Logical Methods in Combinatorics

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Logic and Combinatorics

- 0-1 Laws for various Logics
- Courcelle's Theorem for checking CMSOL properties on graphs of bounded tree-width
- A theorem for checking CMSOL properties on graphs of bounded clique-width
- A theorem on computing generating functions (à la Bürgisser) of CSMOL graph properties.

We want to study the effect of Logic on the existenc of linear (sometimes modular) recurrence relations.

Outline

Lecture 1: Linear recurrences of graph polynomials

- Graph polymials
- Motivating examples
- Monadic Second Order Logic and Graph polymials

Lecture 2: Modular linear recurrences for density functions

- Counting finite topologies
- Density functions
- Liner recurrence relations
- Modular liner recurrence relations

Lecture 1

Linear Recurrences for Graph polynomials

Main theme

We deal with a purely **graph theoretical** problem:

Given a **regularly constructed** indexed family of graphs G_n such as
the paths P_n, the circles C_n, the wheels W_n, the cliques K_n, the grids $Grid_{m,n}$

and a graph polynomial $\mathfrak{P}$, such as
the matching , Tutte, clique, cover polynomial

compute all the values $\mathfrak{P}(G_n)$.

Often we have a (linear) recurrence relation, i.e. there is $q \in \mathbb{N}$,
and polynomials $p_1, \ldots, p_q \in \mathbb{Z}[\bar{X}]$ such that for sufficiently large n

$$\mathfrak{P}(G_{n+q+1}) = \sum_{i=1}^{q} p_i \cdot \mathfrak{P}(G_{n+i})$$

When is this the case?

We shall see that methods from LOGIC help clarifying the situation.

Graph polynomials

Let $\mathfrak{G}$ be the class of all finite graphs, and

$\mathbb{Z}[\bar{X}]$ be a polynomial ring over $\mathbb{Z}$ with $\bar{X} = (X_1, \ldots, X_m)$.

A graph polynomial is a map

$$\mathfrak{P} : \mathfrak{G} \to \mathbb{Z}[\bar{X}]$$

which is invariant under graph isomorphisms.

There are obvious generalizations to

- vertex-labeled and edge-labeled (signed) graphs;
- hypergraphs; and to
- relational structures.
- Knot polynomials are defined on signed graphs, the shading graphs of knot diagrams, where invariance is additionally required under the Reidemeister moves.

Examples of graph polynomials

Well studied graph polynomials are:

- The chromatic polynomial;
  (W.T. Tutte, 1954)
- The Tutte polynomial and its colored versions
  (W.T. Tutte 1954, B. Bollobas and O. Riordan, 1999);
- The various matching polynomials;
  (O.J. Heilman and E.J. Lieb, 1972)
- Various clique and independent set polynomials
  (I. Gutman and F. Harary 1983)
- The Farrel polynomials,
  (E.J. Farrell, 1979)
- The various cover polynomials for digraphs
  (F.R.K. Chung and R.L. Graham, 1995)

The matching polynomial, I

For a graph G, the **matching polynomial** $\mu(G, X) \in \mathbb{Z}[X]$ is defined by

$$\mu(G, X) = \sum m_k(G) \cdot X^k$$

where $m_k(G)$ is the number of k-matchings of G.

We compute $\mu(P_n, X)$:

We use auxiliary polynomials

$$\mu^+(P_n, X) = \sum m_k^+(P_n) \cdot X^k$$

and

$$\mu^-(P_n, X) = \sum m_k^-(P_n) \cdot X^k$$

where $m_k^+(P_n)$ and $m_k^-(P_n)$ is the number of k-matchings of P_n which **includes**, respectively **excludes** the last vertex.

Clearly we have $m_k(P_n) = m_k^+(P_n) + m_k^-(P_n)$ hence

$$\mu(P_n, X) = \mu^+(P_n, X) + \mu^-(P_n, X)$$

The matching polynomial, II

It is easy to see that

$$\mu^-(P_{n+1}) = \mu^-(P_n) + \mu^+(P_n)$$
$$\mu^+(P_{n+1}) = X \cdot \mu^-(P_n)$$

For $\bar{\mu}_n = (\mu^-(P_n), \mu^+(P_n))^t$ we get

$$A\bar{\mu}_n = \bar{\mu}_{n+1}$$

with

$$a_{1,1} = 1, a_{1,2} = 1, a_{2,1} = X, a_{2,2} = 0$$

The characteristic polynomial of A is

$$det(\lambda \mathbf{1} - A) = \lambda^2 - \lambda - X$$

so we get the linear recurrence relation (independent of n)

$$\mu(P_{n+2}) = \mu(P_{n+1}) + X \cdot \mu(P_n)$$

The vertex-cover polynomial

For a graph G, the **vertex-cover polynomial** $vc(G, X) \in \mathbb{Z}[X]$ is defined by

$$vc(G, X) = \sum vc_k(G) \cdot X^k$$

where $vc_k(G)$ is the number of k-vertex-covers of G.

