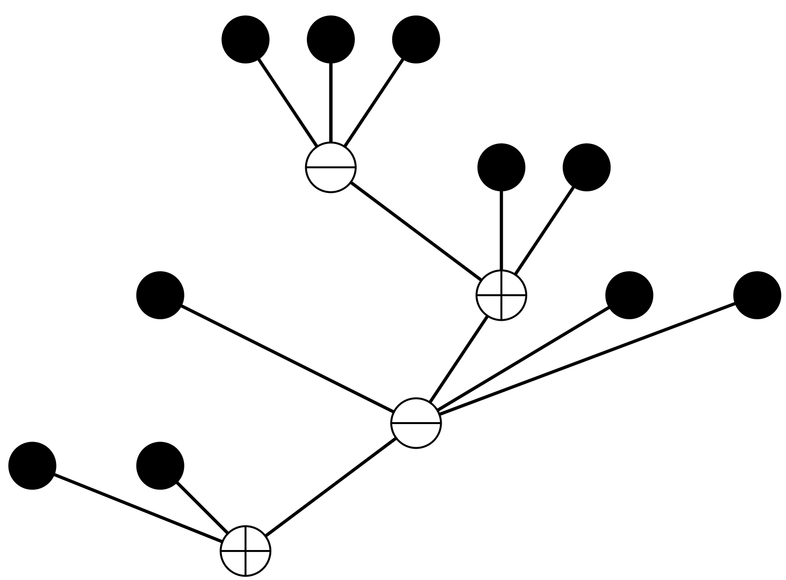
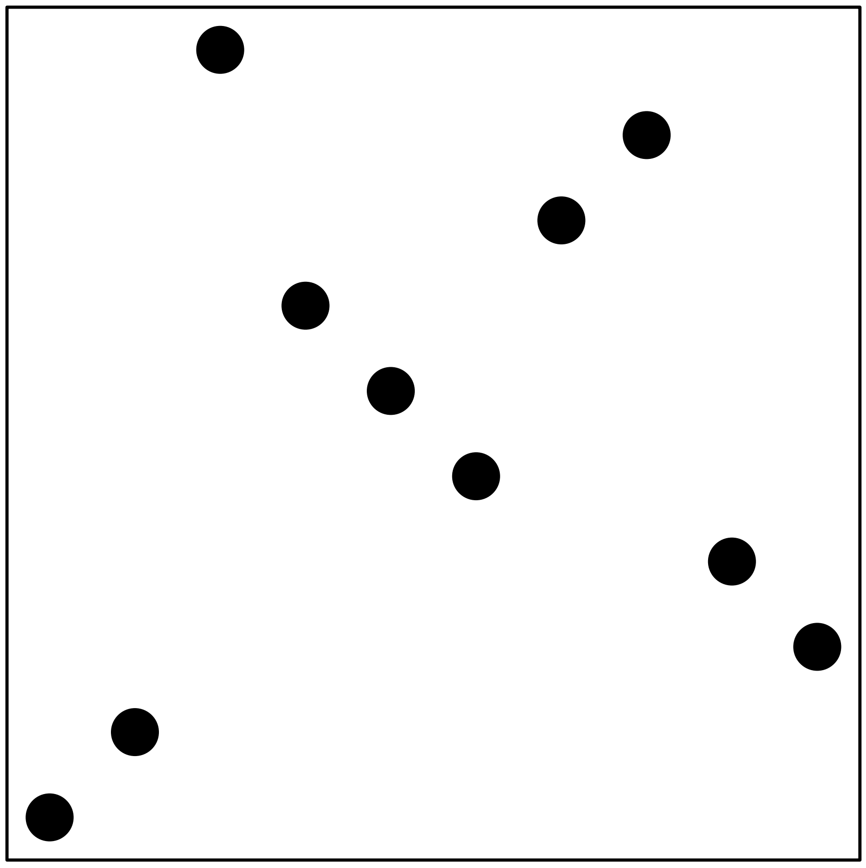
Theorem (Hoppen *et. al.* '2013, BBFGMP '2017)
The random permutons (μ_{σ_n}) converge in distribution to μ iff for every k, $\mathrm{perm}_k(\sigma_n) \xrightarrow[n\to\infty]{d} \mathrm{perm}_k(\mu)$.

II - Patterns and the tree encoding

A subpermutation of σ_n can be read on a reduced tree of t_n





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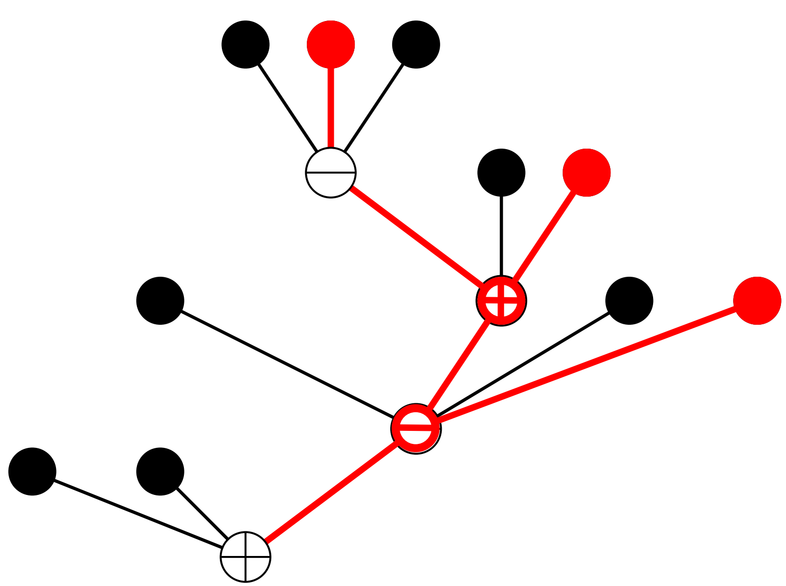

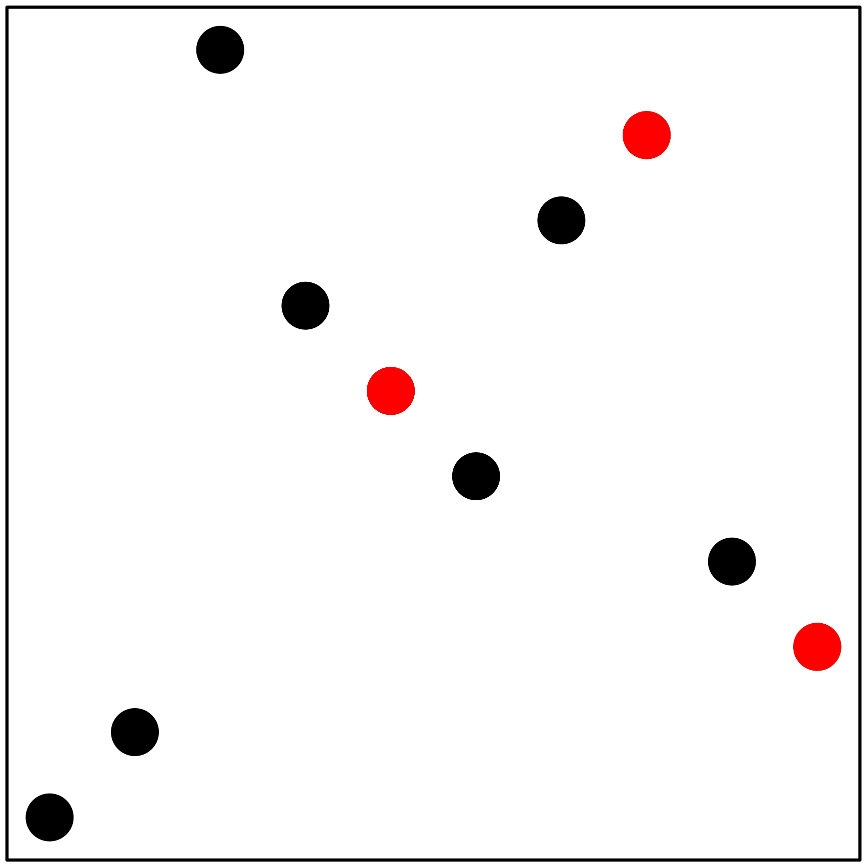

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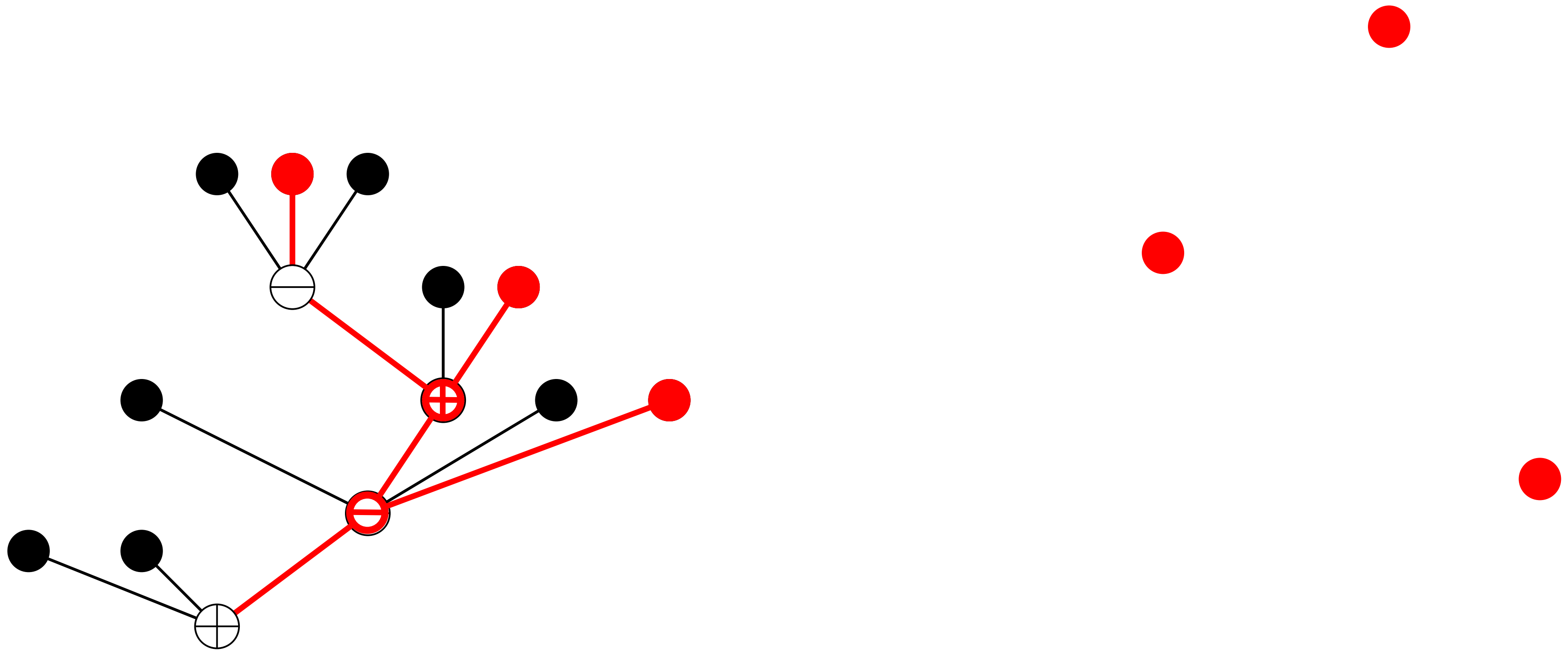

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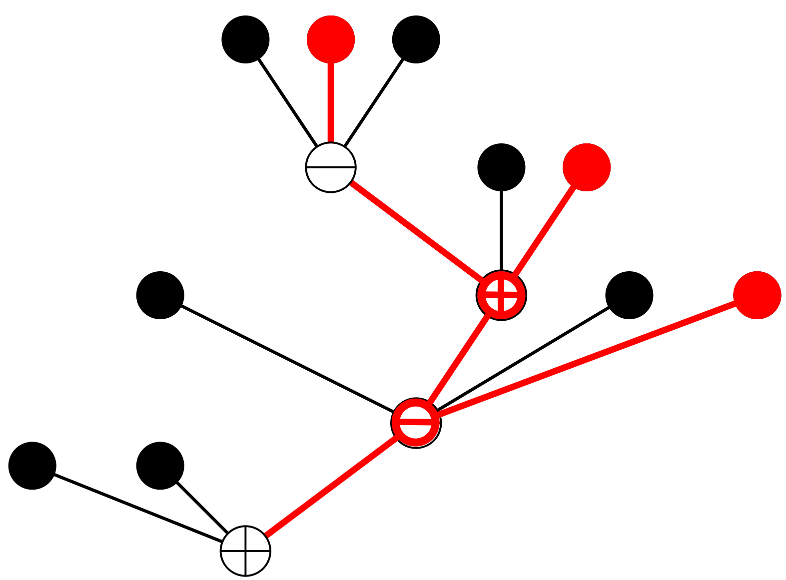
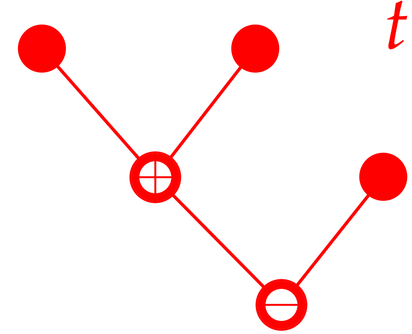
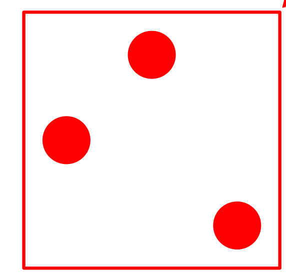


$t_{n|I_n^k}$



$\mathrm{pat}_{I_n^k}(\sigma_n)$



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A subpermutation of σ_n can be read on a reduced tree of t_n

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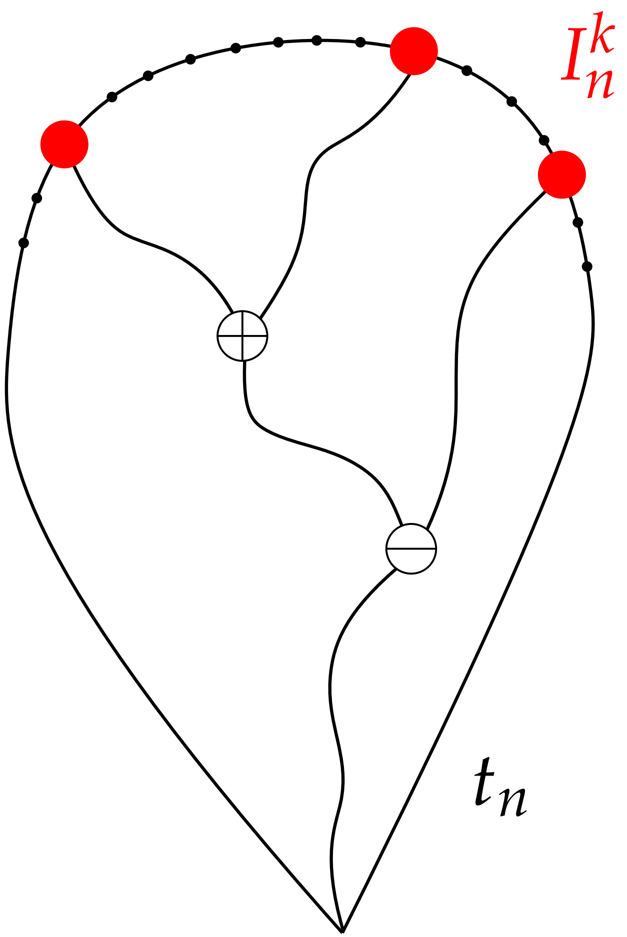

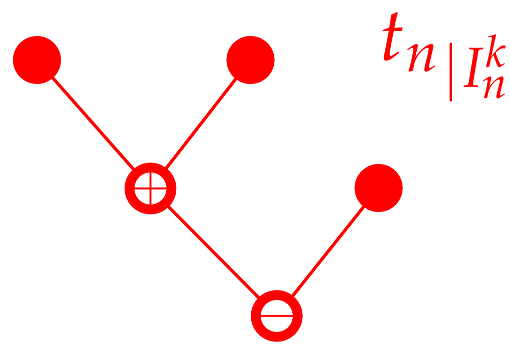

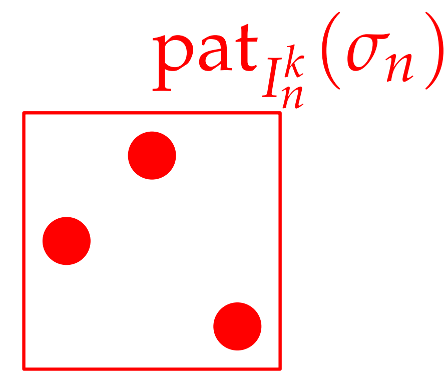

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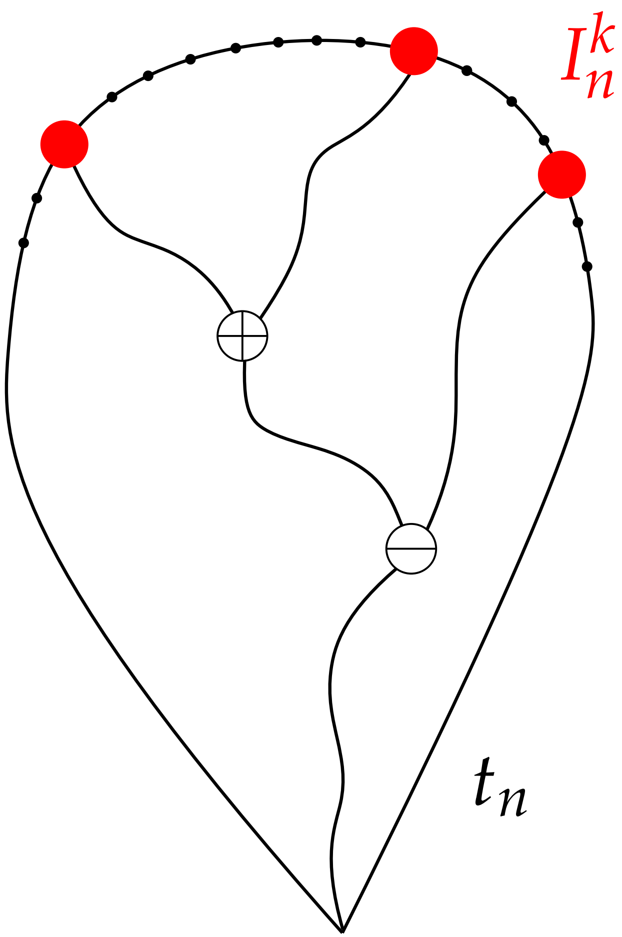

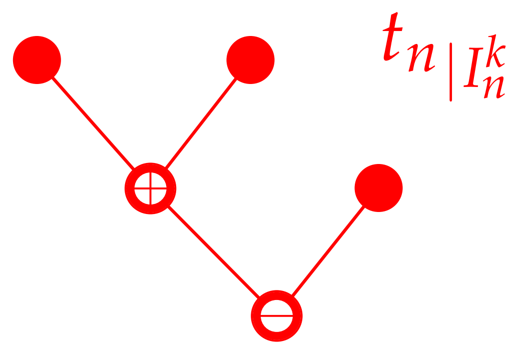

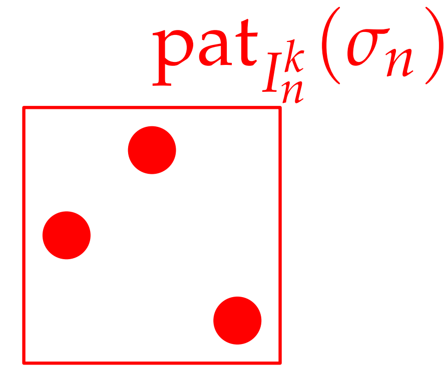

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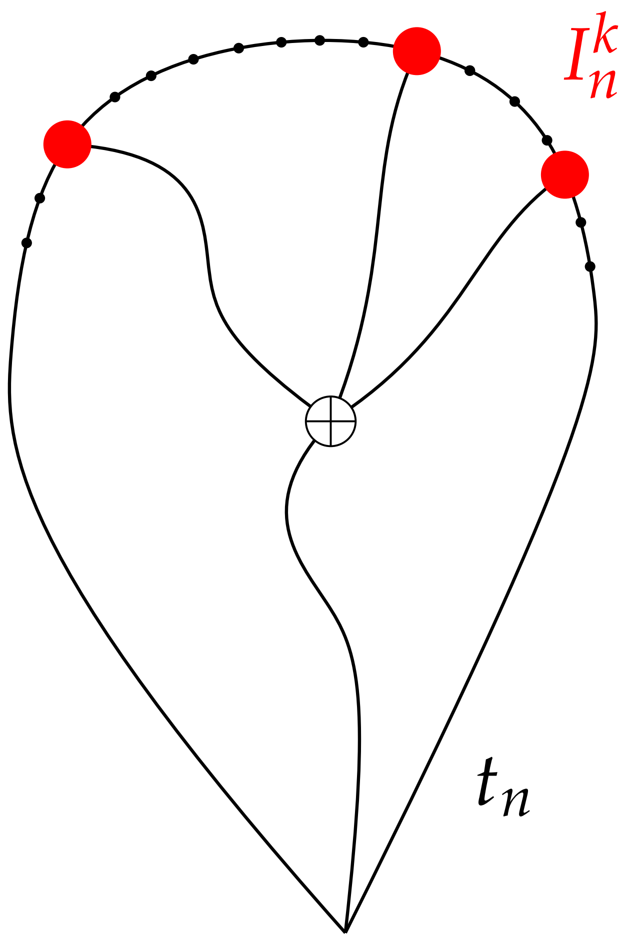

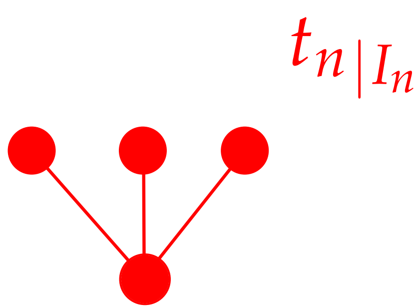

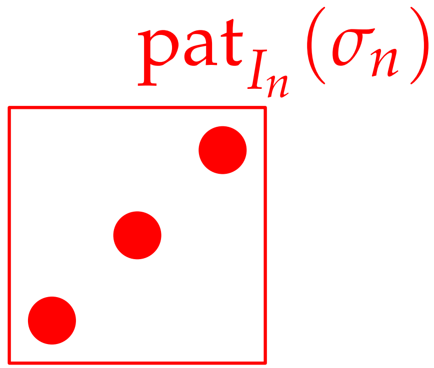

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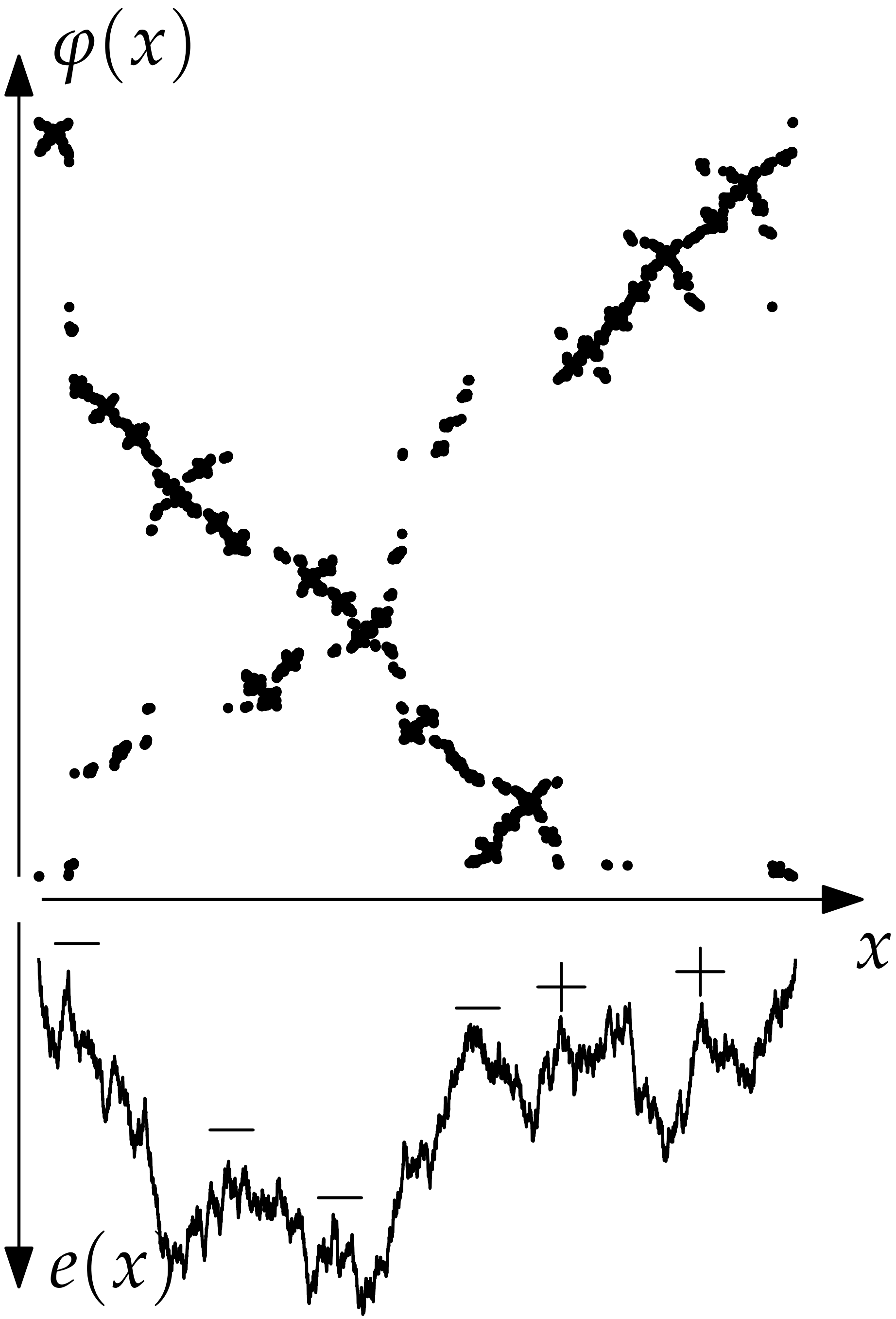
III - Patterns in the Brownian permuton