- $vc(P_{n+1}, X) = X \cdot vc(P_n, X) + X \cdot vc(P_{n-1}, X)$
- $vc(C_{n+1}, X) = X \cdot vc(C_n, X) + X^2 \cdot vc(C_{n-2}, X)$
- Let L_n be the graph which consists of n isolated loops.
  $vc(L_{n+1}, X) = X \cdot vc(L_n, X) = X^n$
- For the wheel graph W_n we have
  $vc(W_{n+1}, X) = X \cdot vc(W_n, X) + X^n = X \cdot vc(W_n, X) + X \cdot vc(L_n, X)$
  hence, using the characteristic polynomial of the matrix, $A = (a_{i,j})$ with
  $a_{1,1} = a_{1,2} = a_{2,2} = X$ and $a_{2,1} = 0$
  $vc(W_{n+1}, X) = 2X \cdot vc(W_n, X) - X^2 \cdot vc(W_{n-1}, X)$

F.M. Dong, M.D. Hendy, K.L. Teo and C.H.C. Little, The vertex-cover polynomial of a graph, Discrete Mathematics 250 (2002), 71-78

$\mathfrak{P}$-recursive families of graphs, I

Let $\mathfrak{P}$ be a graph polynomial and $\mathcal{G} = \{G_n : n \in \mathbb{N}\}$ be a family of graphs.

$\mathcal{G}$ is said to be **$\mathfrak{P}$-recursive** if there is $q \in \mathbb{N}$,
and polynomials $p_1, \ldots, p_q \in \mathbb{Z}[\bar{X}]$ such that for sufficiently large n

$$\mathfrak{P}(G_{n+q+1}) = \sum_{i=1}^{q} p_i \cdot \mathfrak{P}(G_{n+i})$$

Let P_n be the path on n vertices. We get, for sufficiently large n,

- for the chromatic polynomials: $c(P_{n+1})(\lambda) = c(P_n, \lambda)(\lambda) \cdot (\lambda - 1)$.
- for the clique polynomials:
  $cl(P_{n+1})(X) = (1 + X) + cl(P_n)(X)$,
  $cl(K_{n+1})(X) = \sum_k^{n+1} \binom{n}{k} X^k = (X+1)^{n+1} = (X+1) \cdot cl(K_n)(X)$
- for the matching polynomials: $\mu(P_{n+1})(X) = X \cdot \mu(P_{n-1})(X) + \mu(P_n)(X)$,
- for the Tutte polynomials: $T(P_{n+1})(X, Y) = Y \cdot T(P_n)(X, Y)$.
- for the vertex-cover polynomials: $vc(P_{n+1}, X) = X \cdot vc(P_n, X) + X \cdot vc(P_{n-1}, X)$

Previous work, I

N.L. Biggs, R.M. Damerell and D.A. Sands, 1972

In 1972 N.L. Biggs, R.M. Damerell and D.A. Sands introduced recursive families of graphs.

These are our T-**recursive families of graphs**
where T is the Tutte polynomial.

They show that several families of graphs are recursive (in their sense).
Among them there are:

cycles, ladders and wheels

All these families have in common that they can be constructed from an initial graph by the repeated application of a fixed graph operation

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N.L. Biggs, R.M. Damerell and D.A. Sands,
Recursive families of graphs, J. Combin. Theory Ser. B 12 (1972), 123-131

Previous work, II

M. Noy and A. Ribó, 2004

In 2004 M. Noy and A. Ribó study
which graph families G_n,
constructed from an initial graph G_0,
by the repeated application of a fixed graph operation $F(G)$,
are T-**recursive families of graphs**.

They introduce a notion of **recursively constructible families of graphs**, and show that every such family is T-recursive.

Their notion is reminiscent of certain graph grammars.

M. Noy and A. Ribó, Recursively constructible families of graphs, Advances in Applied Mathematics 32 (2004) 350-363.

Our work

We use the finite model theory
of Monadic Second Order Logic (MSOL)
to extend these results in several ways:

- We prove that for every $\mathfrak{P}$ from a wide class of graph polynomials, the MSOL-definable graph polynomials, every recursively constructible family G_n is $\mathfrak{P}$-recursive.
- We extend the result to the class of iteration families of graphs which is proper extension of the class of recursively constructible families.
- We extend the result to signed graphs and knot diagrams and to various knot polynomials.
- We extend the result to hypergraphs and relational structures.

Recursively constructible families, I

In the absence of the formalisms of graph grammars Noy and Ribó give an adhoc definition of

repeated fixed succession
of elementary operations

which can be applied to a graph with a *context*, i.e. a labeled graph.

Let F denote such an operation.

Given a graph (with context) G, we put

$$G_0 = G, G_{n+1} = F(G_n)$$

Then the family

$$\mathcal{G} = \{G_n : n \in \mathbb{N}\}$$

is called **recursively constructible** using F, or an F-**iteration family**.

Recursively constructible families, II

Given a graph polynmial $\mathfrak{P}$,

the question now is to find

a characterization of those operations F,

for which a linear recurrence for the polynomials $\mathfrak{P}(G_n)$ holds.

M. Noy and A. Ribó give only a **suffient condition** in the case of the **Tutte polynomial**.

General strategy

We proceed as outlined in the case of the matching and the vertex-cover polynmial.