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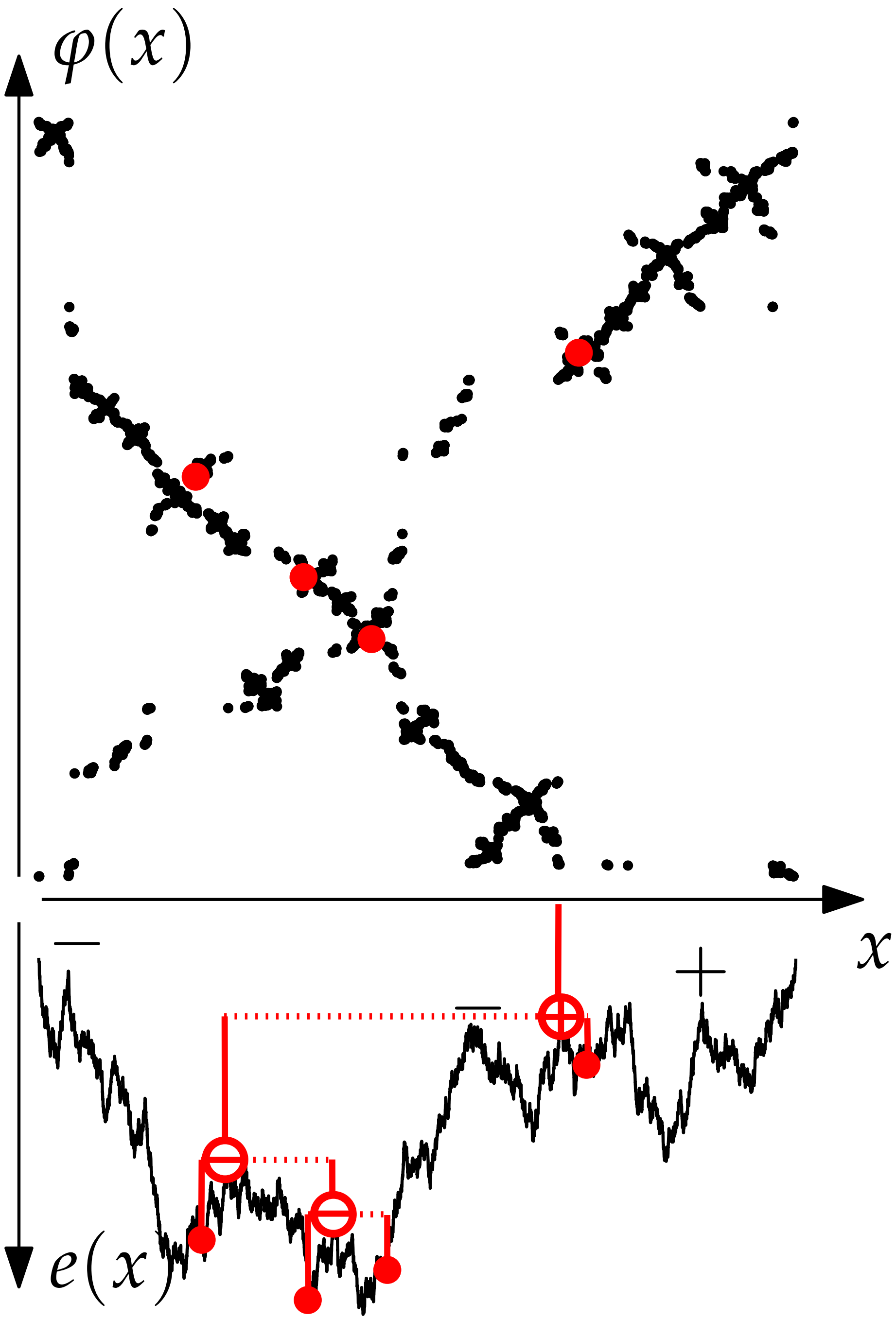
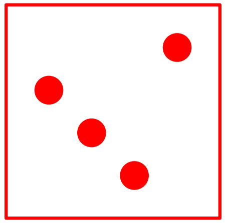
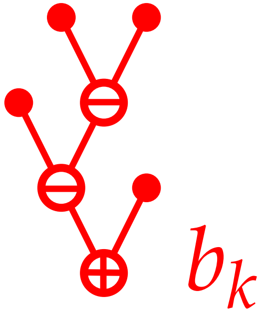


Reduced trees of the Brownian excursion are uniform binary trees (Aldous '93, Le Gall '93)





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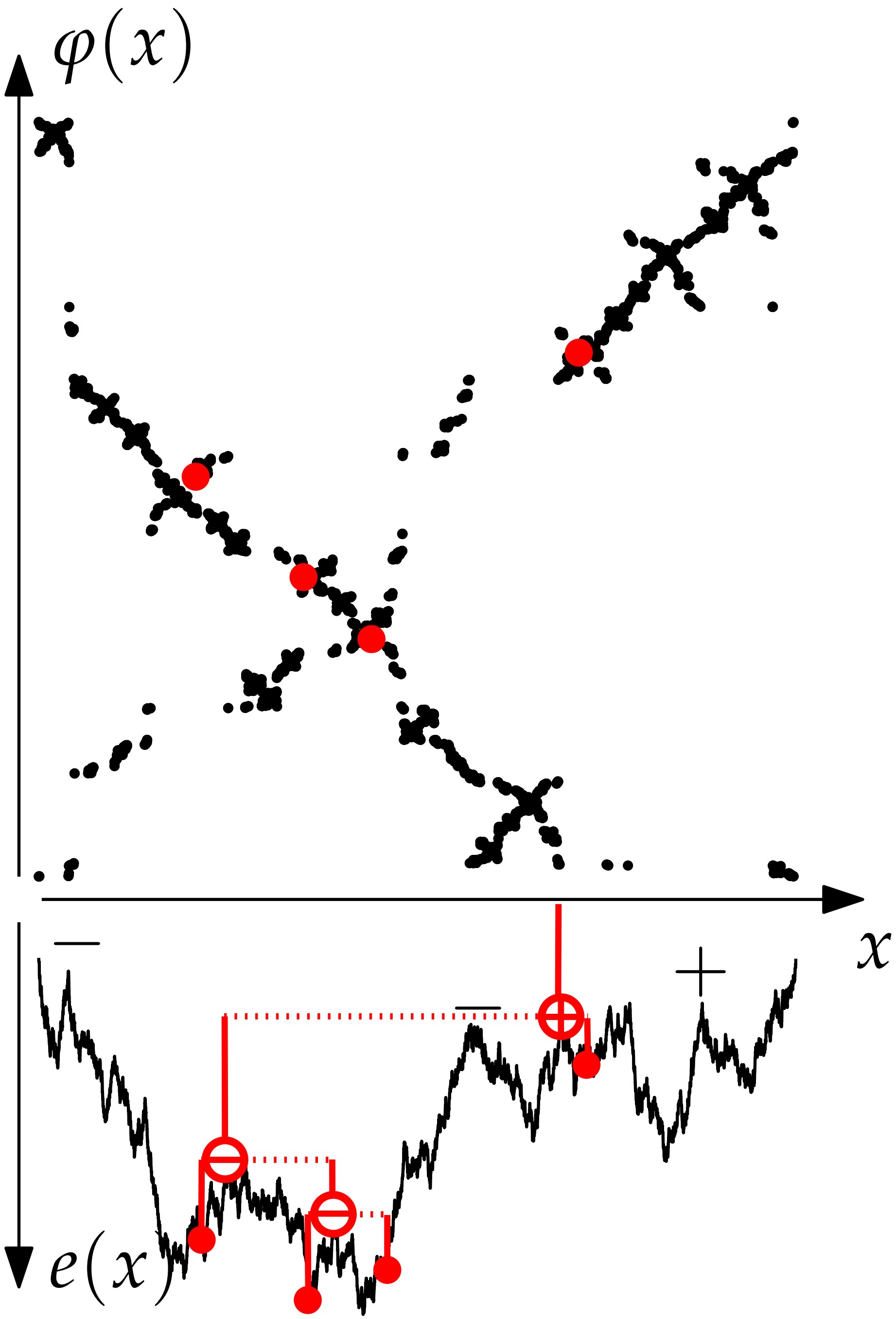

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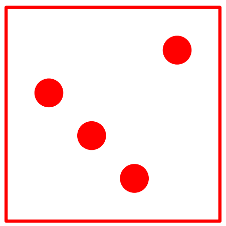
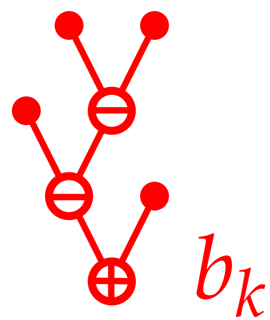


Hence $\mathrm{perm}_k(\mu)$ has the distribution of $\mathrm{perm}(b_k)$ where b_k is a uniform signed binary tree with k leaves.



Summing up

Fix a signed binary tree τ with k leaves. We need only show that

$$\frac{\#\{\text{Schröder trees of size } n \text{ with } k \text{ marked leaves inducing } \tau\}}{\#\{\text{Schröder trees of size } n \text{ with } k \text{ marked leaves}\}}$$

converges to

$$\mathbb{P}(b_k = \tau) = \frac{1}{2^{k-1}\mathrm{Cat}_{k-1}}.$$

IV - Analytic combinatorics

Let $(a_n)_n$ be a nonnegative sequence and $A(z) = \sum_n a_n z^n$ its generating function of radius ρ

Transfer Theorem (Flajolet & Odlyzko) If

- A is defined on a Δ-domain at $\rho > 0$ (e.g. is algebraic)
- $A(z) \underset{z \to \rho}{=} g(z) + (C + o(1))(\rho - z)^{\delta}$ with g analytic, $\delta \notin \mathbb{N}$,

then $a_n \underset{n \to \infty}{=} (\frac{C}{\Gamma(-\delta)} + o(1))\rho^{-n} n^{-1-\delta}$

Proposition (Singular differentiation) Under the same hypotheses, $A'(z) \underset{z \to \rho}{=} g'(z) + \delta(C + o(1))(\rho - z)^{\delta - 1}$

Analytic combinatorics for leaf-counted trees

Recall: nice trees converge to the Brownian CRT.

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Recursive trees counted by number of leaves.
$T(z) = z + F(T(z))$ (Schröder: $F(t) = \sum_{k \geq 2} t^k$).

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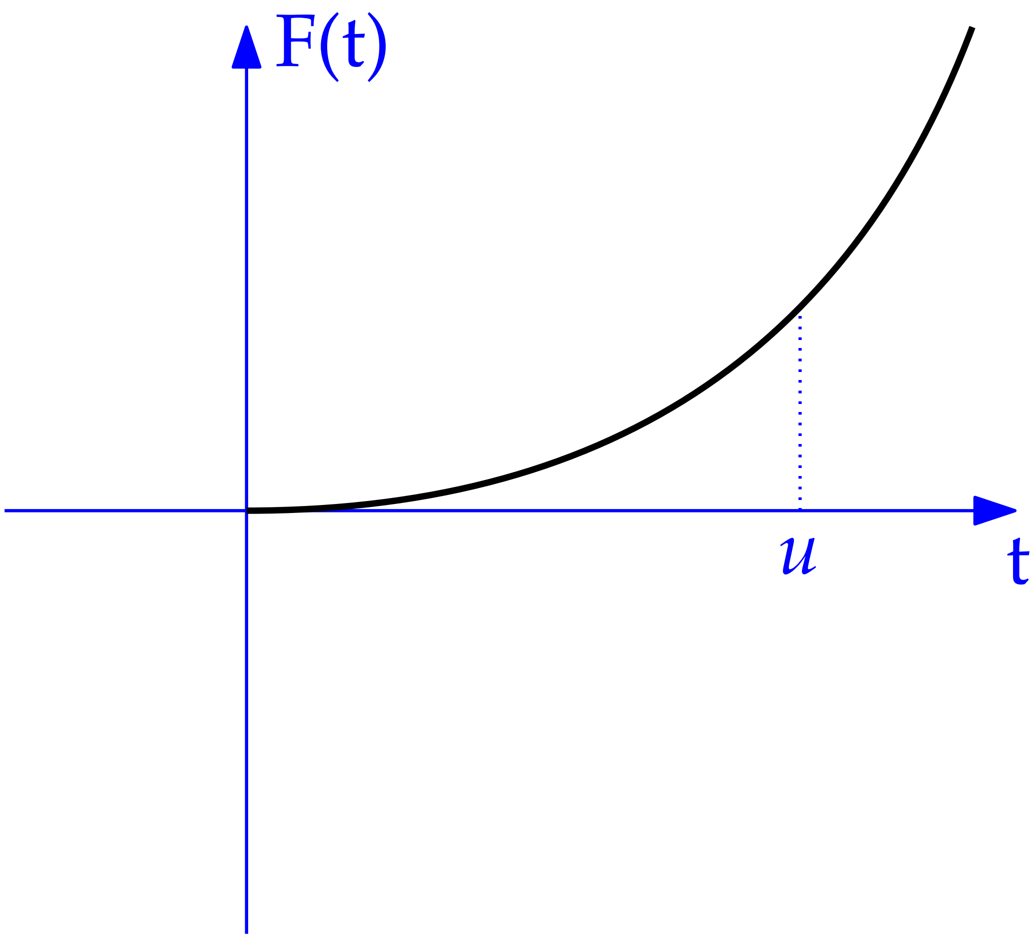
In this case, "very nice" if
$\exists \quad 0 < u < R_F, F'(u) = 1.$
Then T is Δ-analytic at ρ with $T(\rho) = u$ and a square-root singularity (smooth implicit function schema).



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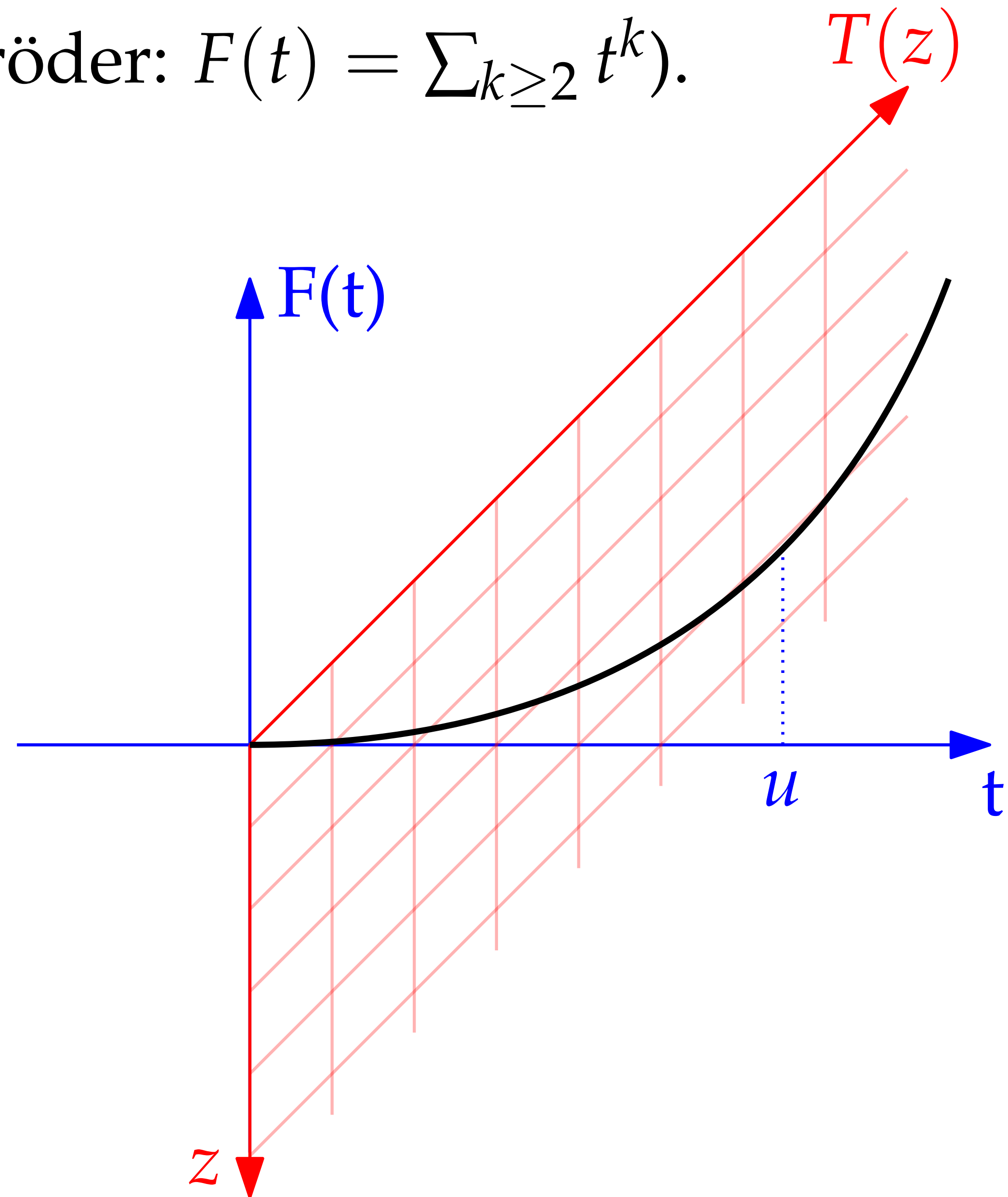

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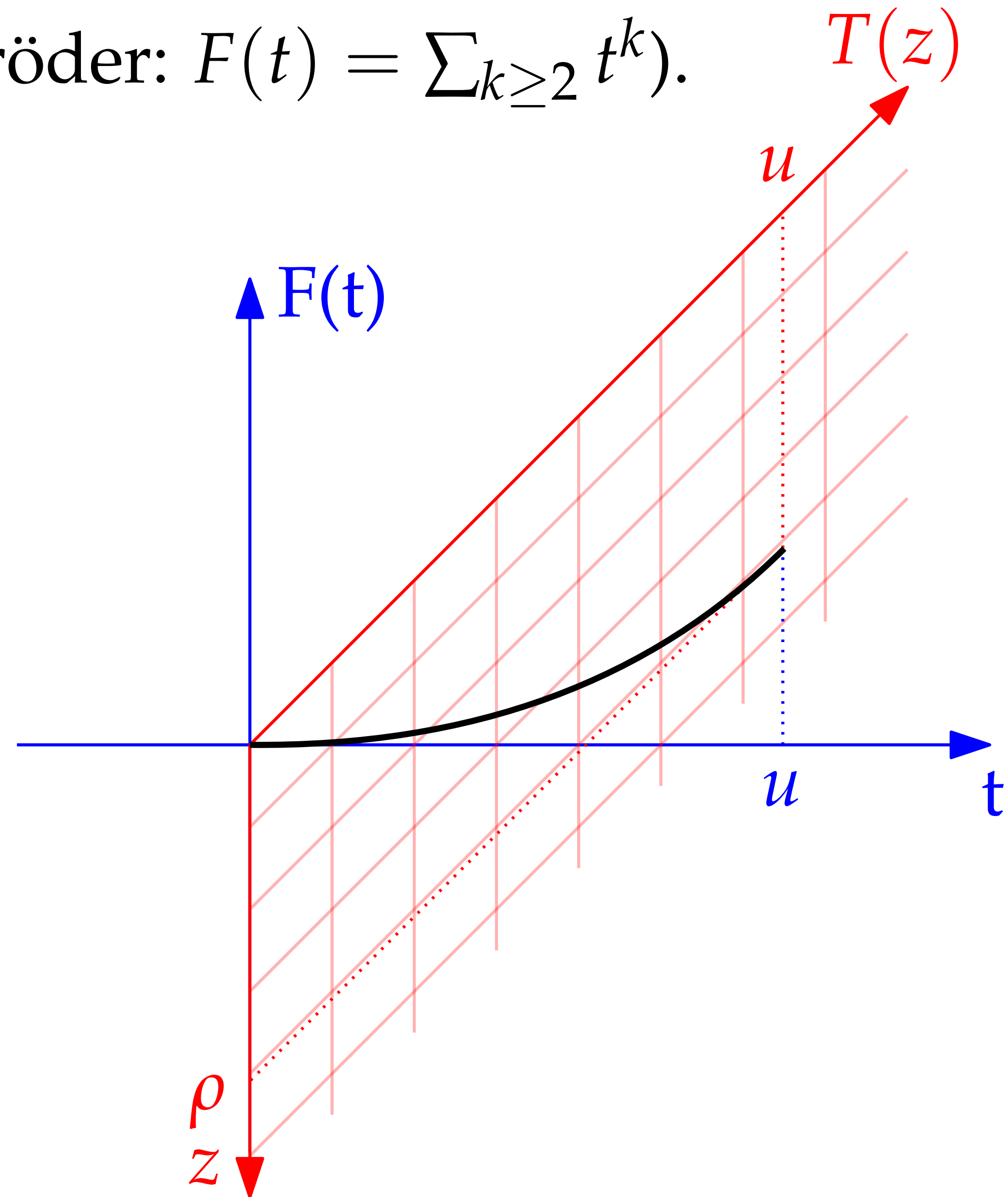

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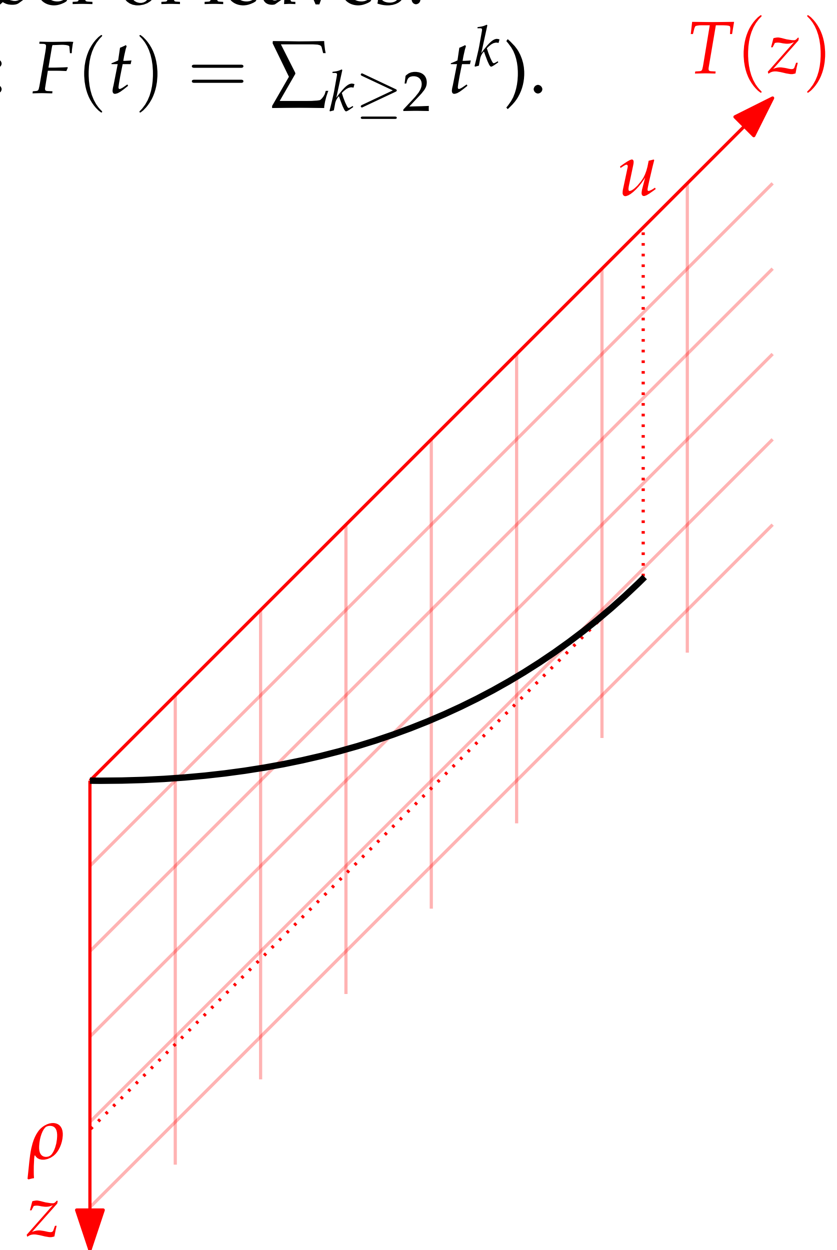