To compute $\mathfrak{P}(G_{n+1})$, we try to find,
depending on $\mathfrak{P}$ and, possibly, on G_0),
but **independently of** n

- an $m \in \mathbb{N}$,
- auxiliary polynomials $\mathfrak{P}_i(G_{n+1}), i \leq m$,
- and a matrix $A = (a_{i,j}) \in \mathbb{Z}[\bar{X}]^{m \times m}$

such that

$$\mathfrak{P}_j(G_{n+1})(\bar{X}) = \sum_i a_{i,j}(\bar{X}) \cdot \mathfrak{P}_i(G_n)(\bar{X})$$

Then we use the **characteristic polynomial of** A

to convert this into a **linear recurrence** relation.

Where logic enters for the polynomials?

The polynomial is of the form

$$\mathfrak{P}(G) = \sum_{(V,E') \in K_1} \left(\prod_{E' \subseteq E} t(\bar{X}) \right)$$

or

$$\mathfrak{P}(G) = \sum_{(V',E|V') \in K_2} \left(\prod_{V' \subseteq V} t(\bar{X}) \right)$$

where $t(\bar{X})$ is a fixed term in the indeterminates $\bar{X}$
and K_1 or K_2 are definable in

Monadic Second Order Logic (MSOL).

We call them $MSOL$-**definable graph polynomials**.

There are more general versions.

$MSOL$-definable polynomials, I

The **vertex-cover polynomials** are of the form

$$\mathfrak{P}(G) = \sum_{(V,E,V') \in K_2} \left(\prod_{V' \subseteq V} X \right)$$

because saying that V' is a vertex-cover of (V, E) is $MSOL$-definable.

Rearranging the terms we get

$$vc(G) = \sum_{(V,E,V') \in VC} \left(\prod_{V' \subseteq V} X \right) = \sum_k vc_k(G) \cdot X^k$$

Note: The second order variable for V' is needed.

$MSOL$-definable polynomials, II

The **matching polynomial** is of the form

$$\mu(G) = \sum_{(V,E') \in Matching} \left(\prod_{E' \subseteq E} X \right)$$

However, being a matching is

- NOT $MSOL$-definable if graphs are represented as $G = (V, E)$.
- but IS $MSOL$-definable, if the graph is represented by its incidence graph $I(G) = (V \cup E, R)$.

For the **Tutte polynomial**, we have to add a linear order on the edges, to make it $MSOL$-definable, and note, that the Tutte polynomial is then indepent of the order on the edges.

Where logic enters in the operation F?

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ-structures. We write $\mathfrak{A} \equiv_q^{MSOL} \mathfrak{B}$,
if $\mathfrak{A}$ and $\mathfrak{B}$ cannot be distinguished
by $MSOL(\tau)$-formulas of quantifier rank q.

An operation F on τ-structures is

$MSOL$-**smooth**

if whenever $\mathfrak{A} \equiv_q^{MSOL} \mathfrak{B}$, then also $F(\mathfrak{A}) \equiv_q^{MSOL} F(\mathfrak{B})$.

The operation F should be $MSOL$-smooth for the presentation of the graphs, for which the polynomial is $MSOL$-definable.

For forming the cliques K_n we need the operation of adding a vertex connected to all previous vertices.

This is $MSOL$-smooth for $G = (V, E)$ but not for $I(G) = (V \cup E, R)$.

k-structures

A vocabulary τ is a set of relation symbols.

A τ-structure $\mathfrak{A}$
is an interpretation of the vocabulary τ over a non-empty universe A.

For $k \in \mathbb{N}$, a k-τ-structure is a τ-structure with k additional unary relations $C_1^A, \ldots C_k^A$, called colours.

We denote by τ_k the vocabulary $\tau \cup \{C_1, \ldots, C_k\}$.

Basic operations on k-τ-structures

$Add_{C_i}(\mathfrak{A})$**:** For $i \leq k$, add a new element to A of colour C_i.

$\rho_{i,j}(\mathfrak{A})$**:** For $i, j \leq k$, recolour all elements of A of colour i with colour j.

$\eta_{R,i_1,\ldots,i_m}(\mathfrak{A})$**:** For $R \in \tau$ an m-ary relation symbol
and for each $a_1 \in C_{i_1}^A, \ldots, a_m \in C_{i_m}^A$
add the tuple $(a_1, \ldots, a_m)$ to R^A.

$\delta_{R,i_1,\ldots,i_m}(\mathfrak{A})$**:** For $R \in \tau$ an m-ary relation symbol
and for each $a_1 \in C_{i_1}^A, \ldots, a_m \in C_{i_m}^A$
delete the tuple $(a_1, \ldots, a_m)$ from R^A.

Quantifierfree transductions:
For each $R \in \tau_k$ of arity $\alpha(R)$
let $\phi_R(x_1, \ldots, x_{\alpha(R)})$ be
a quantifierfree τ_k formula with free variables as indicated.
A quantifier free transduction redefines all the predicates R^A in $\mathfrak{A}$ by ϕ_R^A.

$MSOL$-elementary and $MSOL$-smooth operations

An operation F on τ_k-structures is $MSOL$-elementary if F is a finite composition of basic operations on τ_k-structures.

Proposition:

Let F be $MSOL$-elementary and $\mathfrak{A}$ and $\mathfrak{B}$ two τ_k structures with $\mathfrak{A} \equiv_q^{MSOL} \mathfrak{B}$, then $F(\mathfrak{A}) \equiv_q^{MSOL} F(\mathfrak{B})$,
Hence, F is a $MSOL$-smooth.

Proposition:

Let F be $MSOL$-elementary and $\mathcal{G}$ be an F-iteration family. Then $\mathcal{G}$ is of bounded clique-width.

Corollary:

$I(K_n)$, $Grid_{n,n}$ are not F-iteration families for any F which is $MSOL$-elemntary.

Main Result

THEOREM: Let

- F be an $MSOL$-smooth operation on τ_k-structures.
- $\mathfrak{P}$ be a τ-polynomial which is $MSOL(\tau)$-definable.
- $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be an F-iteration family of τ-structures.

Then $\mathcal{A}$ is $\mathfrak{P}$-recursive, i.e. there is $q \in \mathbb{N}$,
and polynomials $p_1, \ldots, p_q \in \mathbb{Z}[\bar{X}]$ such that for sufficiently large n

$$\mathfrak{P}(G_{n+q+1}) = \sum_{i=1}^{q} p_i \cdot \mathfrak{P}(G_{n+i})$$

Proof ingredients

- For fixed q and a fixed number of free variables, there are, up to logical equivalence, only finitely many $MSOL(\tau)$-formulas of quantifier rank q.

  Let $\bar{\mathfrak{P}} = (\bar{\mathfrak{P}}_1, \dots \bar{\mathfrak{P}}_\alpha)$ be the vector of all $MSOL(\tau)^q$-definable polynomials.

- Feferman-Vaught Theorem for $MSOL$-definable graph polynomials

  J.A. Makowsky, Algorithmic uses of the Feferman-Vaught Theorem, Annals of Pure and Applied Logic, 126 (2004), 159-213

- Bilinear version of the Feferman-Vaught Theorem for graph polynomials.

  With an $MSOL$-elementary operation F and a fixed q there is a matrix M_F such that

$$\bar{\mathfrak{P}}(F(G)) = M_F \cdot \bar{\mathfrak{P}}(G)$$

- Use the characteristic polynomial of M_F.

Lecture 2

The Specker-Blatter Theorem

Counting finite topologies

Let T_n be the number of topologies on the set $\{1, \ldots, n\}$.

$T_1 = 1$, as the underlying set is always open.

$T_2 = 4$, for each singleton, we can decide whether it is open or not.

T_n is bounded by 2^{2^n}, hence $T_5 \leq 2^{32}$.

Two papers

$T_5 = 7181$

A. Shaafat, *On the number of topologies definable for a finite*, J. Australian Mat. Soc., vol 8 (1968), 194-198.

$T_5 = 6942$

J. Evans, F. Harary and M.S. Lynn, *On the computer enumeration of finite toplogies*, Communications of the ACM, 10 (1967), 295-297.

In the course we shall prove

$$7181 \neq 2 \mod 5$$

$$6942 = 2 \mod 5$$

This will allow us to conclude that $T_5 = 7181$ is not possible.

Logic and Combinatorics

- The class of finite topologies is not definable in First Order or even Second Order Logic.
- But the number of topologies on n points is the same as the number of reflexive transitive relations on n points.
- The class of reflexive transitive relations on n points is First order definable.
- Counting First Order definable relations is amenable using techniques from logic.

Graph properties

A simple graph

$$G =< V, E >=< V(G), E(G) >$$

has a set V of vertices and $E \subseteq V^2$ of edges. Simple graphs have no multiple edges. G is finite if V is finite.

A **graph property** $\mathcal{P}$ is a class of (finite) graphs closed under graph isomorphisms.

A graph property $\mathcal{P}$ is **monotone**, if it is closed under subgraphs, and **hereditary**, if it is closed under induced subgraphs.

A graph property $\mathcal{P}$ is $FOL(\tau)$-**definable** if it consists of all (finite) graphs satisfying an $FOL(\tau)$-formula ϕ. (Similarly for $SOL, MSOL$)

Examples and Exercise:

Check for monotonicity, hereditarity and definability:

Planar graphs, regular graphs, Eulerian graphs, Hamiltonian graphs,

connected graphs, 3-colorable graphs, bipartite graphs

Counting graphs: The density function.

- Graphs with n vertices will have $V = \{0, 1, \ldots, n-1\} = [n]$.
- There are $gr(n) = 2^{\binom{n}{2}} = 2^{\frac{n(n-1)}{2}}$ many graphs with n vertices.
- For a property $\mathcal{P}$ denote by $\mathcal{P}^n$ the graphs with n vertices in $\mathcal{P}$, and by $f_{\mathcal{P}}(n) =| \mathcal{P}^n |$, the number of graphs G with $V(G) = [n]$ which are in $\mathcal{P}$. $f_{\mathcal{P}}(n)$ is called the **density function** of $\mathcal{P}$.

  If $\mathcal{P}$ is hereditary, the density function of $\mathcal{P}$ is also called the **speed** $\mathcal{P}$, since it is a monotone increasing function.

What can we say about $f_{\mathcal{P}}(n)$?

Density functions: References

There is rich literature on density functions.

[Rio68] J. Riordan,
Combinatorial identities, Robert E. Krieger, New York, 1968

[HP73] F. Harary and E. Palmer,
Graphical enumeration, Academic Press, 1973.