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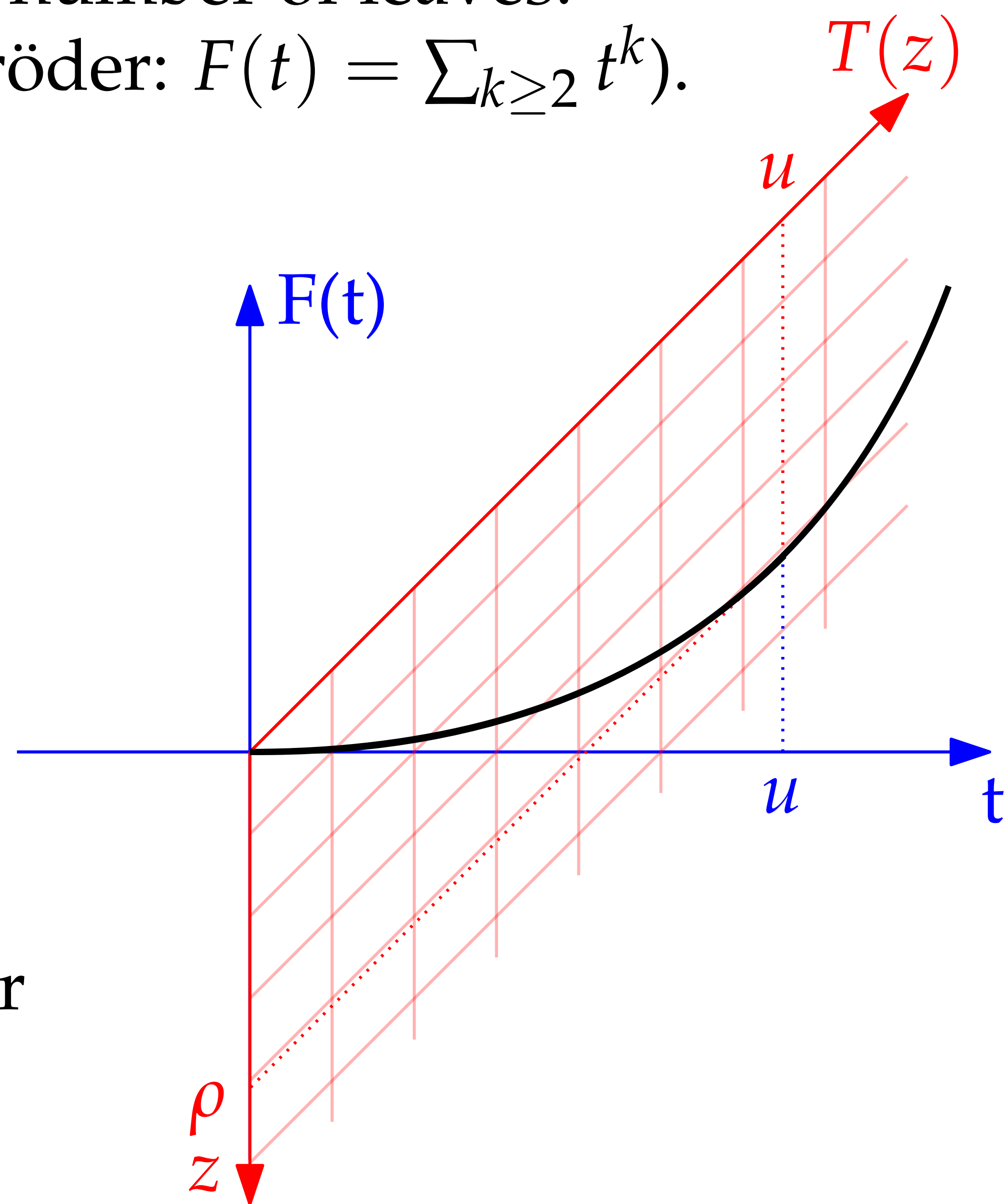
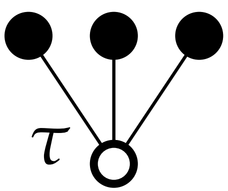


This is the case for Schröder (F rational)

Uniform k-subtree in large unsigned trees

T has square-root singularity at ρ and F analytic at $T(\rho)$. Then, the $g.f$ of trees with k marked leaves that induce the k-tree τ is

$$z^k T'(z) \prod_{v \text{ internal node of } \tau} T'(z)^{\deg(v)} \frac{1}{\deg(v)!} F^{(\deg(v))}(T(z))$$



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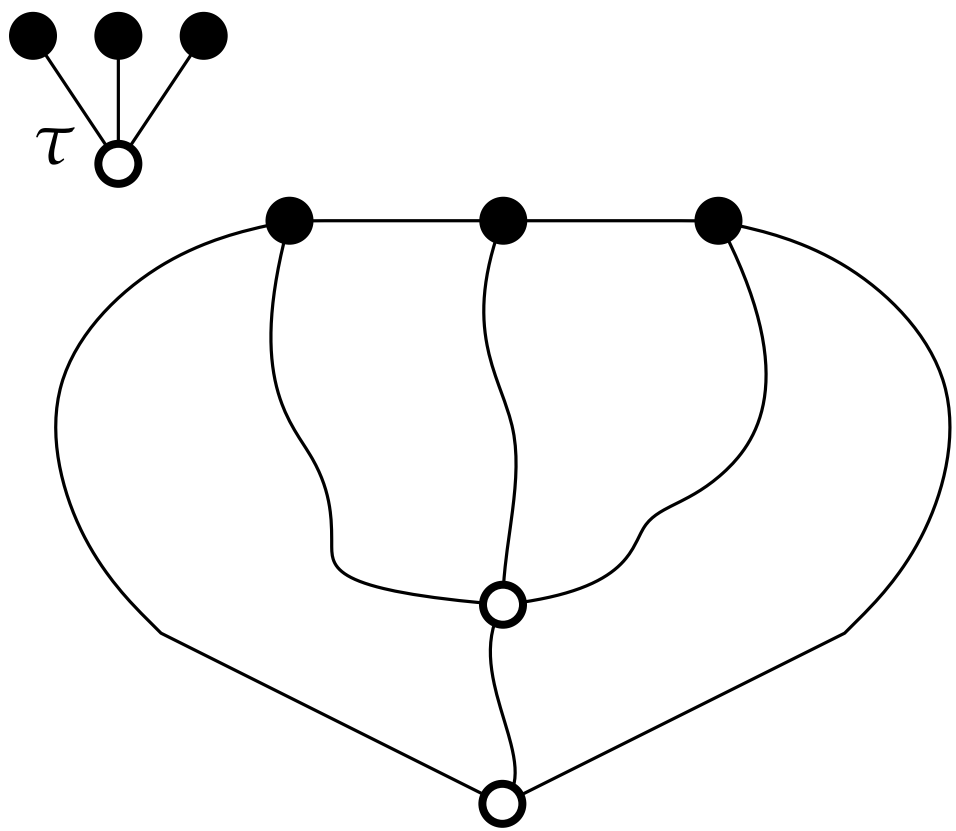

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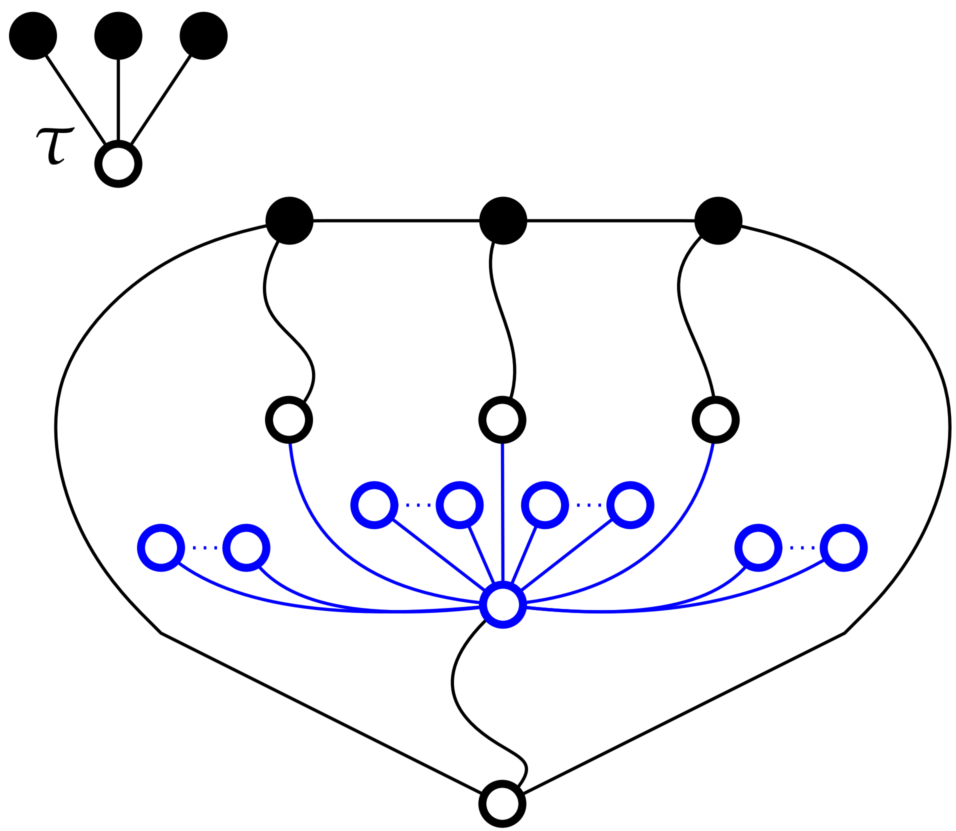

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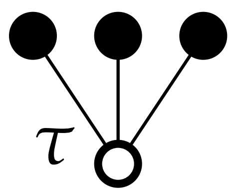

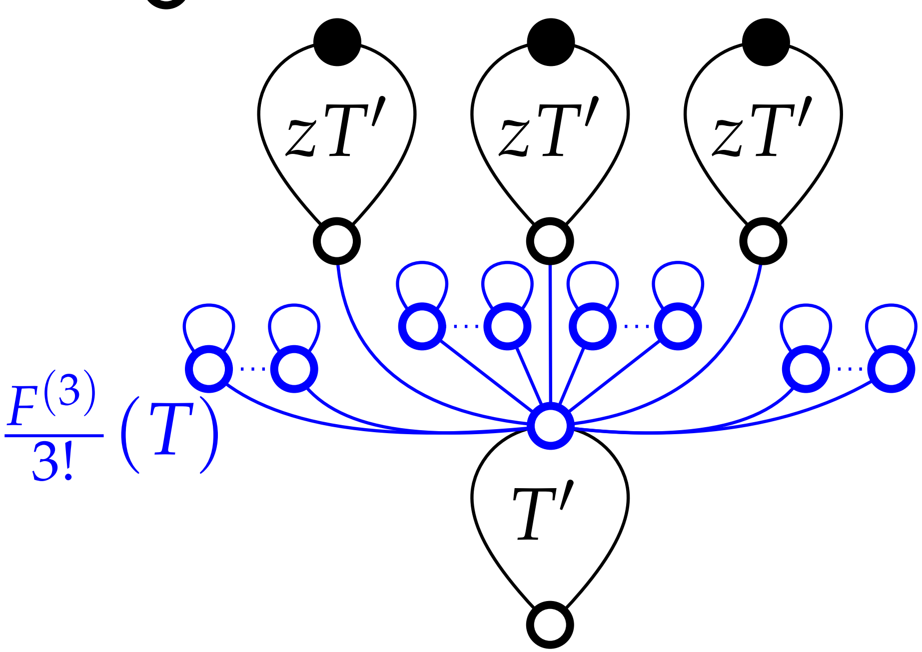

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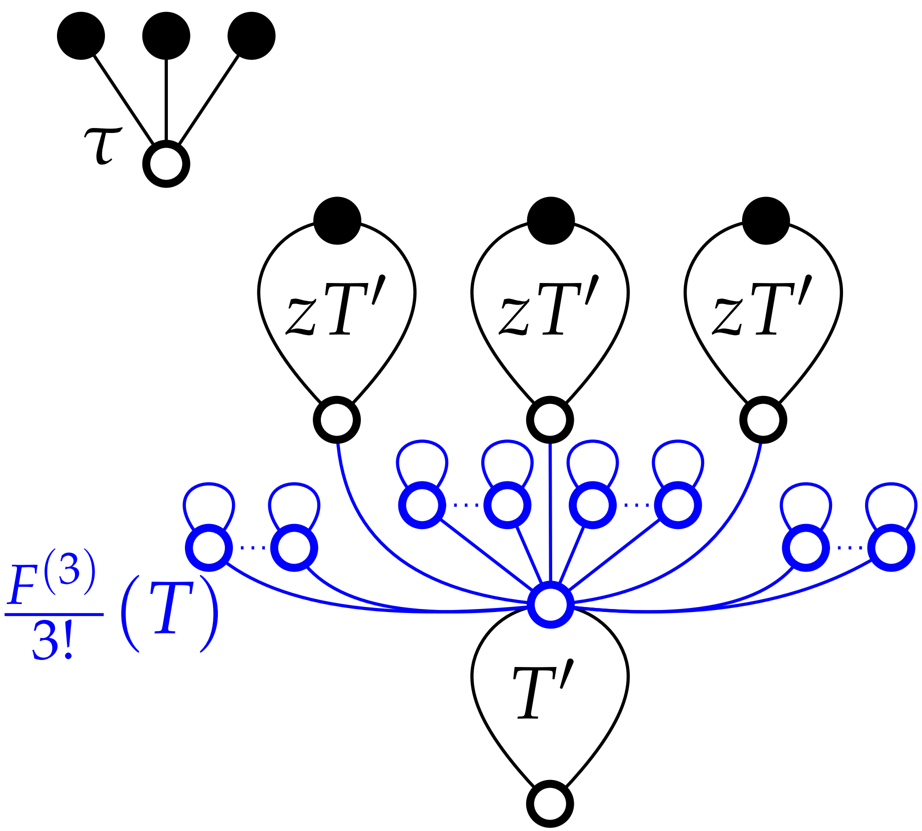

$\sim_\rho C_\tau(\rho - z)^{-\#\{\text{nodes in } \tau\}/2}$.
Dominates when τ binary.
(Then C_τ doesn't depend on τ).
Transfer: $t_{n|I_n^k}$ converges in distribution to a uniform binary tree.

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Counting signed trees that induce a given signed tree τ: adding parity constraints on the height of the marked leaf in the marked trees.

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Hence all signed binary trees have the same asymptotic probability, what whe needed for permuton convergence.

Part 2 - statement

Substitution decomposition

For $\sigma \in \mathfrak{S}_k, \rho_1, \ldots, \rho_k \in \mathfrak{S}$, define $\sigma[\rho_1, \ldots, \rho_k]$ by replacing the i-th dot in σ by π_i.

Example : $132[21, 312, 2413] = 219784635$.

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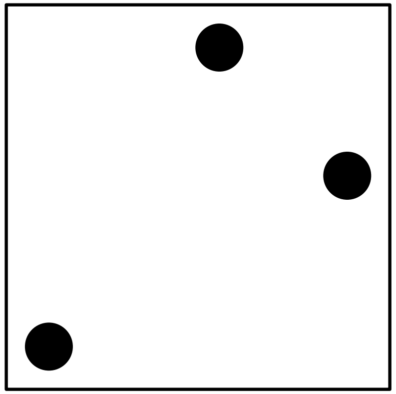

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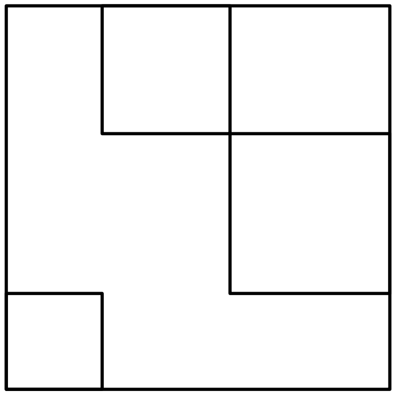

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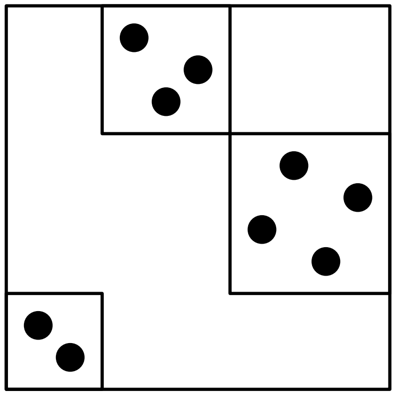

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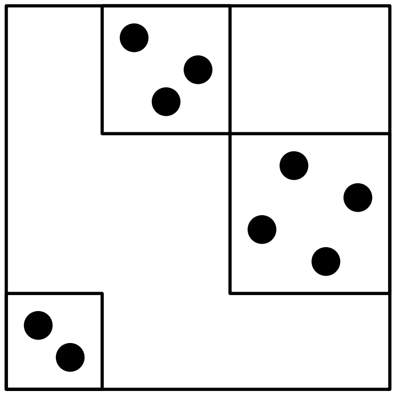

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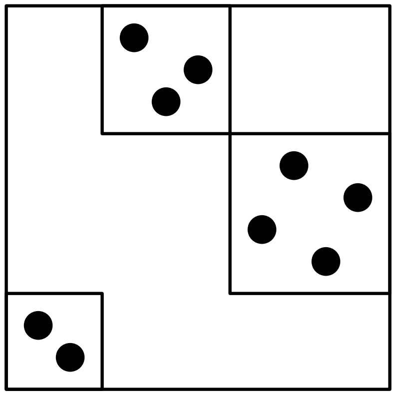

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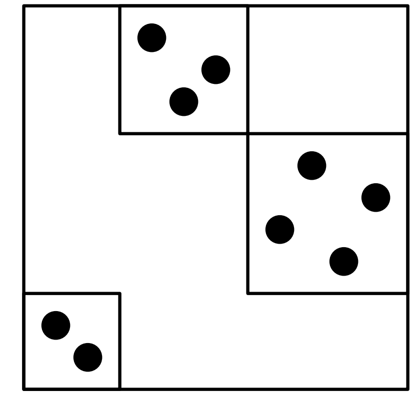

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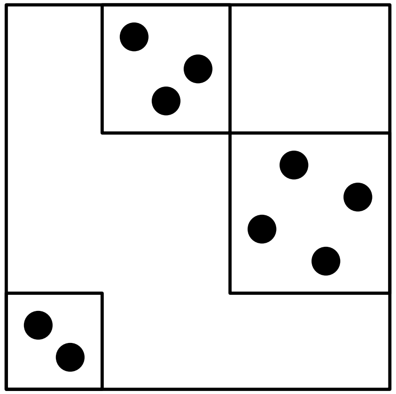
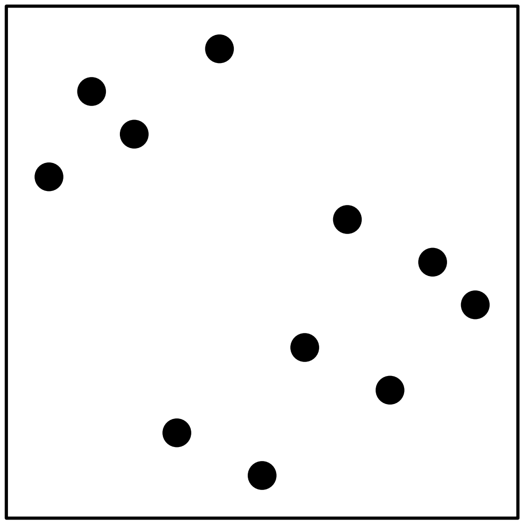


Given σ, either :

- We can find a proper interval mapped to an interval, and then σ can be written as a substitution of smaller permutations
- Or σ can't be decomposed by a nontrivial substitution : σ is a **simple permutation**. Ex : $1, 12, 21, 2413, 3142, 31524, \ldots \sim \frac{n!}{e^2}$.

Substitution decomposition



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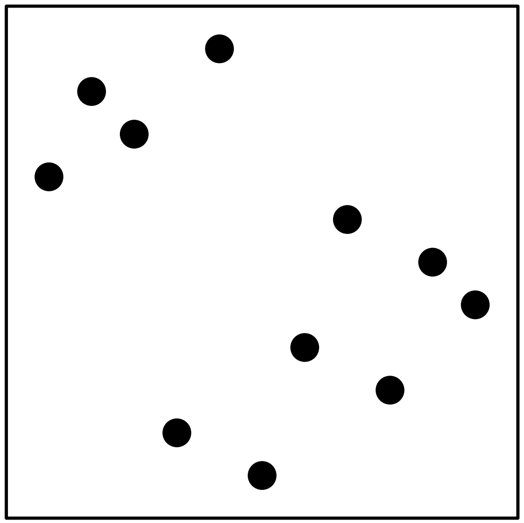

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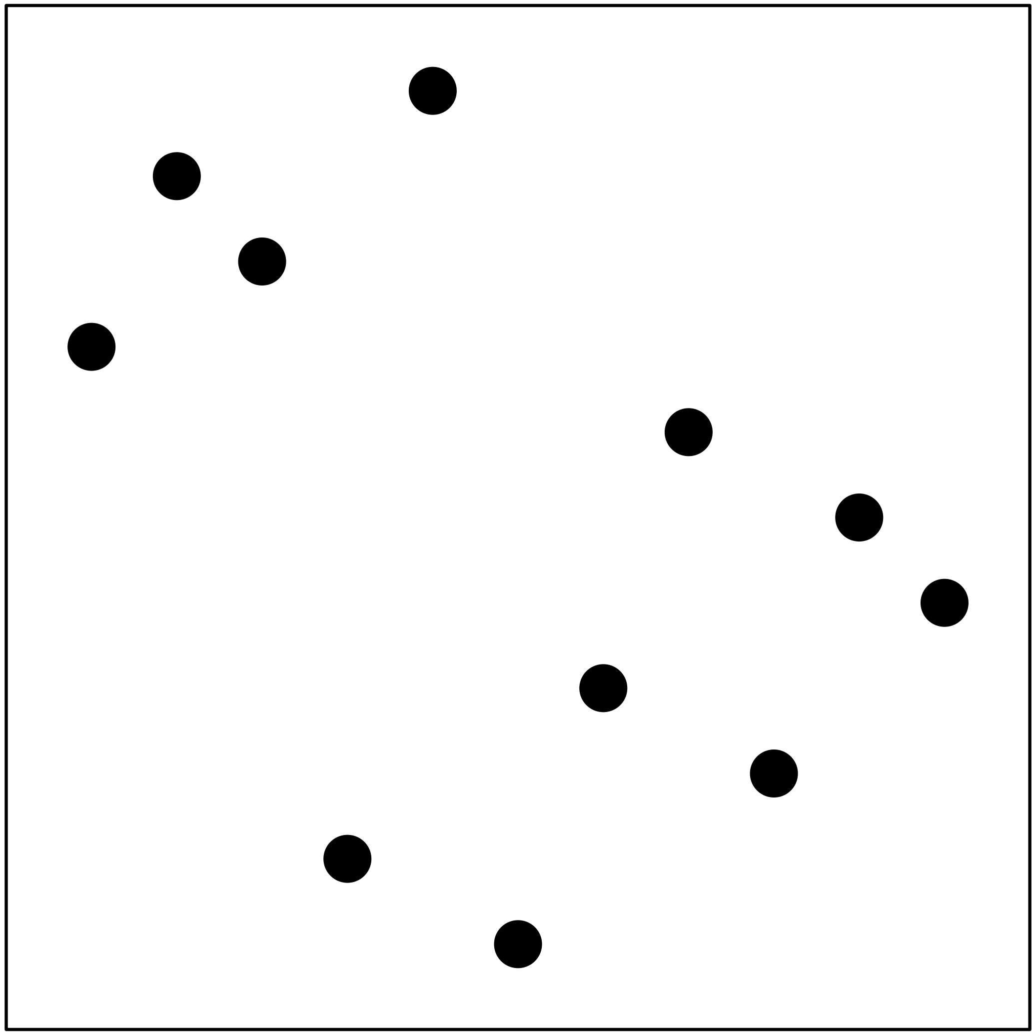

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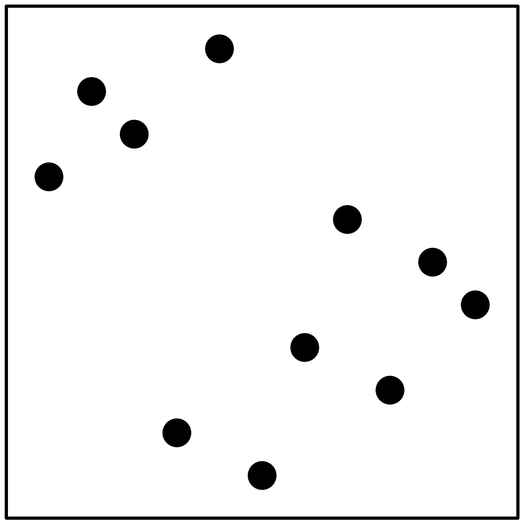