[Wil90] H.S. Wilf,
generatingfunctionology, Academic Press, 1990 (2nd ed. 1994).
Also: `www.math.upenn.edu/~wilf/DownldGF.html`

[BBW02] J. Balogh, B. Bollobás and David Weinreich,
Measures on monotone properties of graphs,
Discrete Applied Mathematics 116 (2002), 17-36.

We shall see examples in a moment.

Some motivating (and confusing) examples

- Trees
- Cliques, stars and disjoint unions of cliques
- Regular graphs

Trees

Let $\mathcal{P}$ be the class of labeled trees.

- $\mathcal{P}$ not hereditary hence not monotone.
  $\mathcal{P}$ is not FOL-definable, but $MSOL$-definable.

- $f_{\mathcal{P}}(n) = n^{n-2}$ (Caley's Theorem, 1889).

Exercise: Compute n^{n-2} modulo m. Show that for fixed m, this gives an ultimately periodic sequence.

Hint: Use Little Fermat: $n^{p-1} = 1 \pmod{p}$ if p does not divide n.

Counting graphs: Cliques

Let $\mathcal{P} = \{K_n : n \in \mathbb{N}\}$ be the cliques (complete graphs).

- $\mathcal{P}$ is hereditary but not monotone.
  $\mathcal{P}$ is FOL-definable.

- $f_{\mathcal{P}}(n) = 1$ and obviously periodic modulo any m.

Counting graphs: Stars

For $\mathcal{P} = \{K_{1,n} : n \in \mathbb{N}\}$, the stars
(complete bipartite graphs with 1 and n vertices),

- $\mathcal{P}$ is not hereditary.
  $\mathcal{P}$ is FOL-definable.
- $f_{\mathcal{P}}(n) = n$ and obviously periodic modulo any m.

If $\mathcal{Q}$ is the closure of $\mathcal{P}$ under induced substructures, we get either stars or sets of isolated points, and $f_{\mathcal{Q}}(n) = n + 1$.

The density function of connected graphs

The class CONN of connected labeled graphs

- is not hereditary,
- is not FOL(R)-definable,
- but it is MSOL(R)-definable using a universal quantifier over set variables.
- It is also definable in Fixed Point Logic (FPL).

Counting labeled connected graphs is treated in [HP74] chapters 1 and 7 and in [Wil90] chapter 3. [HP74] chapter 1, page 7 gives:

$$f_{\mathrm{CONN}}(n) = 2^{\binom{n}{2}} - \frac{1}{n}\sum_{k=1}^{n-1} k\binom{n}{k} 2^{\binom{n-k}{2}} f_{\mathrm{CONN}}(k).$$

This does not look vey useful.

Counting graphs: Two disjoint cliques

Let $\mathcal{P}$ consist of graphs which are a union of two disjoint cliques.

- $\mathcal{P}$ is not hereditary hence not monotone.
  $\mathcal{P}$ is FOL-definable.

- $f_{\mathcal{P}}(n) = 2^{n-1}$ and and ultimately periodic modulo any m
  (using Little Fermat again).

Counting graphs: 1-regular graphs

Let $\mathcal{P}$ consist of graphs which are 1-regular
(disjoint union of non-connected edges, perfect matchings).

- $\mathcal{P}$ is not hereditary.
- $\mathcal{P}$ is FOL-definable.
- $f_{\mathcal{P}}(2m) = \binom{2m}{m} \cdot m!$,
  $f_{\mathcal{P}}(2m+1) = 0$.

The density function for regular graphs

The class REG_r of simple regular graphs where every vertex has degree r is FOL-definable (for fixed r).

The formulas says that every vertex has exactly r different neighbors. The formula grows with r. Regularity without specifying the degree is not FOL-definable, actually not even $CMSOL$-definable.

Counting the number of labeled regular graphs is treated completely in Chapter 7 of [HP74], where an explicit formula is given, essentially due to J.H. Redfield (1927) and rediscovered by R.C. Read (1959).

However, the formula is very complicated.

For cubic graphs, the function is explicitly given: $f_{\mathcal{R}_3}(2n+1) = 0$ and

$$f_{\mathcal{R}_3}(2n) = \frac{(2n)!}{6^n} \sum_{j,k} \frac{(-1)^j (6k-2j)! 6^j}{(3k-j)!(2k-j)!(n-k)!} 48^k \sum_i \frac{(-1)^i j!}{(j-2i)! i!}$$

Where does logic enter?

Theorem: (Blatter and Specker, 1981)

For $\mathcal{P}$ definable in $MSOL(\tau)$,
where τ is relational and has relation symbols of arity at most 2,
and for every $m \in \mathbb{N}$,
the function $f_{\mathcal{P}}(n)$ (mod m) on $\mathbb{Z}_m$
satisfies a linear recurrence relations, i.e
there are $q, n_0, a_1(m), \ldots, a_q(m) \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$f_{\mathcal{P}}(n+q) = \sum_{k=0}^{q-1} a_k(m) \cdot f_{\mathcal{P}}(n+k) \pmod{m}$$

and hence is ultimately periodic.

Theorem: (E. Fischer 2002)

For relations of arity ≥ 4, there are counterexamples.

When does logic enter? (References)

1981 C. Blatter and E. Specker,
Le nombre de structures finies d'une th'eorie à charactère fin, Sciences Mathématiques, Fonds Nationale de la recherche Scientifique, Bruxelles, 1981, pp. 41-44.