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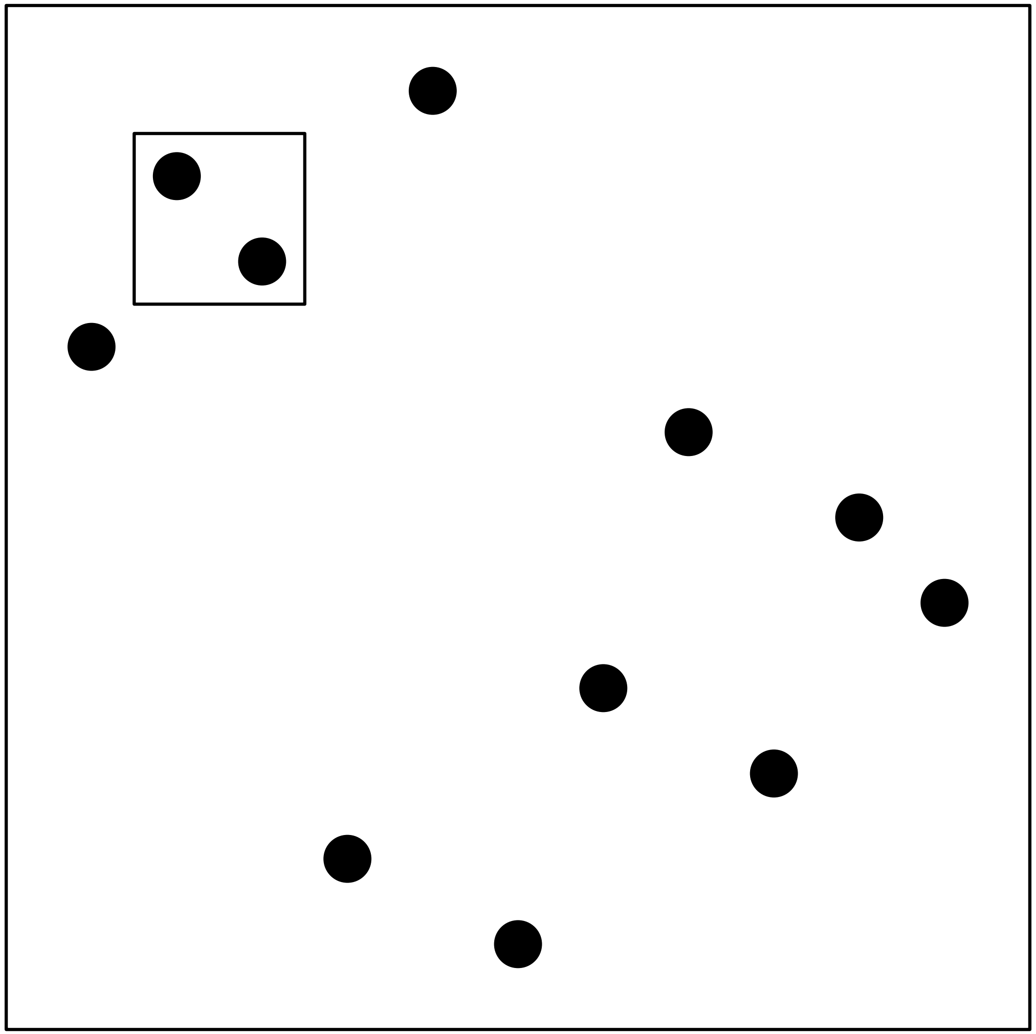

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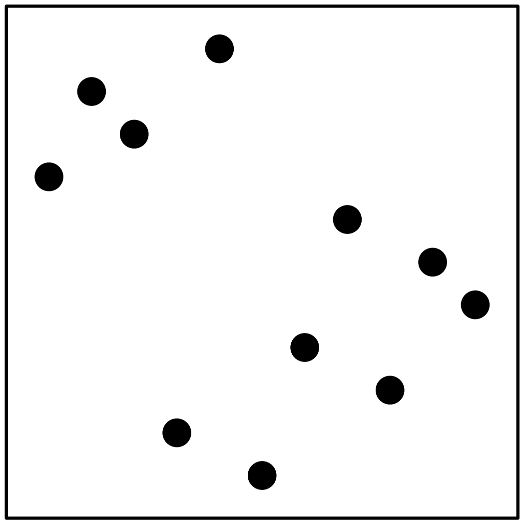

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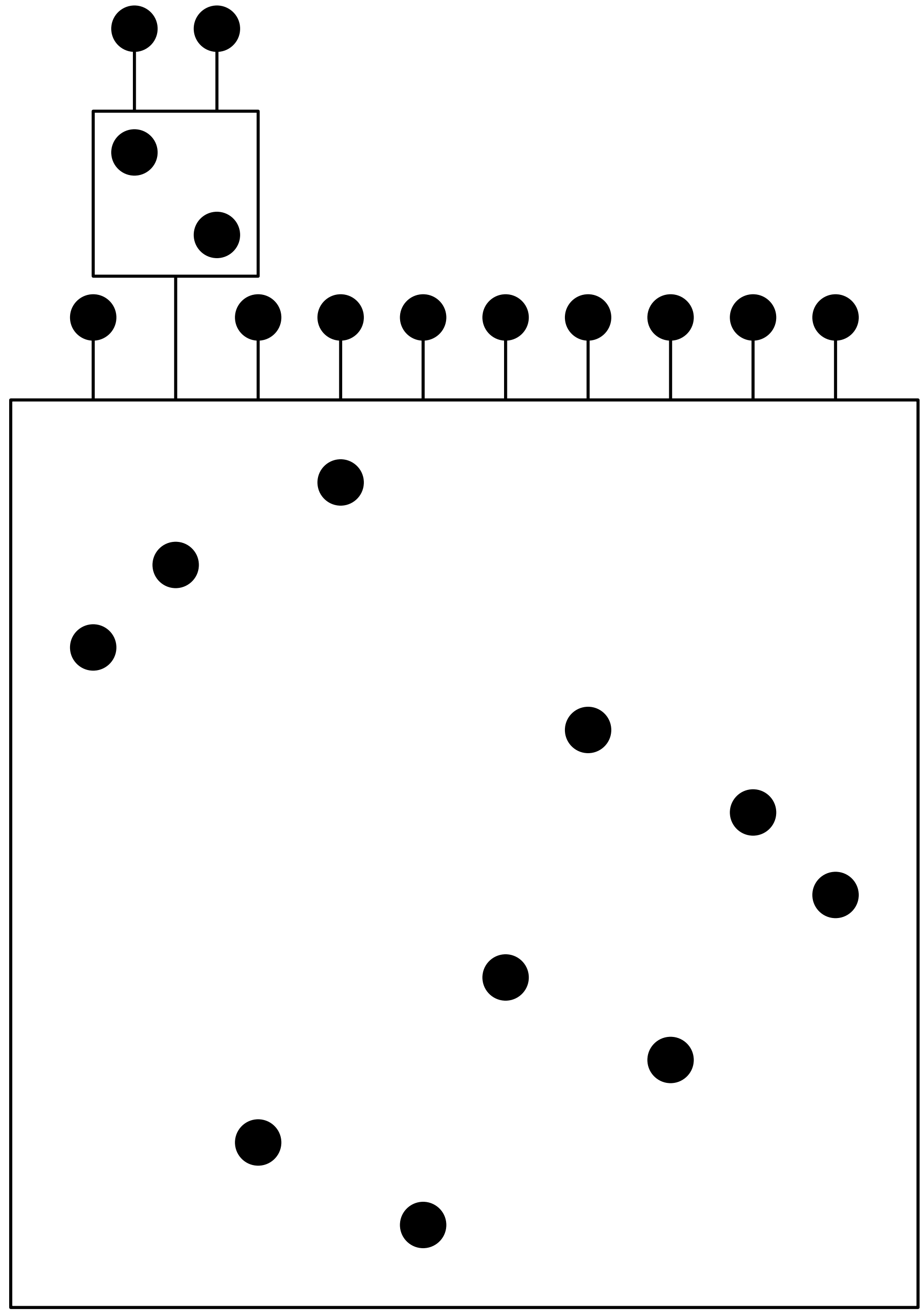

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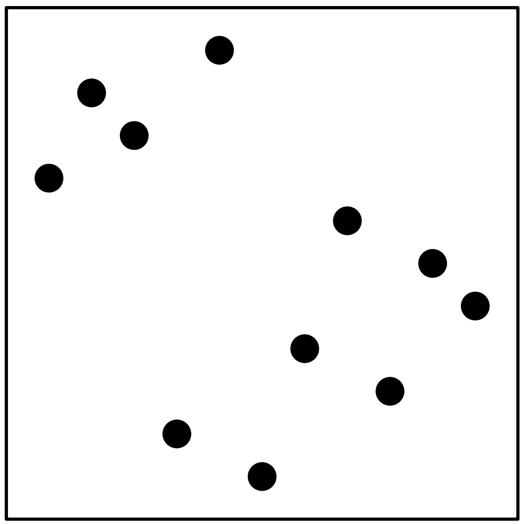

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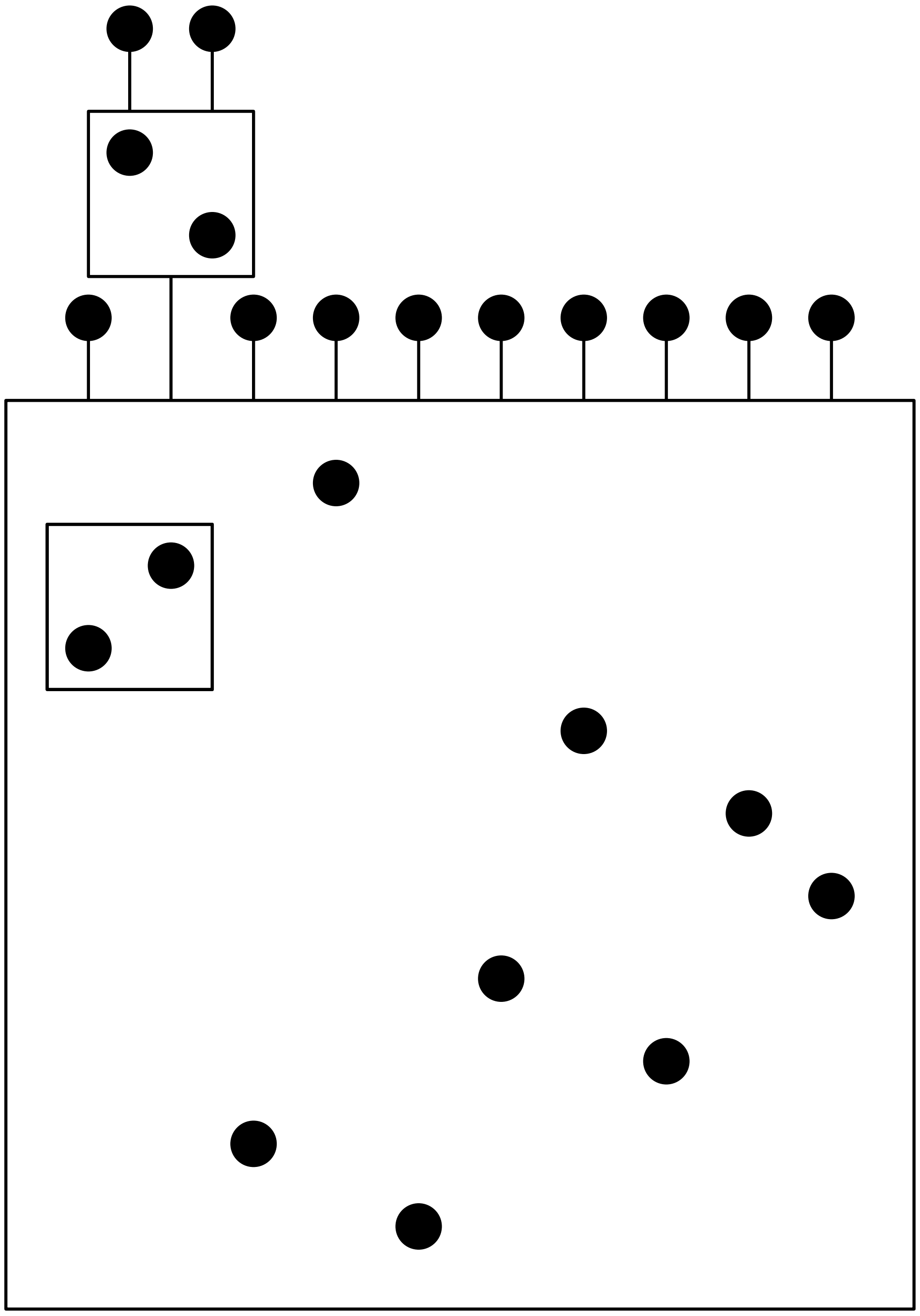

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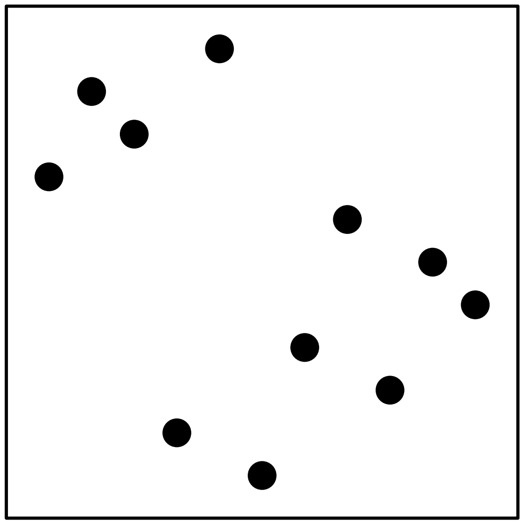

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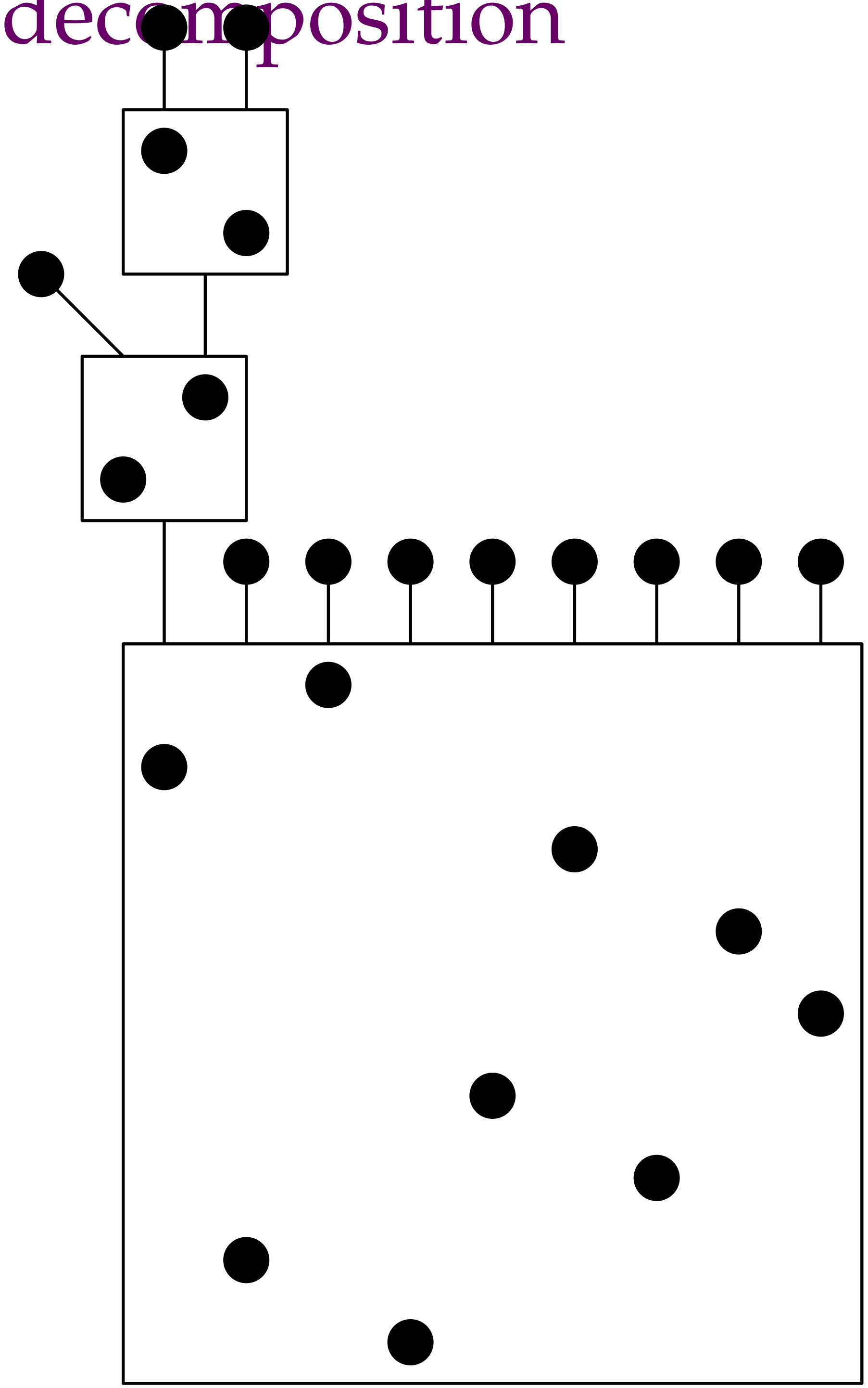

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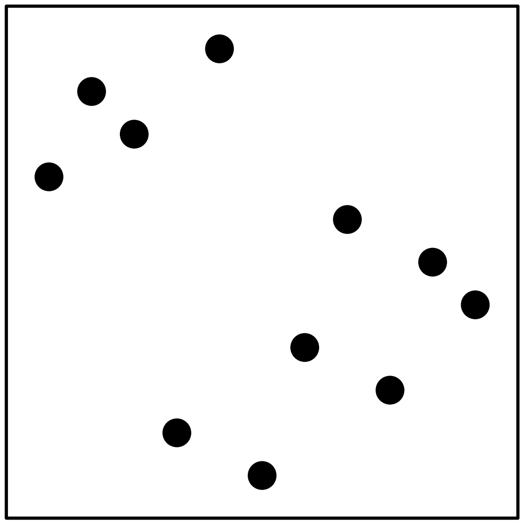

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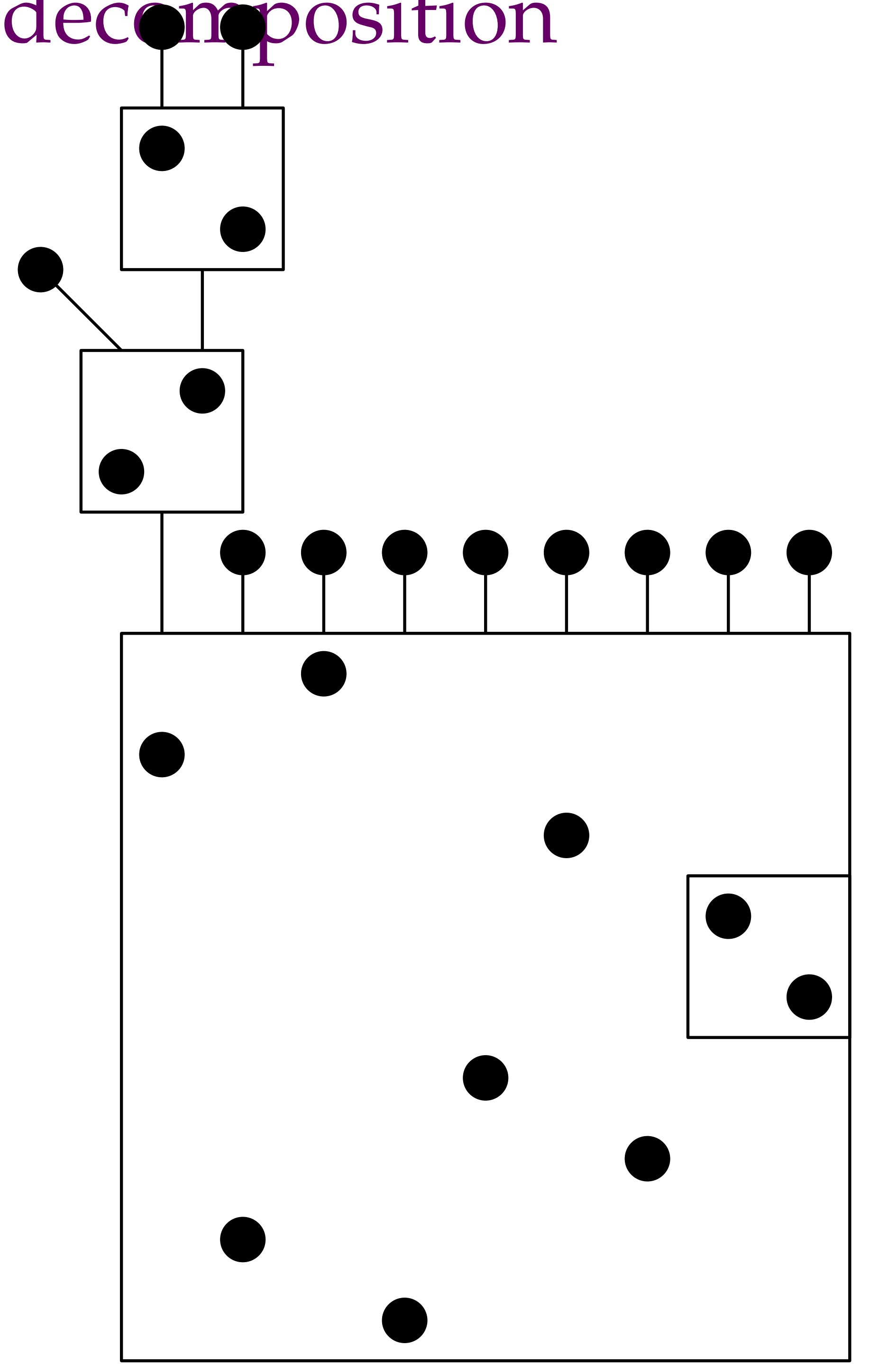

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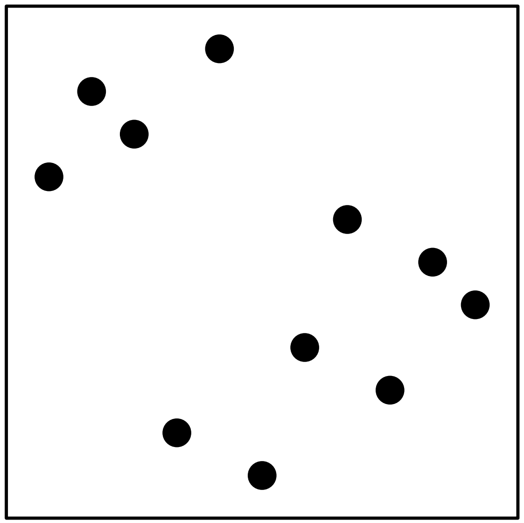

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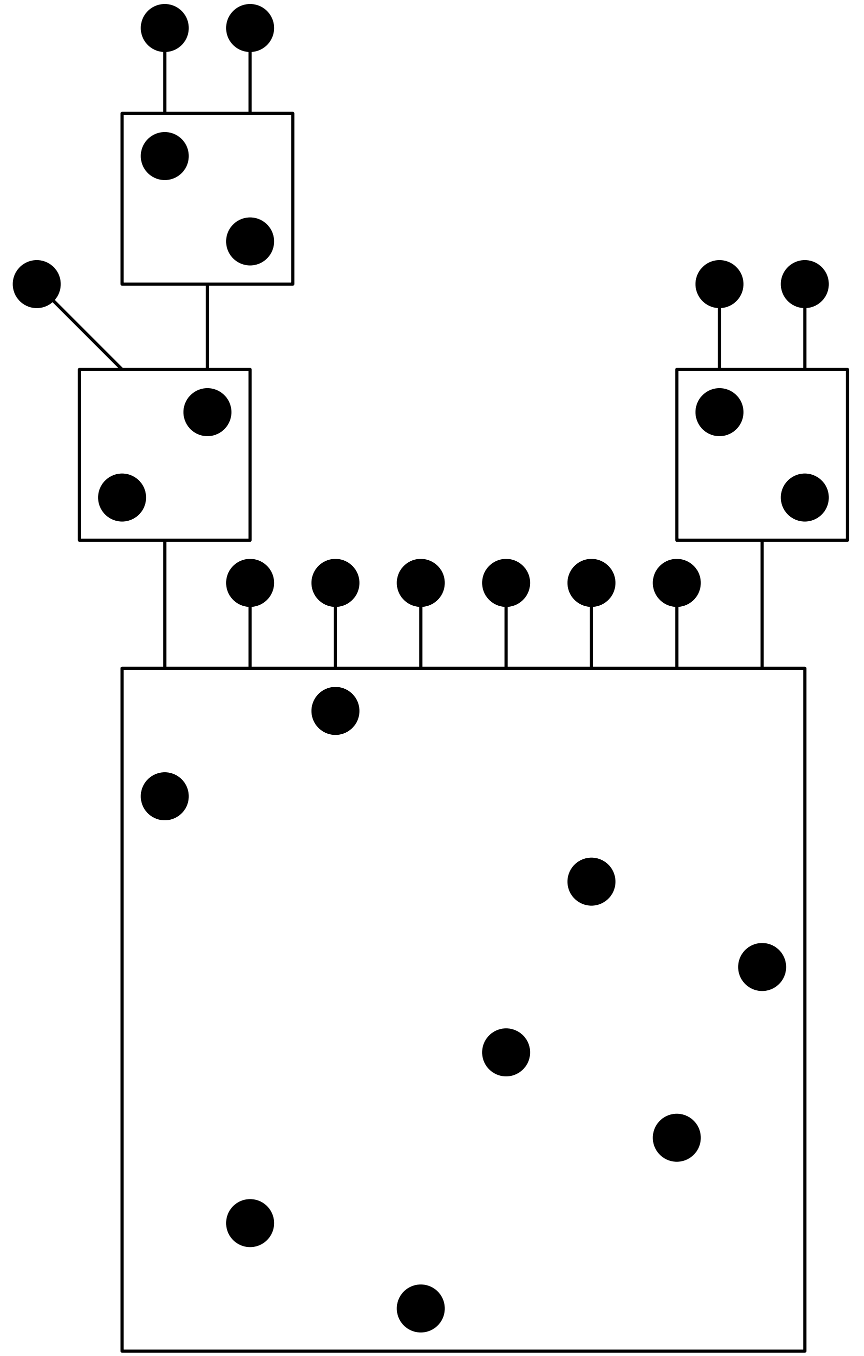