1984 C. Blatter and E. Specker,
Recurrence relations for the number of labeled structures on a finite set, In *Logic and Machines: Decision Problems and Complexity*, E. Börger and G. Hasenjaeger and D. Rödding, Springer Lecture Notes in Computer Science, 171 (1984), pp. 43-61.

From linear recurrences to regular languages

If $d_L(n)$ is a density function of some regular language L over an alphabet Σ, then

- All the values of $d(n)_L$ are non-negative.
- $d(n)_L$ is bounded by $\mid \Sigma \mid^n$.
- $d(n)_L$ satisfies a linear recurrence relation.
- The generating function $f_L(x) = \sum_n d(n)x^n$ is a rational function.

Is every function satisfying the above the density function of a regular language?

E. Barcucci, A. Del Lungo, A. Frosini and S. Rinaldi, From rational functions to regular languages, in *Formal Power Series and Algebraic Combinatorics*, D. Krob, A.A. Mikhalev and A.V. Mikhalev eds., Springer, 2000, pp. 633-644.

Density functions of graph classes

Let P be a graph property, i.e. a class of graphs closed under isomorphisms, and let $d_P(n)$ be its density function for labeled graphs.

- If $P = Graphs$ consists of all simple graphs,

  $$d_{Graphs}(n) = 2^{\binom{n}{2}}$$

  In the unlabeled case the function is rather complicated.

- If $P = LinOrd$ consists of all linear orders.

  $$d_{LinOrd}(n) = n!$$

  In the unlabeled case we have the constant function with value 1.

- If $P = SqGrids$ consists of all square grids,

  $$d_{SqGrids}(n) = \begin{cases} n! & \text{if } n = m^2 \\ 0 & \text{else} \end{cases}$$

  In the unlabeled case we have 1 instead of $n!$.

Lemma 1
Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ a function which satisfies a linear recurrence relation $f(n+1) = \sum_{i=0}^{k} a_i f(n-i)$ over $\mathbb{Z}$ with $a_{max} = \max_i a_i$.
Then there is a constant $c \in \mathbb{Z}$ such that $f(n) \leq 2^{cn}$.

Sketch of Proof:
One can prove this directly by induction with $c = log_2(k \cdot a_{max})$.
Q.E.D.

Corollary 2
For $\mathcal{C} \in \{Graphs, LinOrd, SqGrids\}$,
$f_{\mathcal{C}}(n)$ does not satisfy a linear recurrence over $\mathbb{Z}$.

Modular linear recurrences, I

However we note:

- For every $m \in \mathbb{N}$ and for large enough n we have $n! = 0 \pmod{m}$
  Hence, for $n \geq N(m)$ we have
  $$d_{LinOrd}(n+1) = d_{LinOrd}(n) \pmod{m}$$
  and
  $$d_{SqGrid}(n+1) = d_{SqGrid}(n) \pmod{m}$$

We say that a function $f(n)$ satisfies a **trivial modular recurrence**
if for every m there exists N_m such that
if $n > N_m$ then $g(n) \equiv 0 \pmod{m}$.
This is true in particular, and even equivalent to,
if there exist functions $g(n), h(n)$ with $g(n)$ tending to infinity
such that $f(n) = g(n)! \cdot h(n)$.

Clearly, the two examples above are trivial modular recurrences.

Modular linear recurrences, II

Now we look at

$$d_{Graphs}(n+1) = 2^{\binom{n+1}{2}} = 2^{\binom{n}{2}} \cdot 2^n$$

Hence

$$d_{Graphs}(n+m+1) = d_{Graphs}(n) \cdot \prod_{i=0}^{m} 2^{n+i} = d_{Graphs}(n) \cdot 2^{nm} \cdot \prod_{i=0}^{m} 2^i$$

As $nm = 0 \pmod{m}$ we get

$$d_{Graphs}(n+m+1) = d_{Graphs}(n) \cdot \prod_{i=0}^{m} 2^i \pmod{m}$$

This is a non-trivial recurrence.

It is also different for distinct m and m', in other words, non-uniform in m.

Two equal-sized cliques, I

Let $EQ_2CLIQUE$ the class of graphs which consists of two disjoint unions of equal-sized cliques.

We want to study its density function $d_{EQ_2CLIQUE}(n)$. We have

$$d_{EQ_2CLIQUE}(n) = b_2(n) = \begin{cases} \frac{1}{2}\binom{2m}{m} & \text{for } n = 2m \\ 0 & \text{else} \end{cases}$$

The factor $\frac{1}{2}$ is there because we cannot distinguish the choice of the first clique from the choice of its complement.

Proposition 3 (Lucas, 1878)
For every n which is not a power of 2, we have $b_2(n) \equiv 0 \pmod 2$, and for every n which is a power of 2 we have $b_2(n) \equiv 1 \pmod 2$.
In particular, $b_2(n)$ is not ultimately periodic modulo 2.

A proof may be found as Exercise 5.61 in:
R. Graham, D. Knuth and O. Patashnik,
Concrete Mathematics, 2nd ed., Addison-Wesley 1994

There is a generalization of this for p-many equal-sized cliques.