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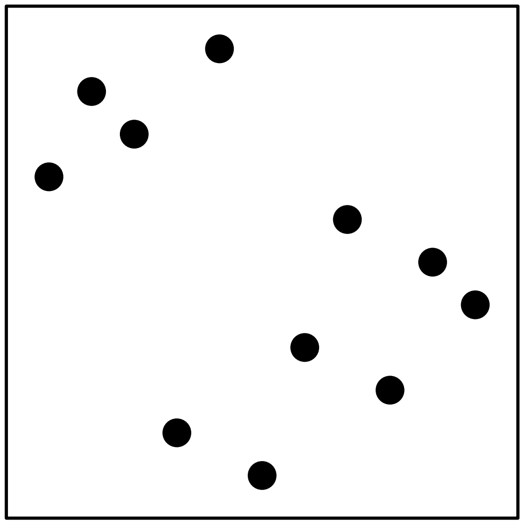

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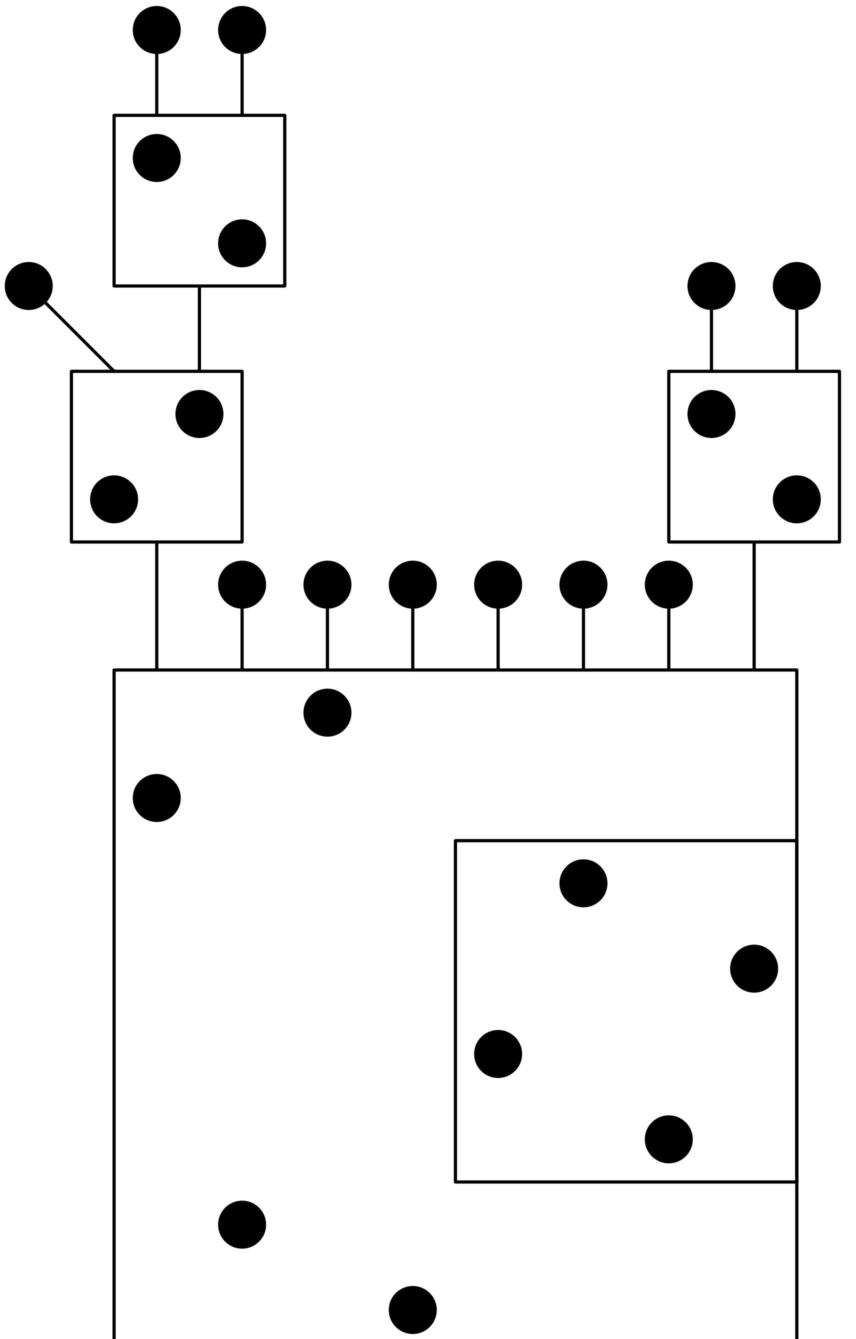

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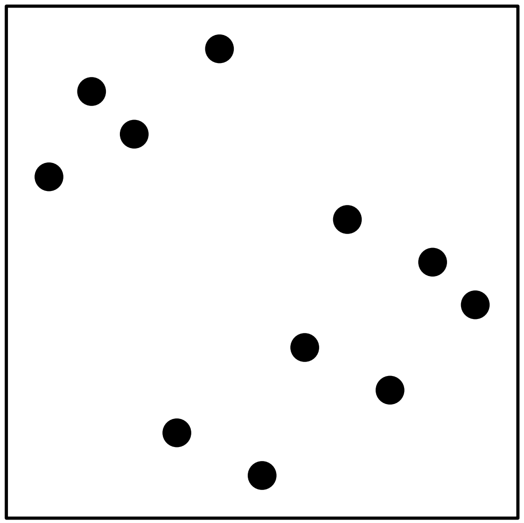

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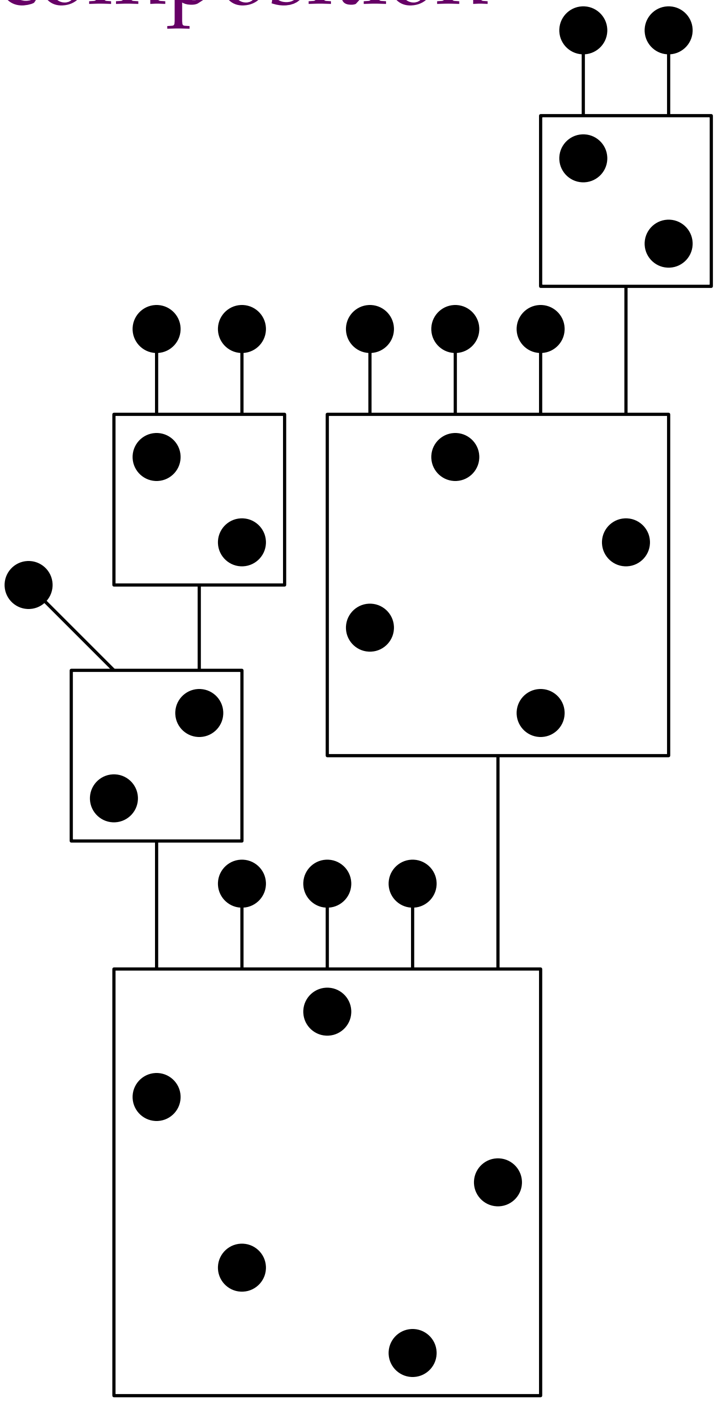

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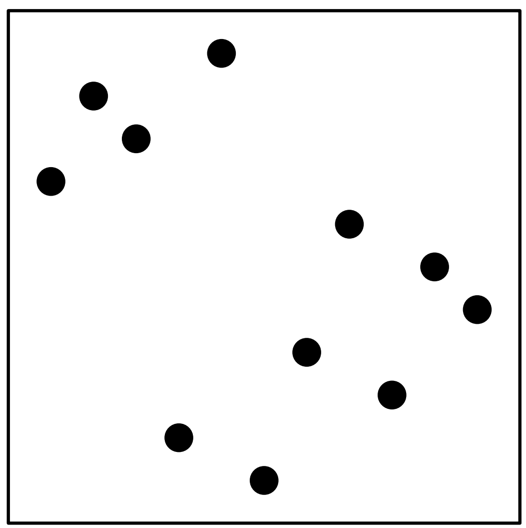

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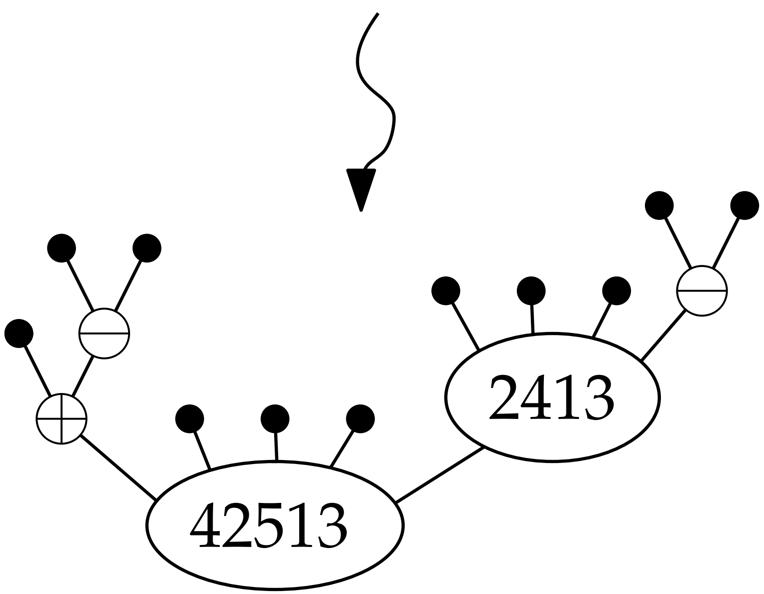

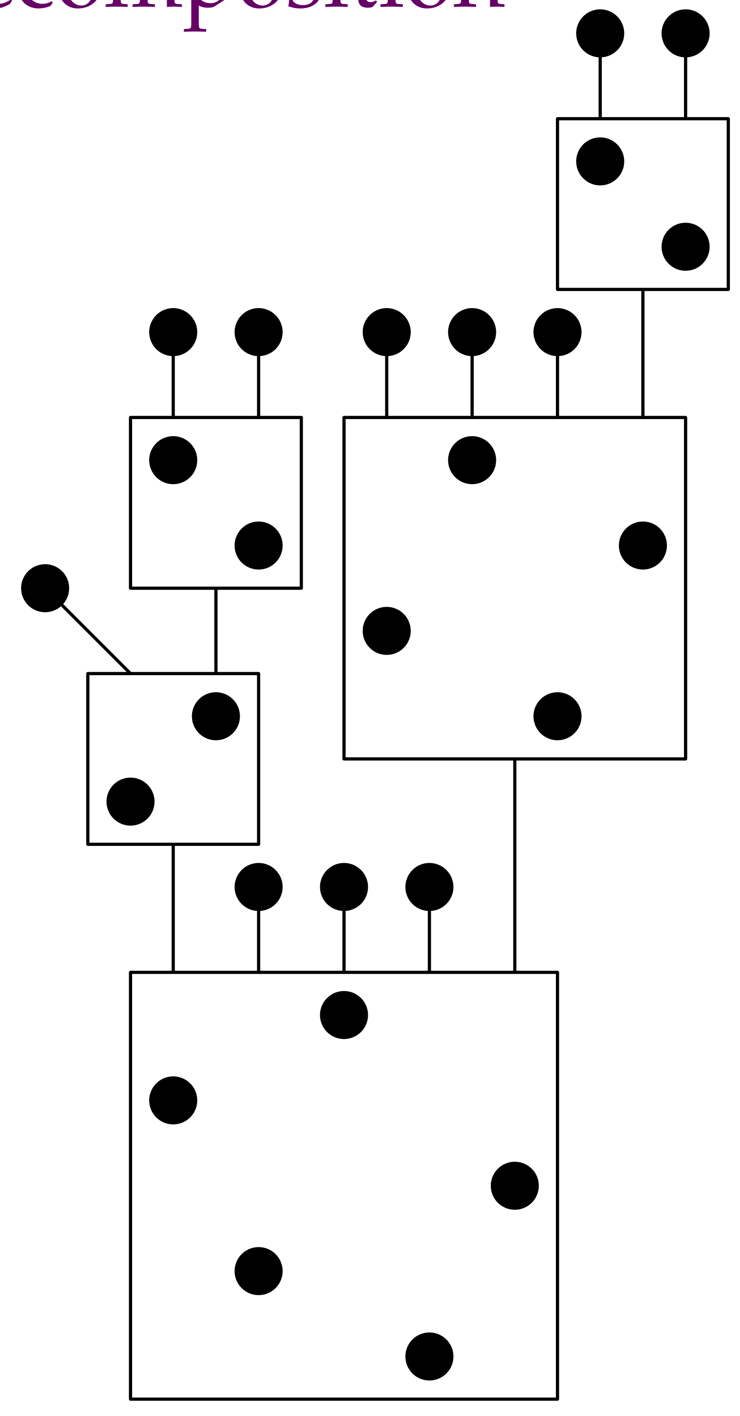

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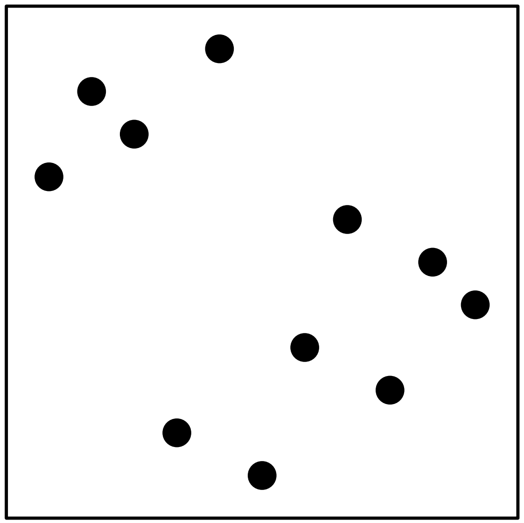

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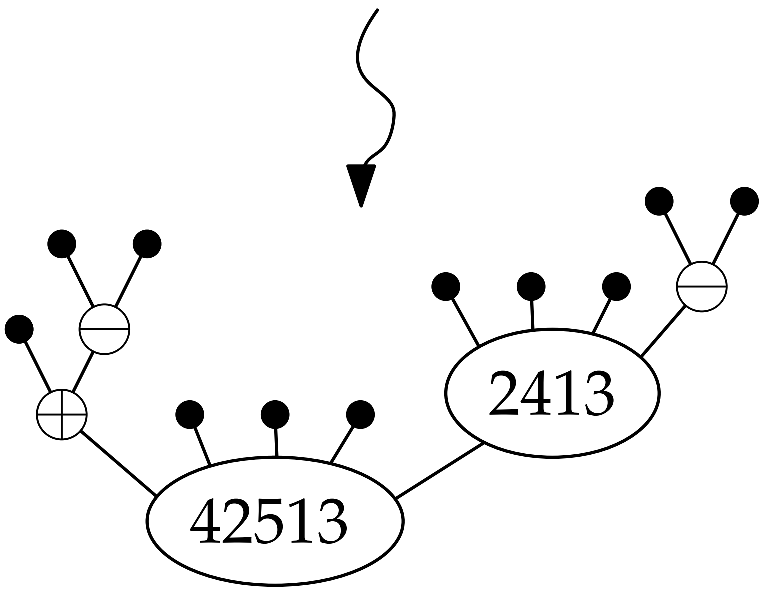

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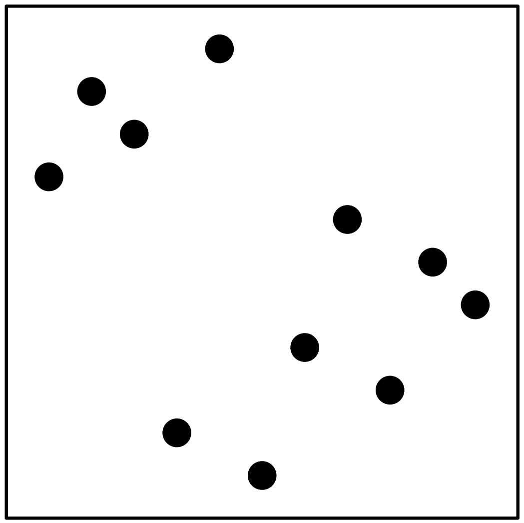

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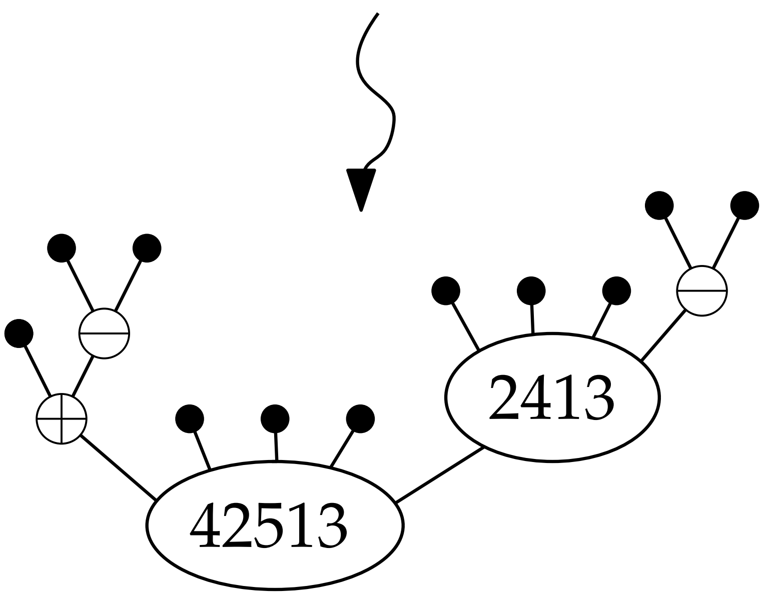

Theorem (Albert, Atkinson 2005): Any permutation can be decomposed into a substitution tree with nodes labeled by simple permutations, unique as long as no $\oplus$ is the left child of a $\oplus$ (same for $\ominus$)

Study classes using substitution

$\mathcal{S} \subset \{$simple permutations $\}$.
$\tilde{\mathcal{S}} = \{$permutations built by substituting simples of $\mathcal{S}\}$.

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This is the case of the separable permutations
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$$\mathcal{T}^{\text{not}\oplus} = \{\bullet\} \uplus \ominus[\mathcal{T}^{\text{not}\ominus}, \mathcal{T}] \uplus \left(\biguplus_{\pi \in \mathcal{S}_{\mathcal{T}}, |\pi| \geq 4} \pi[\mathcal{T}, \dots, \mathcal{T}] \right)$$

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$\rightarrow$ trees coding specification-closed classes are 3-type Galton-Watson trees conditioned on their number of leaves.

In BBFGMP 2017 we treat substitution-closed classes in wider generality

Specifications

Theorem (Bassino, Bouvel, Pivoteau, Pierrot, Rossin 2017)
If $\mathcal{S}_{\mathcal{T}}$ is finite, then there is a finite specification
$\mathcal{T}_i \;=\; \varepsilon_i\{\bullet\} \;\uplus\; \biguplus_{\pi \in \mathcal{S}_{\mathcal{T}}} \; \biguplus_{(k_1,\dots,k_{|\pi|}) \in K_\pi^i} \pi[\mathcal{T}_{k_1}, \dots, \mathcal{T}_{k_{|\pi|}}]$
where $\mathcal{T} = \mathcal{T}_0 \supset \mathcal{T}_1, \dots \mathcal{T}_d$ and $\varepsilon_i \in \{0, 1\}$.
Moreover, there is an algorithm (implemented!) to find it.

$\rightarrow$ system of equations on the generating functions of the specified families, made of analytic functions with nonnegative coefficients.
$\rightarrow$ a Boltzmann sampler for the class.

The case of Av(132)

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\begin{aligned}
\mathcal{T} &= \{\bullet\} \quad \biguplus \quad \oplus[\mathcal{T}^{\text{not}\oplus}, \mathcal{T}_{\langle 21\rangle}] \quad \biguplus \quad \ominus[\mathcal{T}^{\text{not}\ominus}, \mathcal{T}] \\
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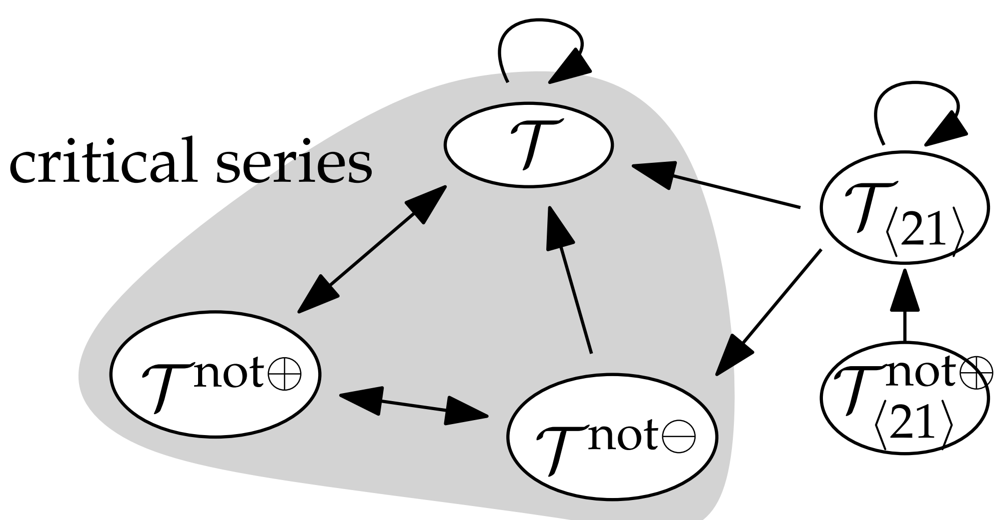
\end{aligned}$$

We plot the dependency graph of the system. In gray, critical families, of maximal growth rate (minimal radius of convergence)



The main theorem

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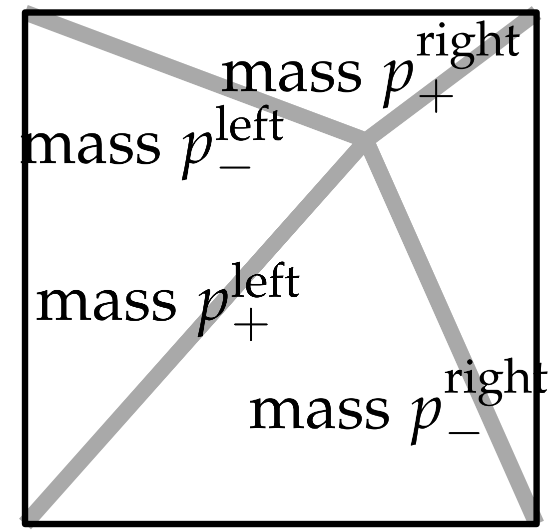

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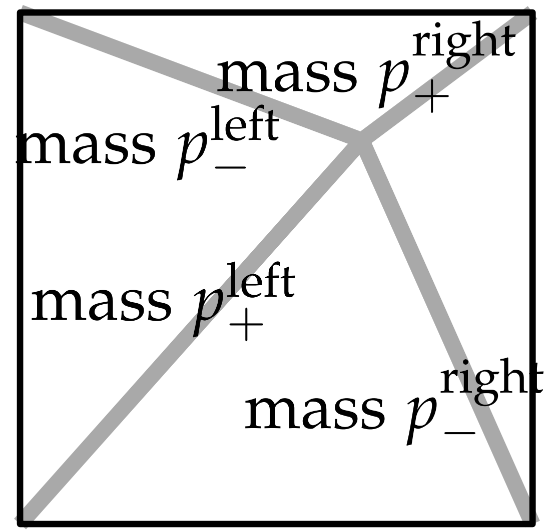
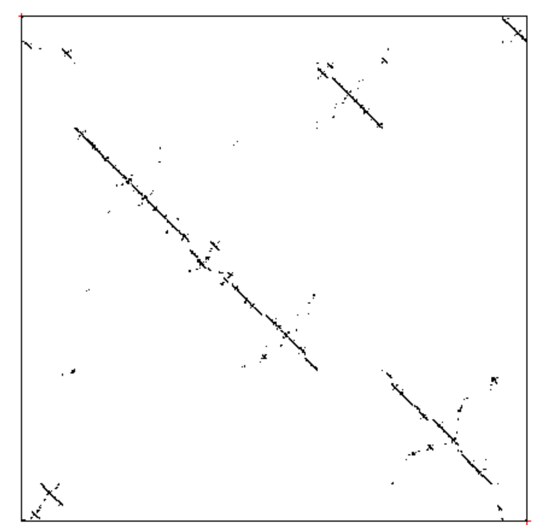


Otherwise, σ_n converges to a biased Brownian permuton of explicit parameter.