Two equal-sized cliques, II

- We can prove (using the various version of the pebble games) that $EQ_2CLIQUE$ is not definable in FOL, or $\mathcal{L}^{\omega}_{\infty,\omega}$.
- One can also prove (using the $MSOL$-version of the pebble games) that $EQ_2CLIQUE$ is not definable in Monadic Second Order Logic $MSOL$.
- However, $EQ_2CLIQUE$ is definable in Second Order Logic SOL.
- With **two** binary E, M relations we can express in FOL that E is the edge relation of a graph in $EQ_2CLIQUE$, and that M is a matching (bijection) between the two cliques.
  Let us call the class so defined $M_2CLIQUES$.

  But for the density function of $M_2CLIQUES$ we have

  $$d_{M_2CLIQUES}(2n) = n! \cdot d_{EQ_2CLIQUES}(2n)$$

  which satisfies the trivial modular recurrence relations.

The Specker-Blatter Theorem revisited, I

Let $\mathcal{P}$ be a graph property which is $MSOL$-definable. and let $d_{\mathcal{P}}(n)$ be its density function.

- (Specker and Blatter, 1981)
  $d_{\mathcal{P}}(n)$ satisfies modular recurrence relations for each m.
- (Specker and Blatter, 1981)
  This remains true with several binary edge relations and unary predicates on the vertices.
- (E. Fischer, 2003) Is false for an FOL-definable class with one quaternary relation.

Relations of bounded degree

Let $\mathcal{A} = \langle A, \bar{R} \rangle$ be a τ-structure.
We define a symmetric relation E_A on $\mathcal{A}$, and call $\langle A, E_A \rangle$ the **Gaifman-graph of** $\mathcal{A}$.

- Let $a, b \in A$. $(a, b) \in E_A$ iff there exists a relation $R \in \bar{R}$ and some $\bar{a} \in R$ such that both a and b appear in $\bar{a}$
(possibly with other members of A as well).

- For any element $a \in A$, the **degree** of a is the number of elements $b \neq a$ for which $(a, b) \in E_A$.

- We say that $\mathcal{A}$ is of **bounded degree** d
if every $a \in A$ has degree at most d.

- We say that $\mathcal{A}$ is **connected**
if its Gaifman-graph is connected.

- For a class of structures $\mathcal{P}$ we say it is of bounded degree d (resp. connected) iff all its structures are of bounded degree d (resp. connected).

The Specker-Blatter Theorem revisited, II

Theorem 4 (E. Fischer and J.A. Makowsky, 2002)
Let $\mathcal{P}$ be a property of τ-structures, which is $MSOL$-definable.
Let $d_{\mathcal{P}}(n)$ be its density function.

- *If $\mathcal{P}$ is of bounded degree d,*
  the function $d_{\mathcal{P}}(n)$ satisfies a modular recurrence relation for every m.

- *Furthermore, if additionally all the models in $\mathcal{P}$ are connected,*
  the function $f_{\mathcal{P}}$ satisfies the trivial recurrence relations for every m.

We have no restrictions on τ, besides not allowing function symbols,

Theorem C and Theorem 4 remain true if we extend $MSOL$ and allow modular counting quantifiers.

Ingredients of the proof of the Specker-Blatter Theorem

- The DU-index of a class of structures.
- The Specker-index of a class of structures.
- The DU-index of a class of structures $\mathcal{P}$ is always smaller or equal to the Specker index.
- Finite DU-index of a class of τ-structures of bounded degree implies modular recurrence relations for all m.
- If τ contains only relation symbols of arity at most 2, finite Specker-index of a class of τ-structures. implies modular recurrence relations for all m.
- $MSOL$-definability of $\mathcal{P}$ (even $CMSOL$-definability) implies finite DU-index.
- If τ contains only relation symbols of arity at most 2, the definability assumption implies finite Specker index.

DU-index of $\mathcal{P}$

We denote by $\mathfrak{A} \sqcup \mathfrak{B}$ the disjoint union of two τ-structures $\mathfrak{A}$ and $\mathfrak{B}$.

Let $\mathcal{P}$ be a class of τ-structures.

(i) We say that $\mathfrak{A}_1$ is $DU(\mathcal{P})$-equivalent to $\mathfrak{A}_2$,
denoted by $\mathfrak{A}_1 \sim_{DU(\mathcal{P})} \mathfrak{A}_2$,
if for every τ-structure $\mathfrak{B}$, $\mathfrak{A}_1 \sqcup \mathfrak{B} \in \mathcal{P}$ if and only if $\mathfrak{A}_2 \sqcup \mathfrak{B} \in \mathcal{P}$.

(ii) The DU-index of $\mathcal{P}$ is the number of $DU(\mathcal{P})$-equivalence classes.

(iii) A class of structures $\mathcal{P}$ is a **Gessel class** if for every $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \sqcup \mathfrak{B} \in \mathcal{P}$ iff both $\mathfrak{A} \in \mathcal{P}$ and $\mathfrak{B} \in \mathcal{P}$.

Basics on the DU-index

- The class of forests is a Gessel class.
- If $\mathcal{P}$ is hereditary and closed under disjoint unions, it is a Gessel class.
- Every Gessel class has DU-index at most 2.
- If $\mathcal{P}$ is a class of connected graphs, $\mathcal{P}$ has DU-index at most 2, but is not a Gessel class.
- If $\mathcal{P}_1$ and $\mathcal{P}_2$ have finite DU-index, so do $\mathcal{P}_1 \cup \mathcal{P}_2$, $\mathcal{P}_1 \cap \mathcal{P}_2$, and the complement $\bar{\mathcal{P}}_1$.