Examples: linear case

The V-shape class from earlier:

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\mathcal{T}_0 &= \{\bullet\} \uplus \oplus[\mathcal{T}_1,\mathcal{T}_2] \uplus \oplus[\mathcal{T}_1,\mathcal{T}_3] \uplus \oplus[\mathcal{T}_4,\mathcal{T}_2] \uplus \ominus[\mathcal{T}_5,\mathcal{T}_0] \uplus 3142[\mathcal{T}_1,\mathcal{T}_1,\mathcal{T}_1,\mathcal{T}_6]\\
\mathcal{T}_1 &= \{\bullet\} \uplus \ominus[\mathcal{T}_7,\mathcal{T}_1]\\
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\mathcal{T}_7 &= \{\bullet\}\\
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\mathcal{T}_9 &= \{\bullet\} \uplus \oplus[\mathcal{T}_1,\mathcal{T}_7]\\
\mathcal{T}_{10} &= \oplus[\mathcal{T}_1,\mathcal{T}_1] \uplus 3142[\mathcal{T}_1,\mathcal{T}_1,\mathcal{T}_1,\mathcal{T}_1]\\
\mathcal{T}_{11} &= \oplus[\mathcal{T}_1,\mathcal{T}_2] \uplus \oplus[\mathcal{T}_1,\mathcal{T}_3] \uplus \oplus[\mathcal{T}_4,\mathcal{T}_2] \uplus \ominus[\mathcal{T}_{10},\mathcal{T}_{11}] \uplus \ominus[\mathcal{T}_{10},\mathcal{T}_1] \uplus \ominus[\mathcal{T}_7,\mathcal{T}_{11}]\\
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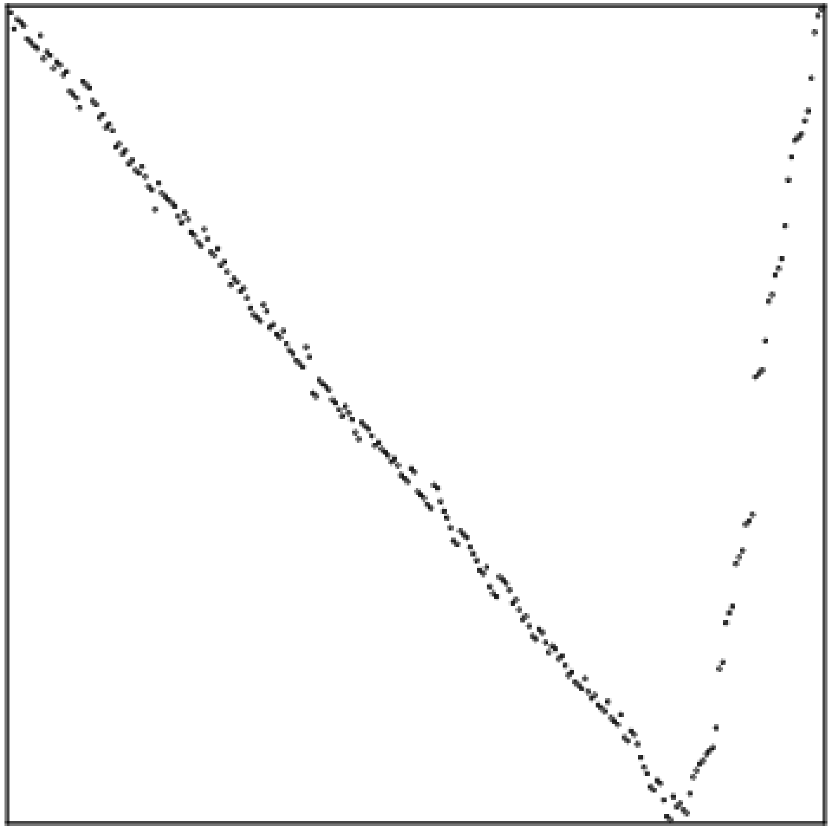
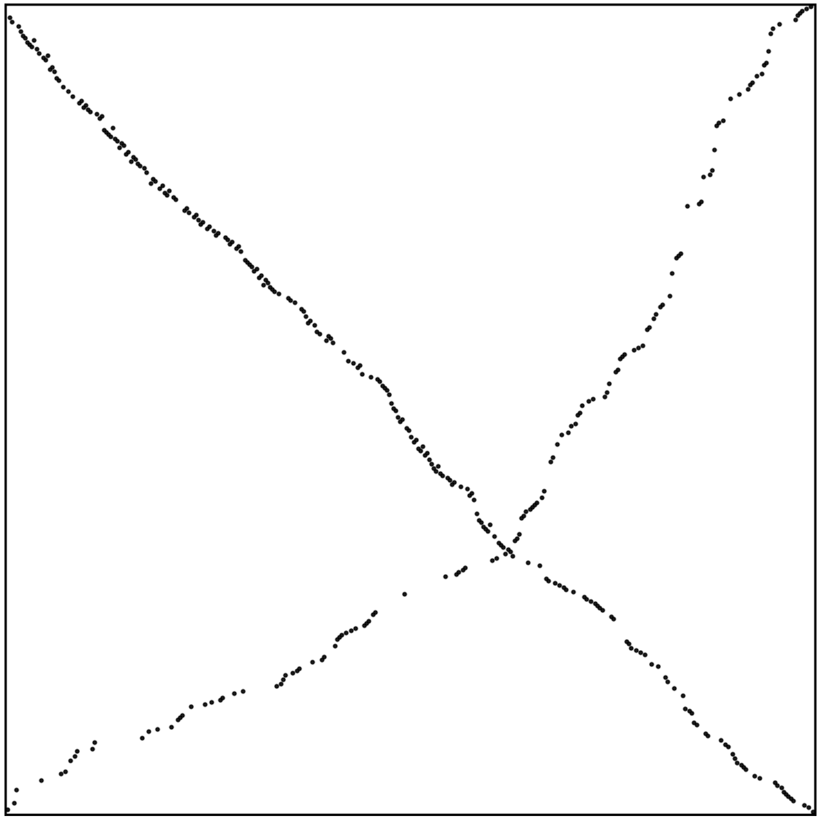
\mathcal{T}_1 &= \{\bullet\} \uplus \ominus[\mathcal{T}_7, \mathcal{T}_1] \\
\mathcal{T}_2 &= \{\bullet\} \uplus \oplus[\mathcal{T}_7, \mathcal{T}_2] \\
\mathcal{T}_3 &= \oplus[\mathcal{T}_8, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_9, \mathcal{T}_6] \\
\mathcal{T}_4 &= \ominus[\mathcal{T}_{10}, \mathcal{T}_{11}] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_1] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_{11}] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\
\mathcal{T}_5 &= \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_1] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1] \\
\mathcal{T}_6 &= \{\bullet\} \uplus \oplus[\mathcal{T}_{12}, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_9, \mathcal{T}_6] \\
\mathcal{T}_7 &= \{\bullet\} \\
\mathcal{T}_8 &= \ominus[\mathcal{T}_9, \mathcal{T}_6] \\
\mathcal{T}_9 &= \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_7] \\
\mathcal{T}_{10} &= \oplus[\mathcal{T}_1, \mathcal{T}_1] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1] \\
\mathcal{T}_{11} &= \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_{11}] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_1] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_{11}] \\
&\quad \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\
\mathcal{T}_{12} &= \{\bullet\} \uplus \ominus[\mathcal{T}_9, \mathcal{T}_6]
\end{aligned}
$$

Critical series are $\mathcal{T}_0, \mathcal{T}_4, \mathcal{T}_{11}$. The critical system is not strongly connected, but aae permutation of $\mathcal{T}_0$ is in $\mathcal{T}_{11}$. Removing $\mathcal{T}_0$ we can apply the theorem.

Examples: linear case

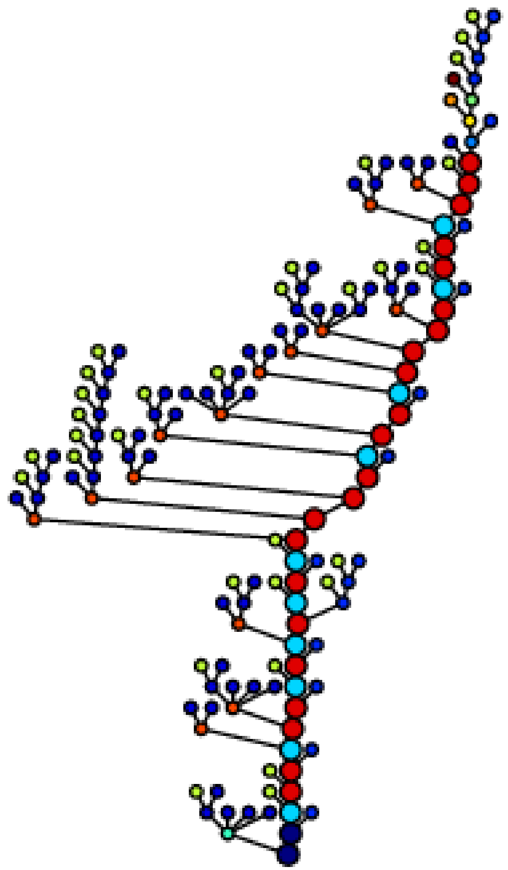


$Av(2413, 1243, 2341, 41352, 531642)$



$Av(2413, 3142, 2143, 34512)$

Examples: linear case



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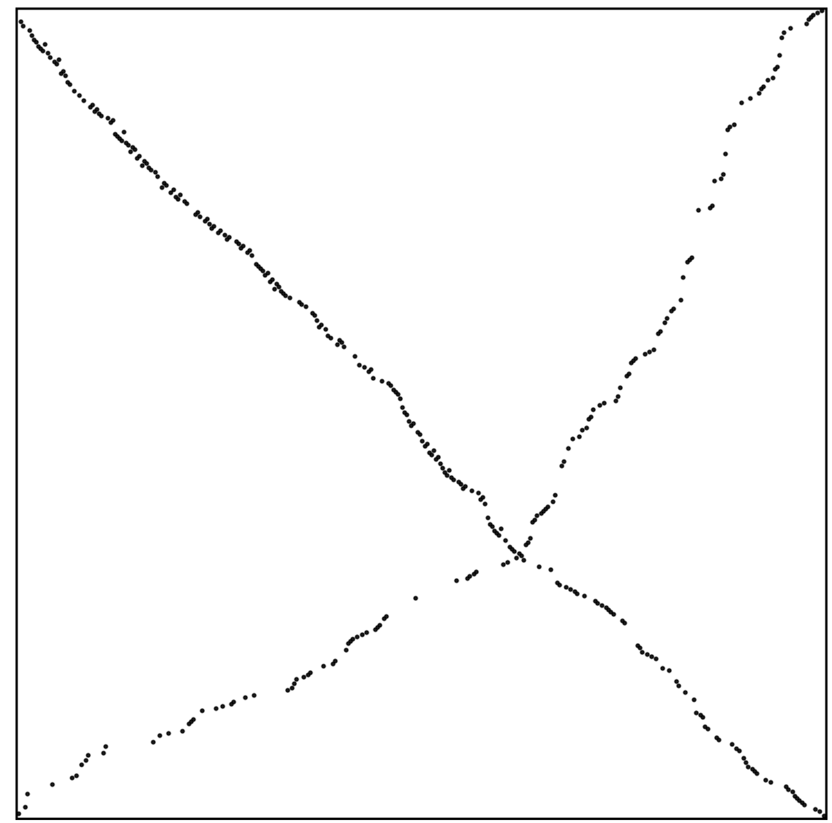

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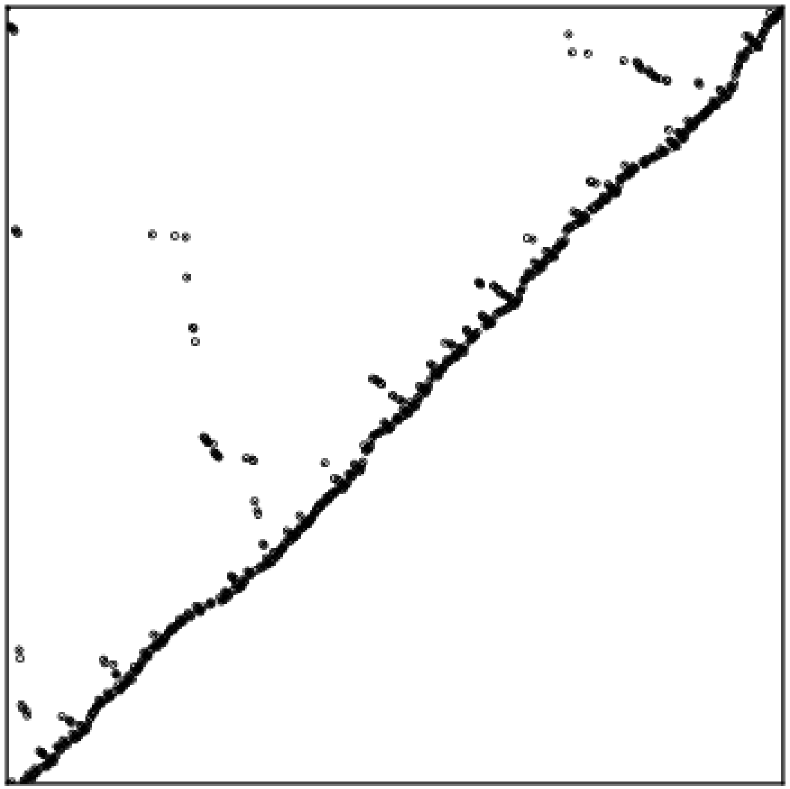
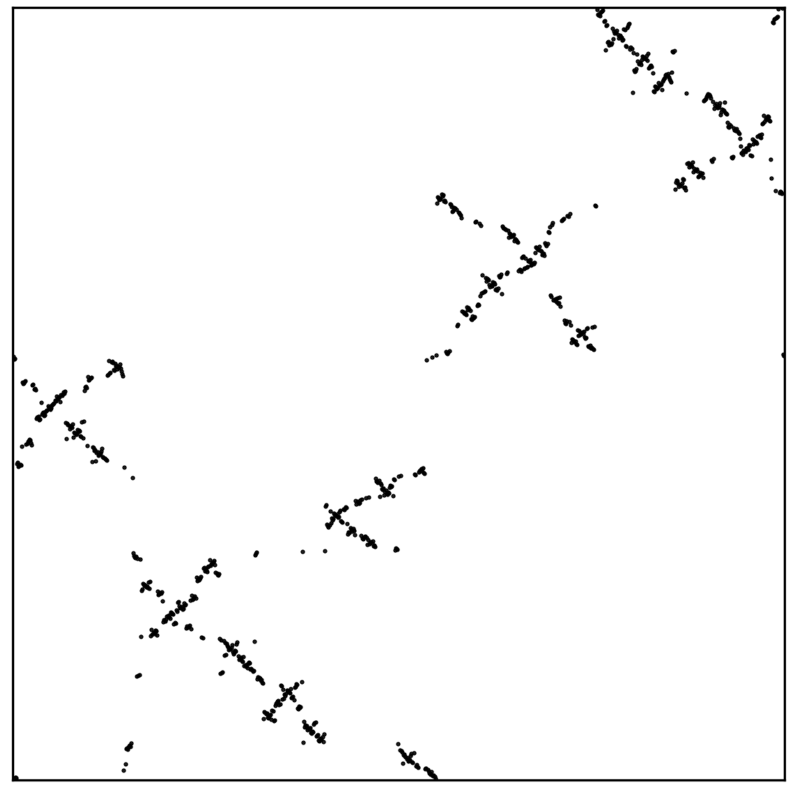
Examples: nonlinear case.



$Av(132)$

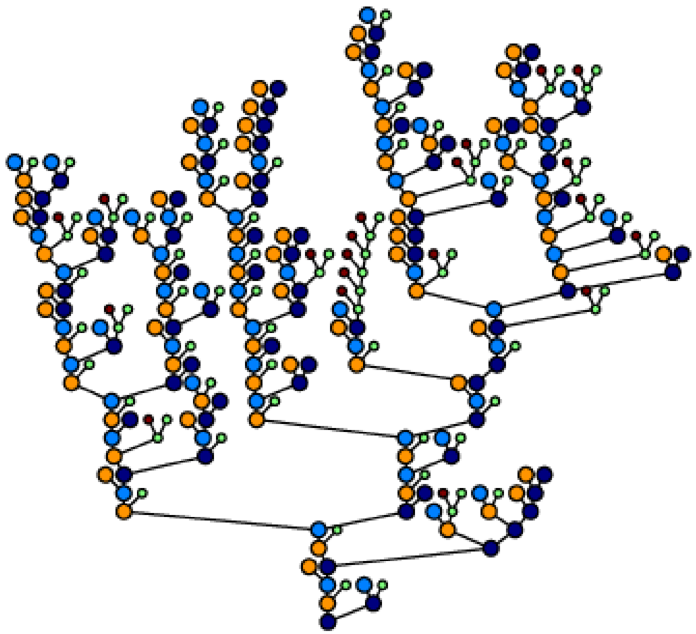
$p = 1$



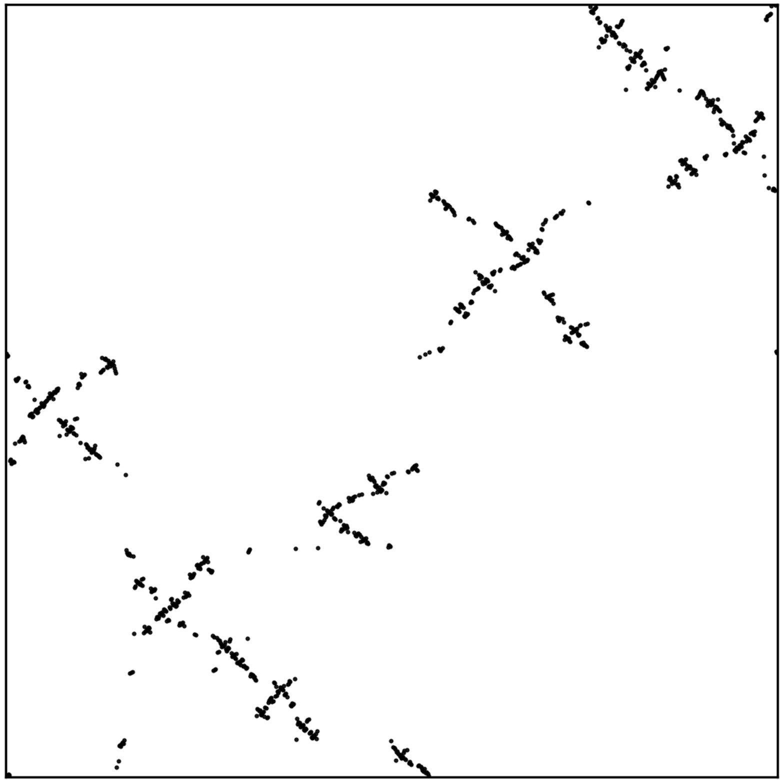
$Av(2413, 31452, 41253, 531642, 41352)$

$p \approx 0.47$ is algebraic of degree 9.

Examples: nonlinear case.



$Av(132)$
$p = 1$

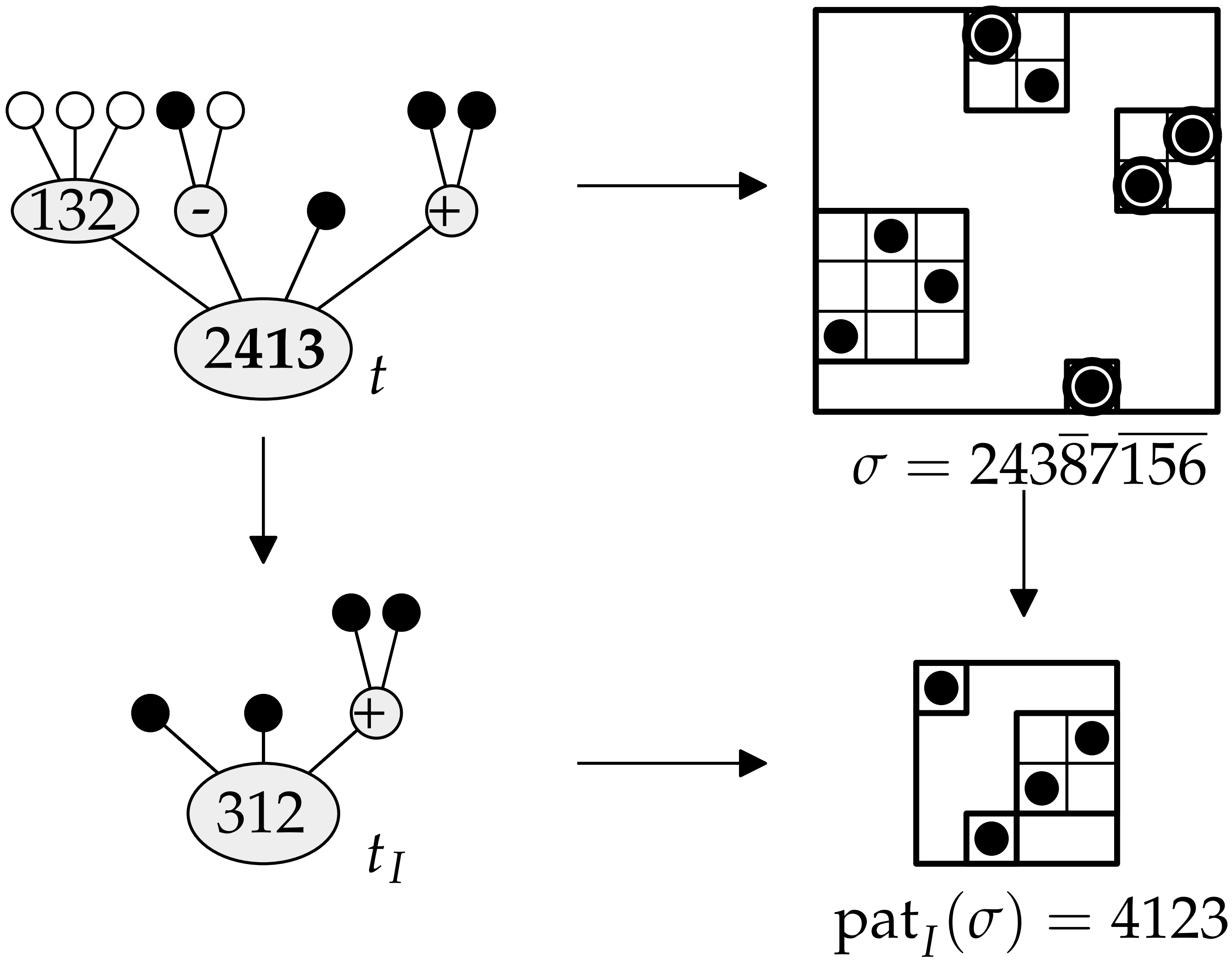


$Av(2413, 31452, 41253, 531642, 41352)$
$p \approx 0.47$ is algebraic of degree 9.

Part 3 - proof of the main theorem

(in the nonlinear case)

Substitution decomposition and patterns



Our goal

Fix a signed binary tree τ with k leaves. We need only show that

$$\frac{\#\{\text{trees in } \mathcal{T} \text{ of size } n \text{ with } k \text{ marked leaves inducing } \tau\}}{\#\{\text{trees in } \mathcal{T} \text{ of size } n \text{ with } k \text{ marked leaves}\}}$$

converges to

$$\mathbb{P}(b_k^p = \tau) = \frac{p^{\#\oplus}(1-p)^{\#\ominus}}{\mathrm{Cat}_{k-1}}.$$

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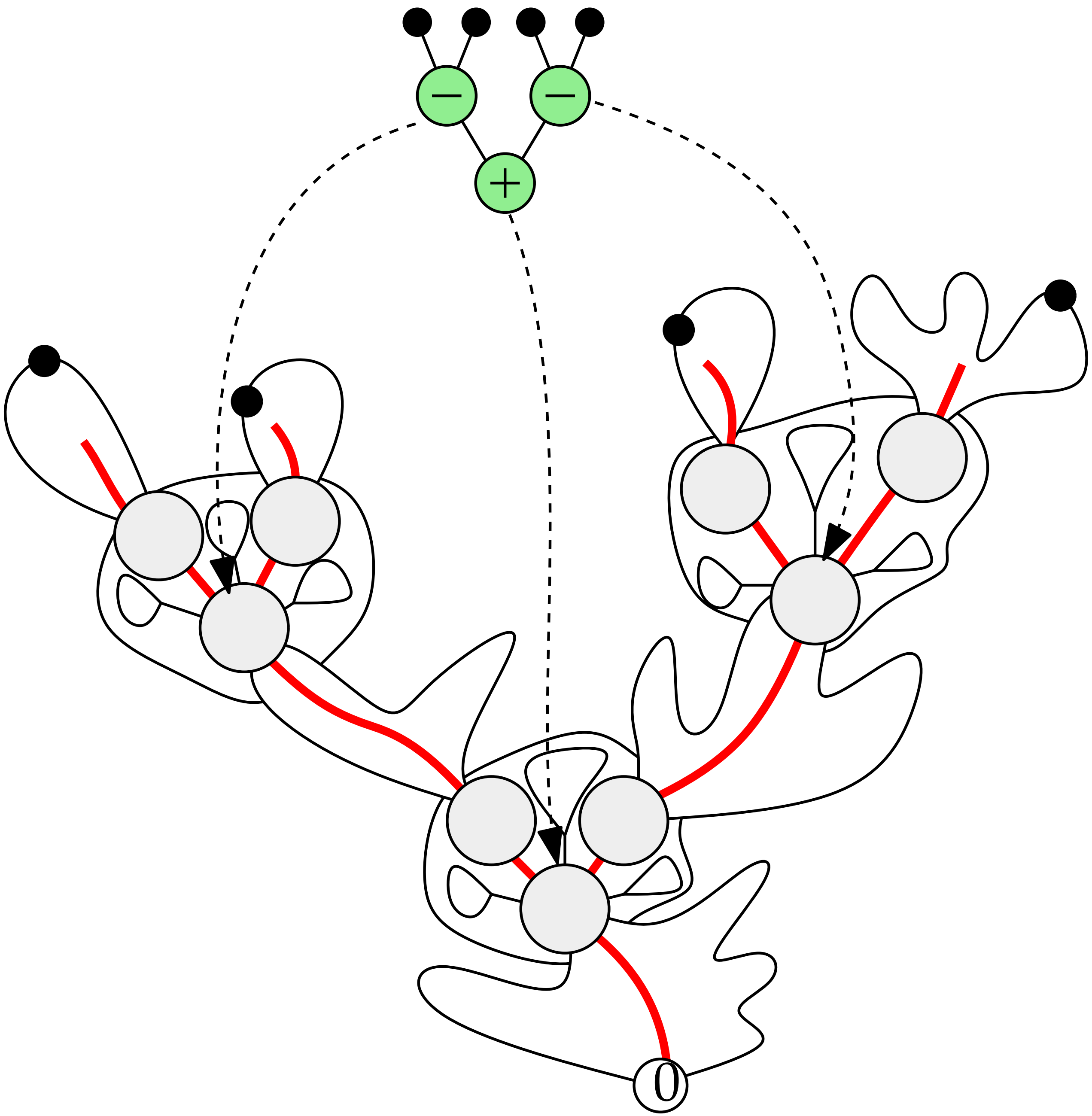
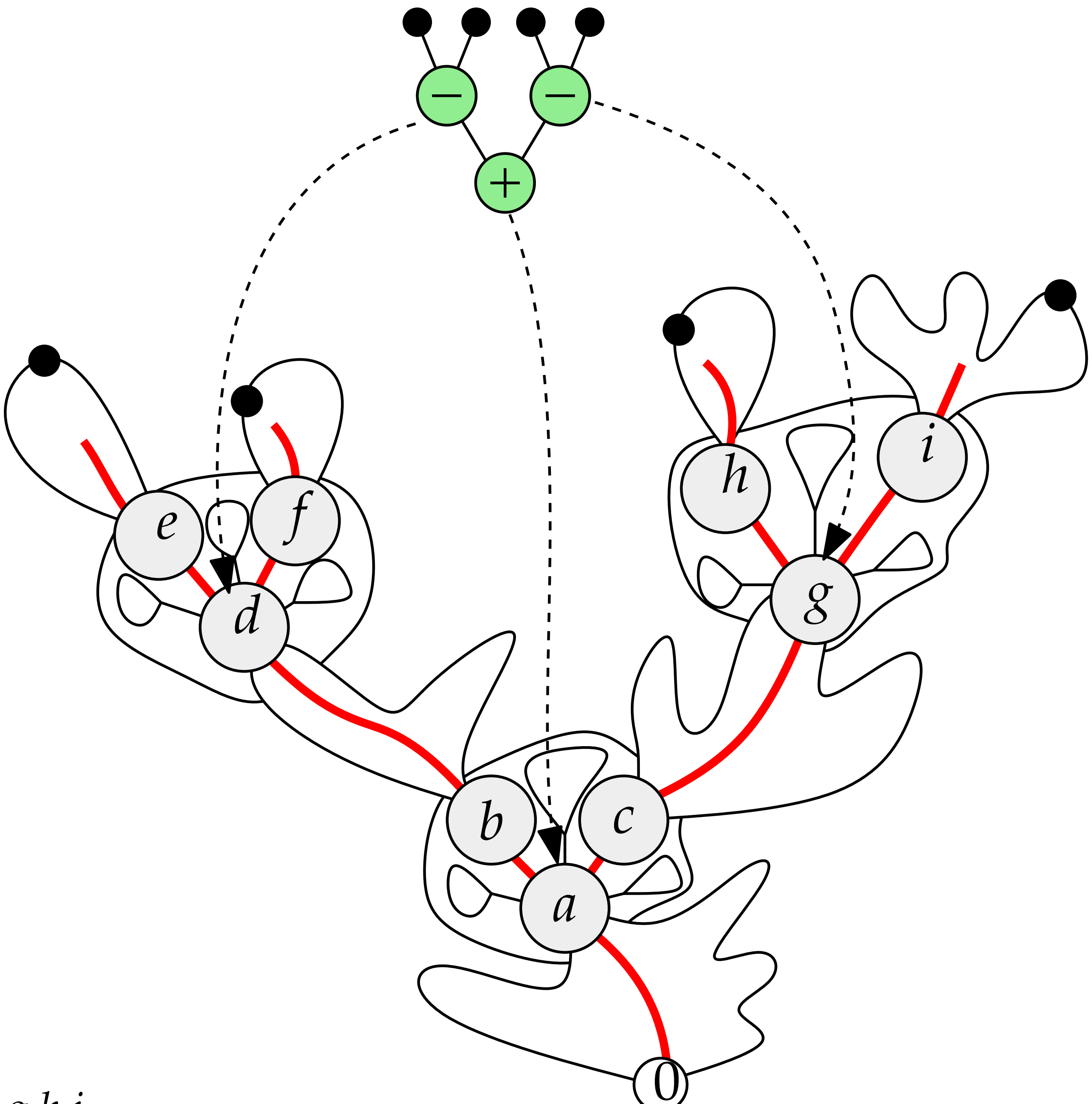
The denominator is $[z^{n-k}]T_0^{(k)}$.

G.F. of the numerator

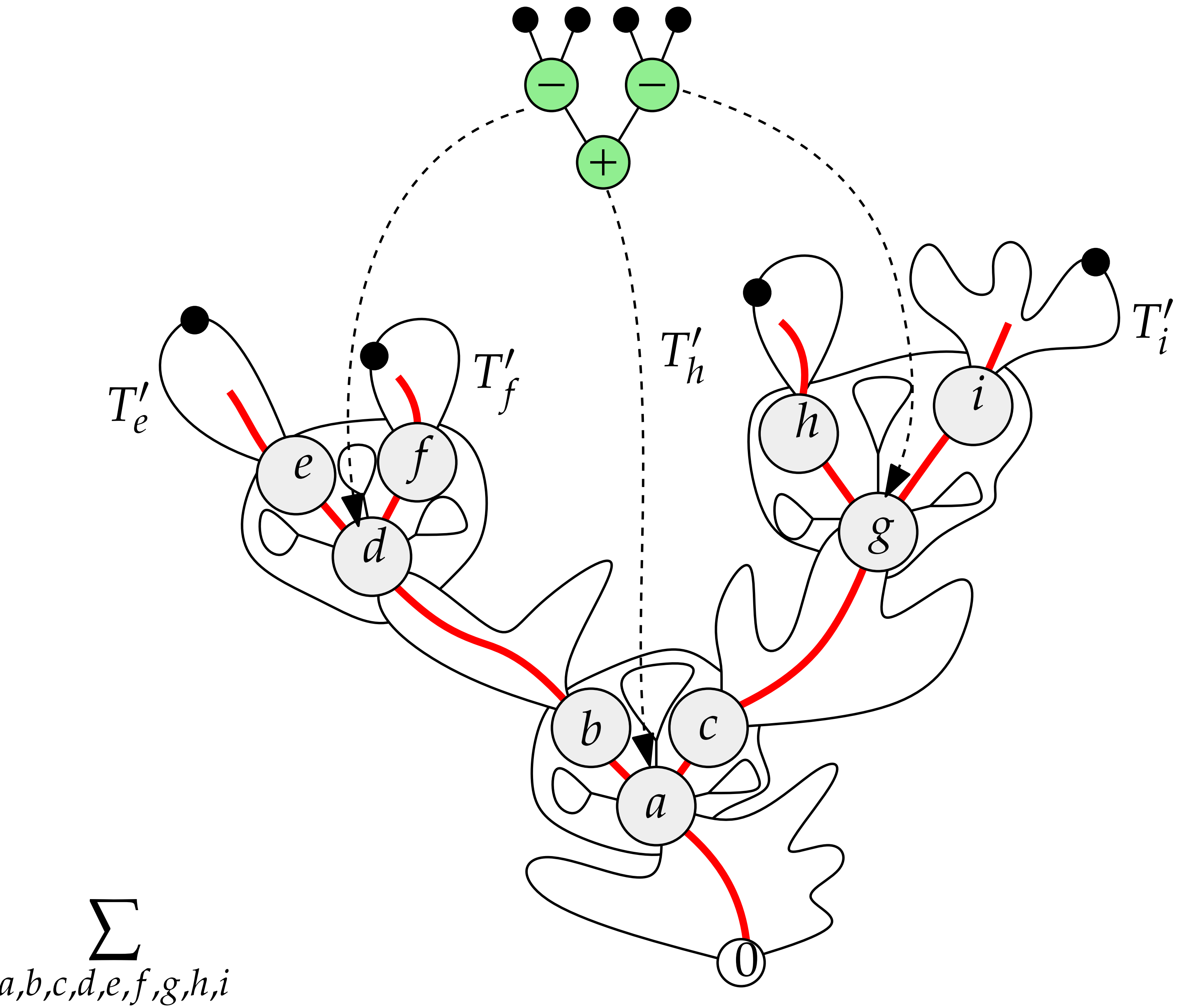


G.F. of the numerator



$$\sum_{a,b,c,d,e,f,g,h,i}$$

G.F. of the numerator



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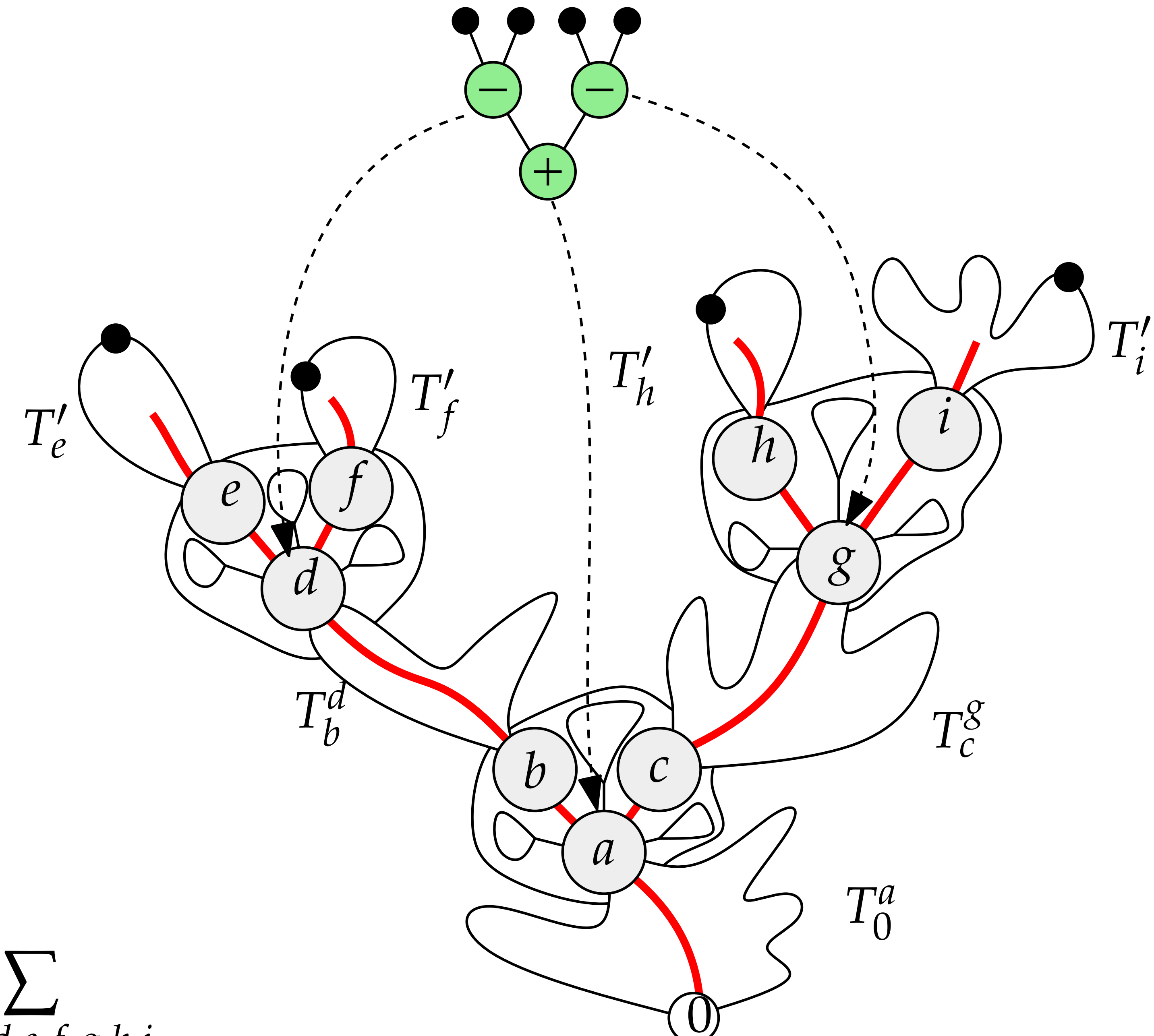
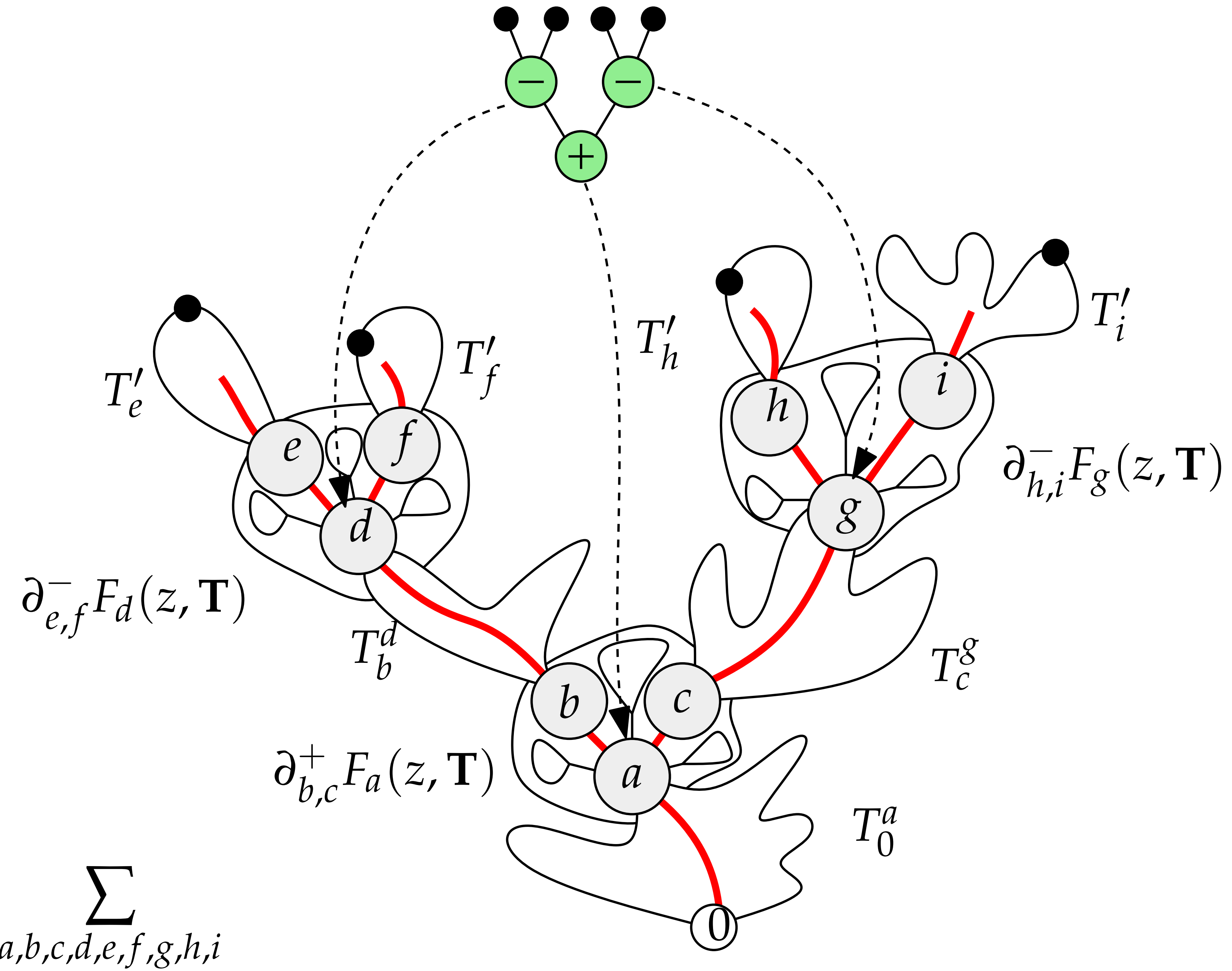


$$\sum_{a,b,c,d,e,f,g,h,i}$$

G.F. of the numerator



DLW Theorem

We can apply the following theorem to our system of equations, partially applied in the subcritical series.

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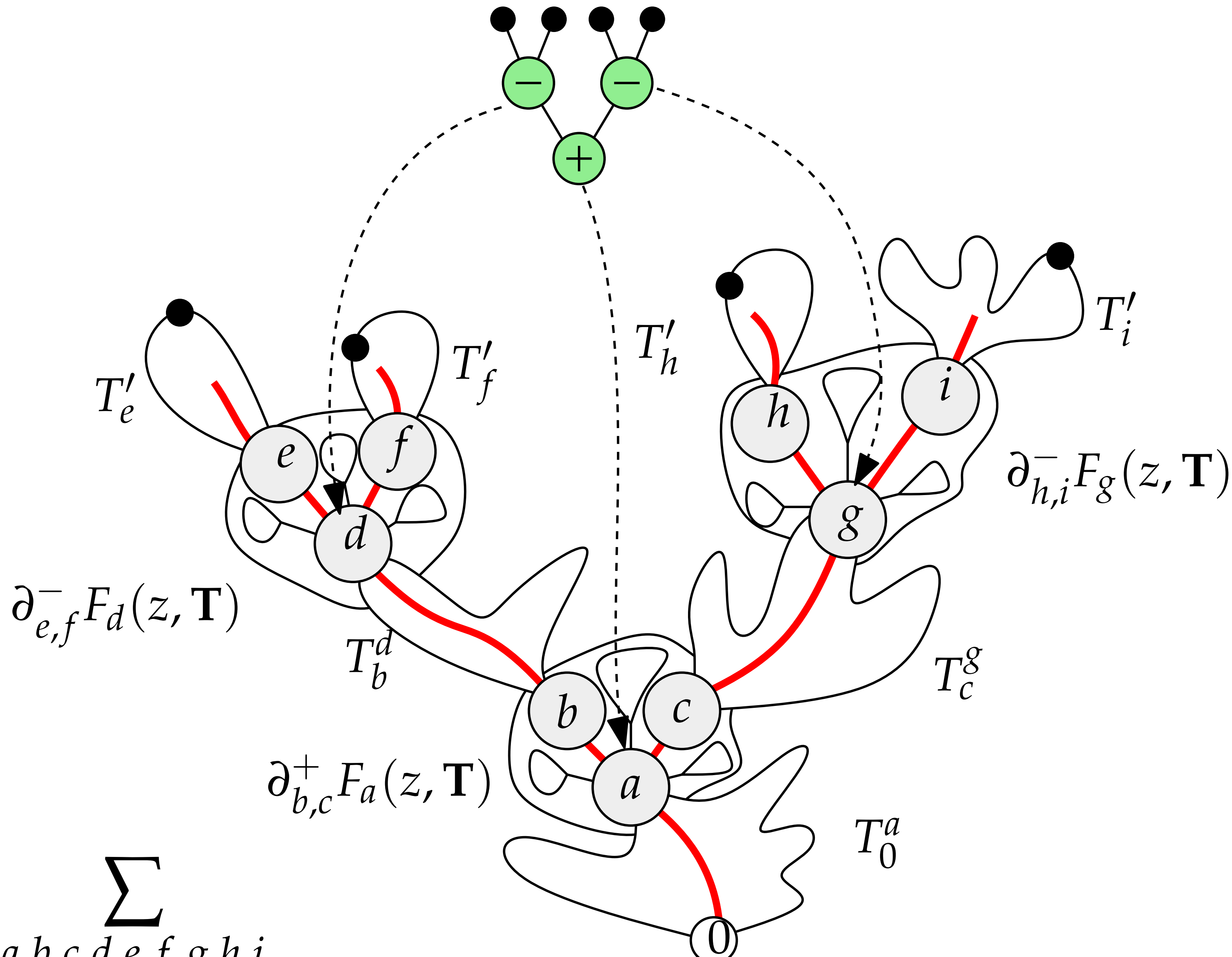
Theorem (Drmota 2009) Let $\mathbf{T} = \mathbf{\Phi}(z, \mathbf{T})$ be a system of equations, $\mathbf{\Phi} = \mathbf{\Phi}(z, \mathbf{t})$ with nonnegative coefficients and no constant term or t_i term. Assume that $\mathbf{\Phi}$ is analytic in z with radius $> \rho$, polynomial and nonlinear in $\mathbf{T}$. Assume the graph of dependence is strongly connected. Then

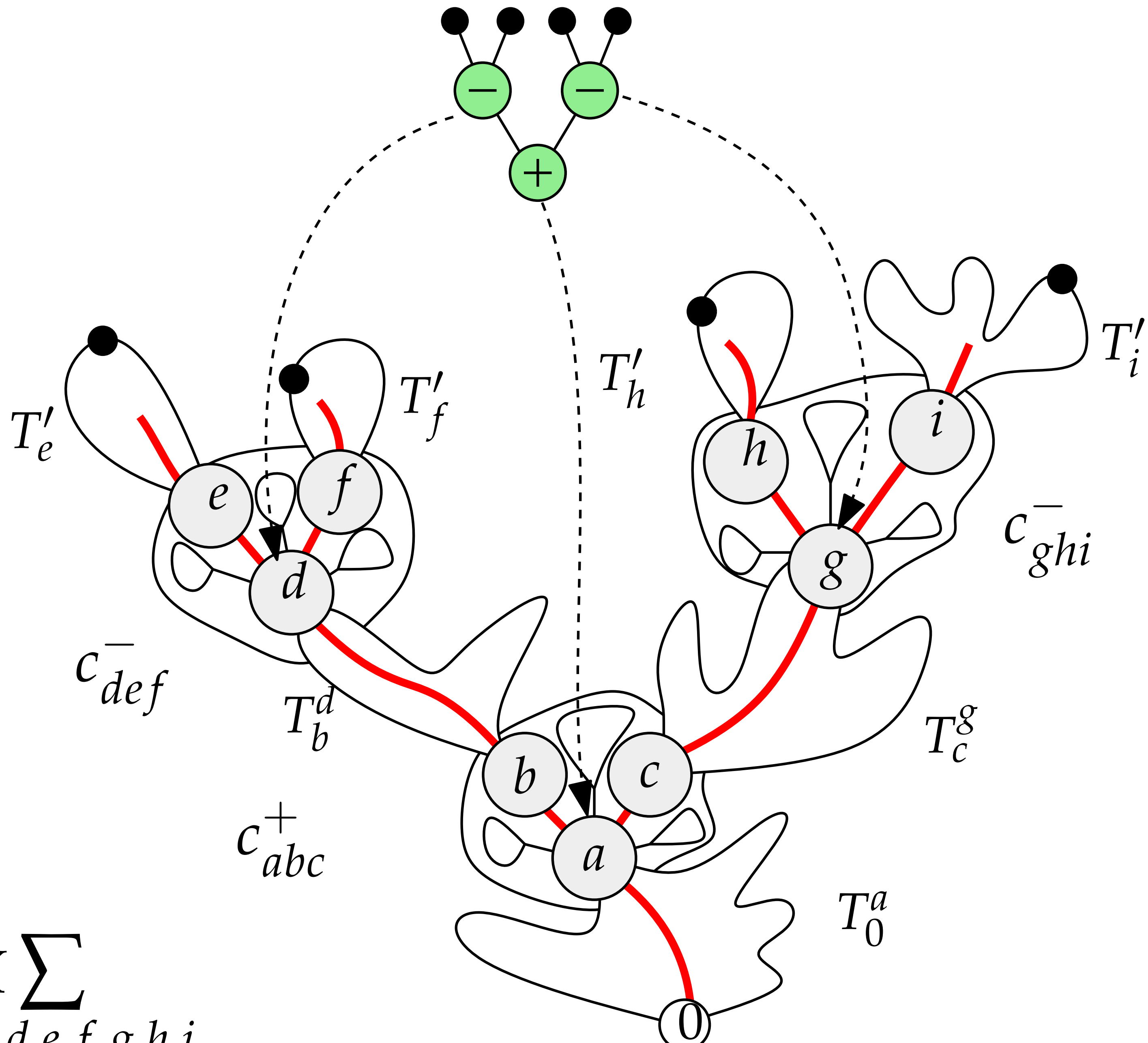
1. All T_i have a square root singularity at ρ
   $\mathbf{T}(z) = \mathbf{T}(\rho) - c(\mathbf{v} + o(1))\sqrt{z - \rho}$.
2. Defining $(\mathbf{M}_{i,j}(z))_{i,j} = \mathrm{Jac}_{\mathbf{T}}\Phi(z, \mathbf{T}(z))$, then $\mathbf{M}(\rho)$ has Perron eigenvalue 1 with left and right eigenvectors $\mathbf{u}$ and $\mathbf{v}$. Moreover
   $(T_i^j)_{i,j} = (\mathrm{Id} - \mathbf{M}(z))^{-1} \sim_{z \to \rho} C\mathbf{v}\mathbf{u}^T \frac{1}{\sqrt{z-\rho}}$.

Asymptotics of numerator



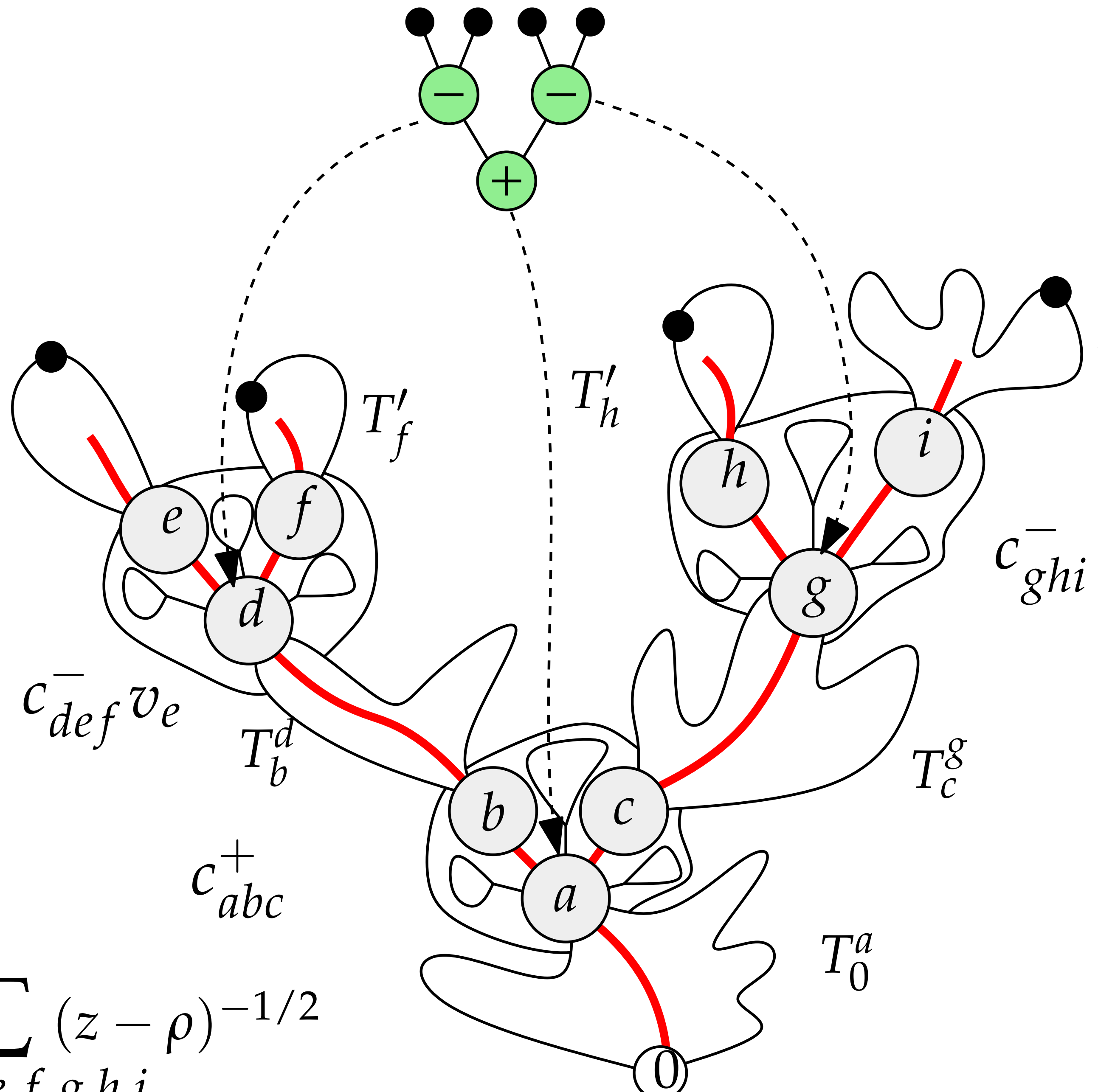
$$\sum_{a,b,c,d,e,f,g,h,i}$$

Asymptotics of numerator



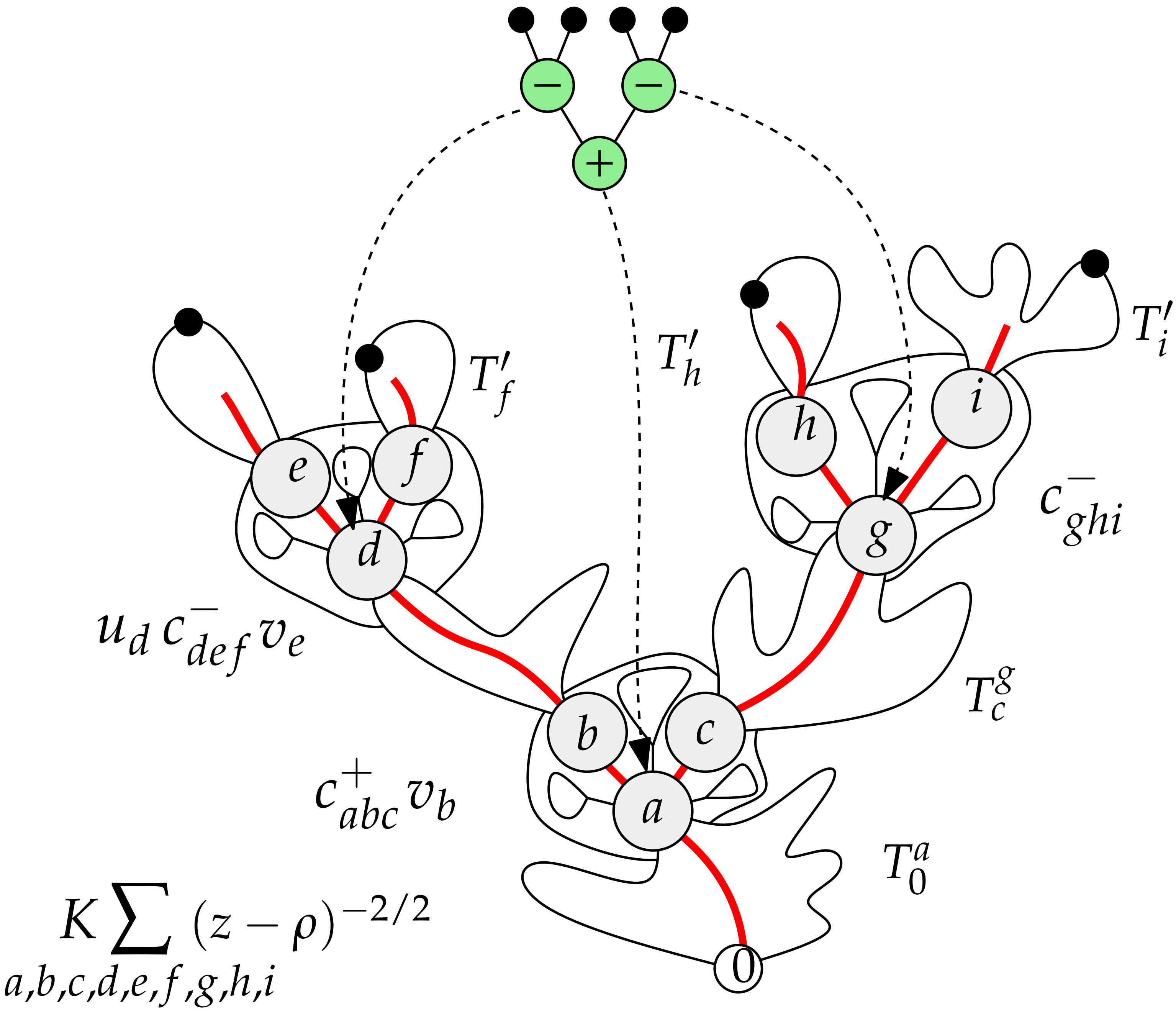
$$K \sum_{a,b,c,d,e,f,g,h,i}$$

Asymptotics of numerator

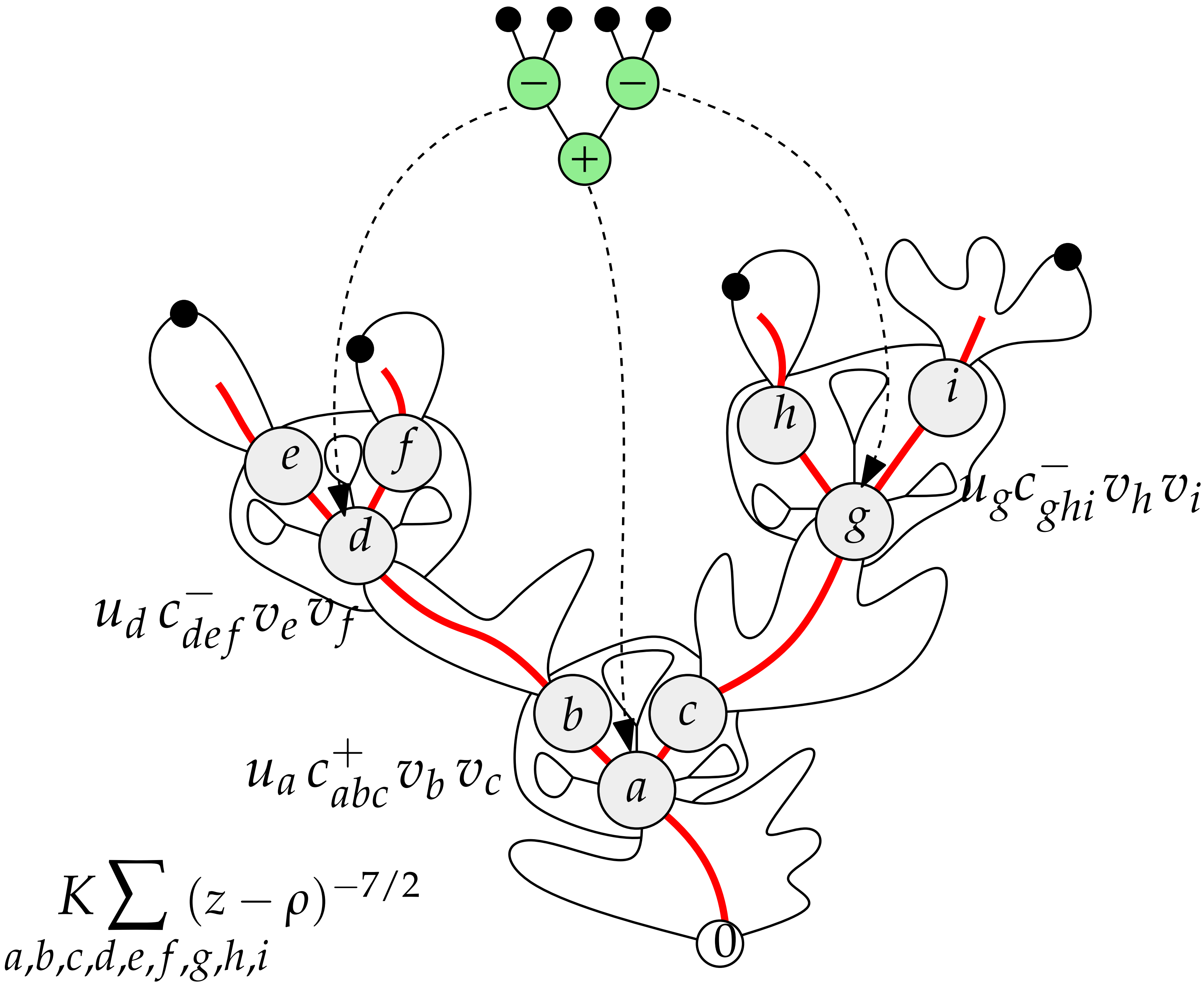


$$K \sum_{a,b,c,d,e,f,g,h,i} (z-\rho)^{-1/2}$$

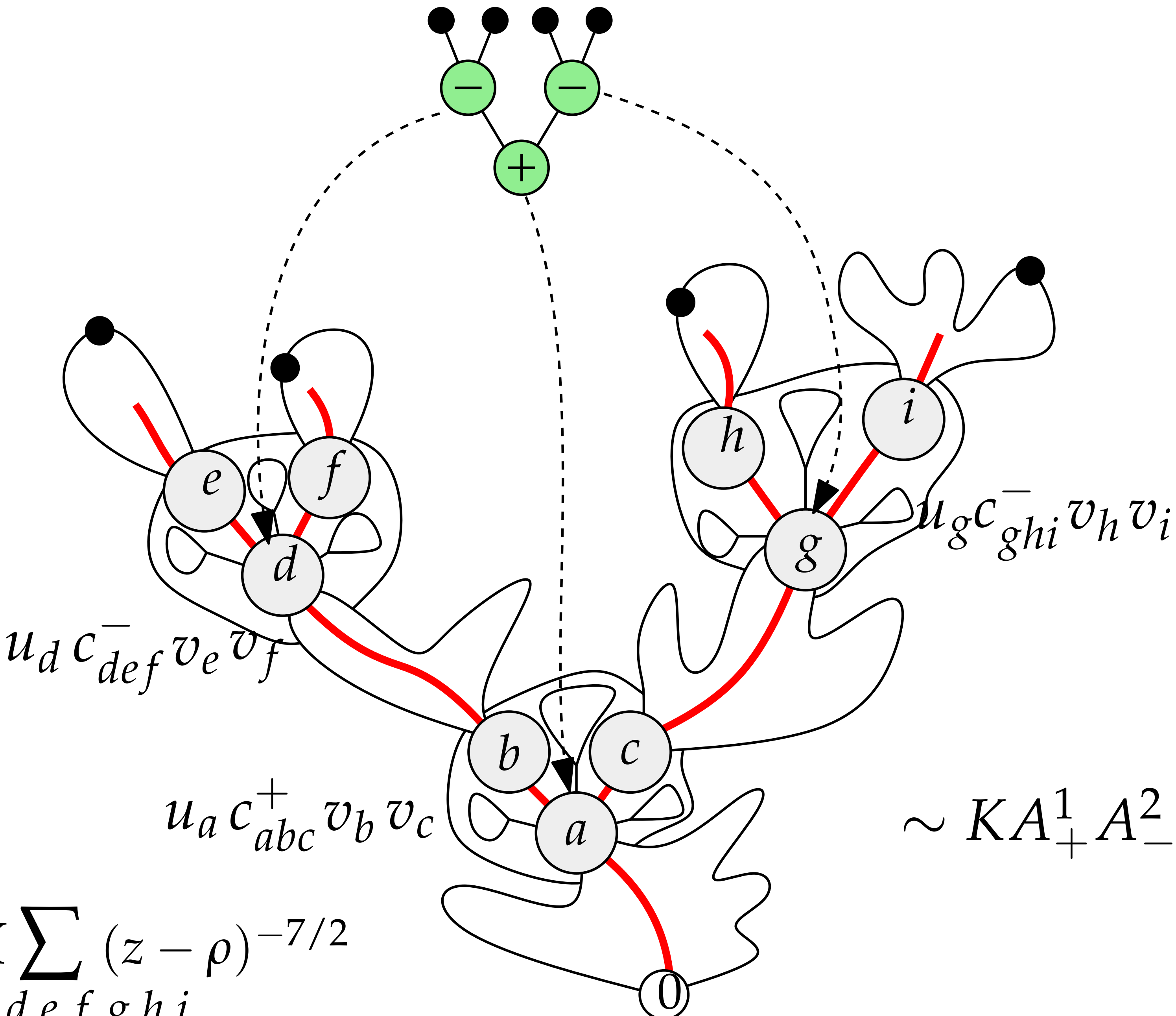
Asymptotics of numerator



Asymptotics of numerator



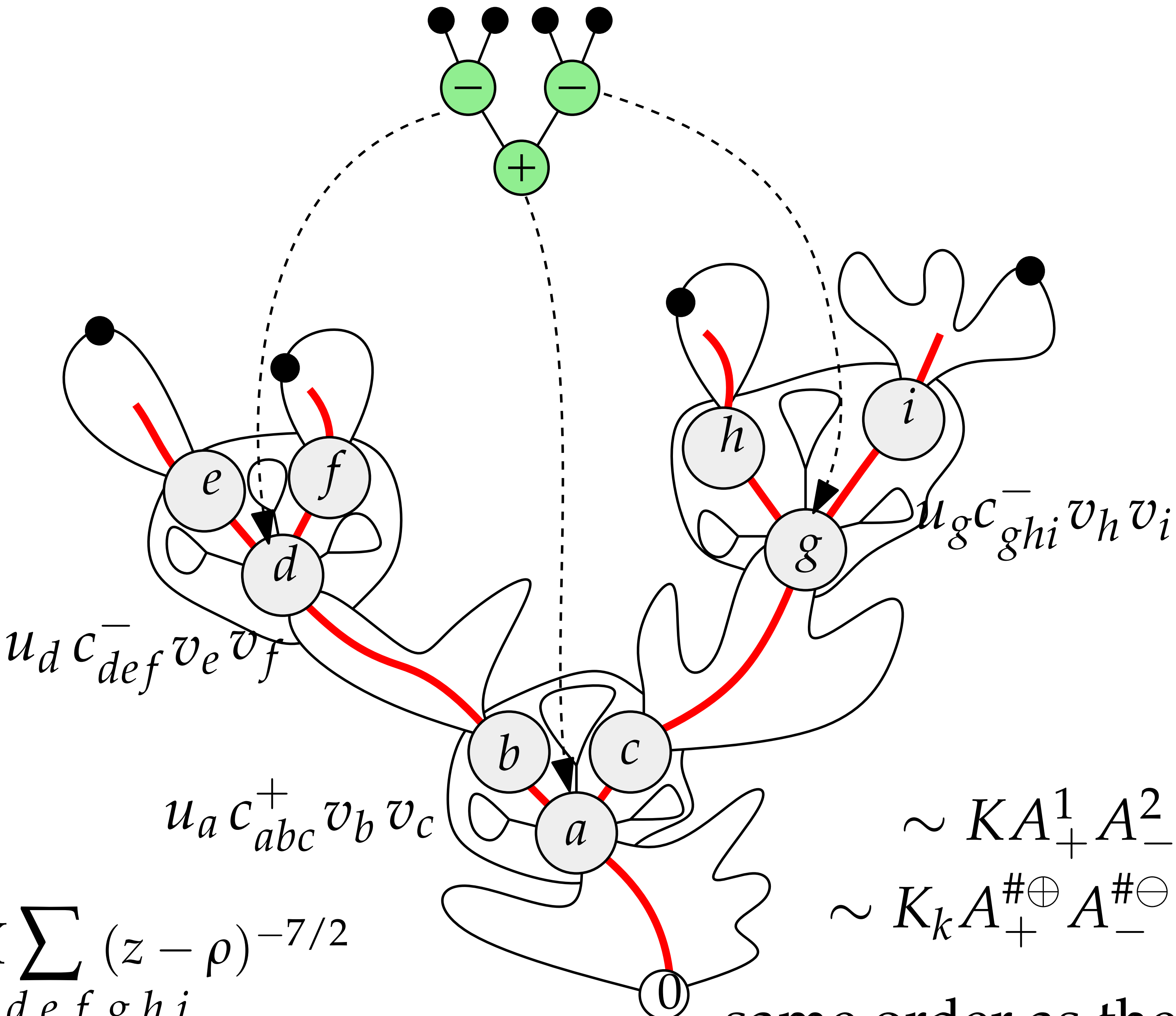
Asymptotics of numerator



$$\sim KA^1_+ A^2_- (z-\rho)^{-7/2}$$

$$K \sum_{a,b,c,d,e,f,g,h,i} (z-\rho)^{-7/2}$$

Asymptotics of numerator



$$K \sum_{a,b,c,d,e,f,g,h,i} (z-\rho)^{-7/2}$$

$$\sim K A^1_+ A^2_- (z-\rho)^{-7/2}$$

$$\sim K_k A^{\#\oplus}_+ A^{\#\ominus}_- (z-\rho)^{-1/2-k}$$

same order as the denominator !

Part 4 - what’s the point ?

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of scaling-limit results for pattern-avoiding permutations ?

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of scaling-limit results for pattern-avoiding permutations ?
On a continuous limiting object, we can compute things, then recover results on the discrete objects !

Some previous work

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- Extremal combinatorics: Presutti-Stromquist (2009) introduced permutons to provide a lower bound for the packing density of (2413) (conjectured tight)

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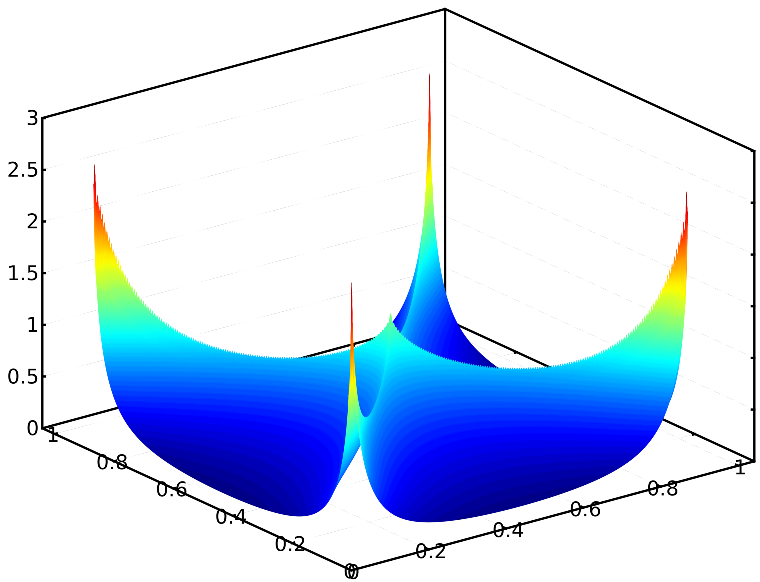
- Extremal combinatorics: Presutti-Stromquist (2009) introduced permutons to provide a lower bound for the packing density of (2413) (conjectured tight)
- Joint convergence of all pattern densities is automatic.
- Asymptotics of the number of cycles of fixed length (Mukherjee '16), of the length of the longest increasing subsequence (Mueller, Starr,'13) and of the total displacement (Bevan, Winkler, '19) in Mallows permutations using the permuton limit + regularity of convergence.

Expectation of the permuton

As μ is a random measure, it is natural to compute its average $\mathbb{E}\mu$, which is the limit of the permuton obtained by stacking all separable permutations of a given size.

Theorem (M. 2017) The permuton $\mathbb{E}\mu$ has density function $\frac{1}{\pi}(\beta(x,y)+\beta(x,1-y)), 0 \leq x \leq \min(y,1-y)$

$$\beta(x,y) = \frac{3xy-2x-2y+1}{(1-x)(1-y)}\sqrt{\frac{1-x-y}{xy}} + 3\arctan\sqrt{\frac{xy}{1-x-y}}.$$



We recover the expected shape of doubly-alternating Baxter permutations. (Dokos-Pak)