Gessel's Theorem (1984)
(without proof, just as an illustration)

Theorem 5 (I. Gessel 1984)

If $\mathcal{C}$ is a Gessel class of directed graphs of degree at most d, then

$$f_{\mathcal{C}}(m+n) \equiv f_{\mathcal{C}}(m) \cdot f_{\mathcal{C}}(n) \pmod{\frac{m}{\ell}}$$

where ℓ is the least common multiple of all divisors of m not greater than d.

In particular, $f_{\mathcal{C}}(n)$ satisfies for every $m \in \mathbb{N}$ the linear recurrence relation

$$f_{\mathcal{C}}(n) \equiv a^{(m)} f_{\mathcal{C}}(n - d!m) \pmod{m}$$

where $a^{(m)} = f_{\mathcal{C}}(d!m)$.

DU-index and Logic

We can use pebble games to prove:

Theorem 6

(i) If $\mathcal{P}$ is FOL-definable, it has finite DU-index.

(ii) If $\mathcal{P}$ is $MSOL$-definable, it has finite DU-index.

(iii) If $\mathcal{P}$ is $CMSOL$-definable, it has finite DU-index.

Many classes of finite DU-index.

There are only countably many classes of structures definable by $MSOL$-formulas.

How many classes are there of finite DU-index?

Proposition 7 (After an idea of Specker, 2002)

(i) There are continuum many classes of structures (closed under isomorphisms) with DU-index ≤ 2.

(ii) There are continuum many Gessel classes.

Bounded DU-index and bounded degree

Theorem 8 (E. Fischer and J.A. Makowsky, 2002)
Let $\mathcal{P}$ be a property of τ-structures, with finite DU-index and all its members of bounded degree d. Then

- $d_{\mathcal{P}}(n)$ *satisfies a modular recurrence relation for every m.*
- *Furthermore, if additionally all the models in $\mathcal{P}$ are connected, (hence the DU-index is 2, the function $f_{\mathcal{P}}$ satisfies the trivial recurrence relations for every m.*

Example $\overline{EQ_2CLIQUES(A)}$ shows that bounded degree cannot be dropped, even for DU-index 2 (and connected structures).

For structures of unbounded degree one needs a stronger assumption, the finiteness of the Specker-index, to be discussed later (if time permits).

Orbit of a structure

Let $\mathcal{G} \subseteq \mathcal{S}_n$ be a subgroup of the full permutation group of $[n] = \{0, \ldots, n-1\}$.

For a τ-structure $\mathfrak{A}$ over the universe $A = [n]$

- the orbit $Orb_{\mathcal{G}}(\mathfrak{A})$ is the set of different labeled structures $\sigma(\mathfrak{A})$ obtained using relabelings from $\mathcal{G}$, i.e.

$$Orb_{\mathcal{G}}(\mathfrak{A}) = \{\sigma(\mathfrak{A}) : \sigma \in \mathcal{G}\}$$

- If $\mathcal{G} = \mathcal{S}_n$, we omit it and write $Orb(\mathfrak{A})$

- $Aut_{\mathcal{G}}(\mathfrak{A})$ is the set of τ-automorphisms of $\mathfrak{A}$ which are in $\mathcal{G}$.

As $Aut_{\mathcal{G}}(\mathfrak{A}) \subseteq \mathcal{G}$ is a subgroup, we have the fundamental identity

Proposition 9
For a τ-structure $\mathfrak{A}$ over the universe $A = [n]$ and $\mathcal{G}$ a subgroup of $\mathcal{S}_n$ we have

$$| Aut_{\mathcal{G}}(\mathfrak{A}) | \cdot | Orb_{\mathcal{G}}(\mathfrak{A}) | = | \mathcal{G} |$$

Density functions and orbits via a subgroup

Let $\mathcal{G}$ be subgroup of $\mathcal{S}_n$.
$\mathcal{G}$ induces an equivalence relation on $\mathcal{P}$:

$$\mathfrak{A} \sim_{\mathcal{G}} \mathfrak{A}' \text{ iff there is } \sigma \in \mathcal{G} \text{ with } \sigma(\mathfrak{A}) = \mathfrak{A}'$$

We denote by $\mathcal{P}/\mathcal{G}$ the set of these equivalence classes in $\mathcal{P}^n$ and denote its equivalence classes by $[\mathfrak{A}]_{\mathcal{G}}$.

Lemma 10
The orbits $Orb_{\mathcal{G}}(\mathfrak{A})$ and the density function $d_{\mathcal{P}}(n)$ are related by the following formula:

$$d_{\mathcal{P}}(n) = \sum_{[\mathfrak{A}]_{\mathcal{G}} \in \mathcal{P}^n/\mathcal{G}} Orb_{\mathcal{G}}(\mathfrak{A})$$

To show that $d_{\mathcal{P}}(n) = 0 \pmod{m}$ it suffices to show that for each $\mathfrak{A} \in \mathcal{P}^n$ we have $Orb_{\mathcal{G}}(\mathfrak{A}) = 0 \pmod{m}$.

Degrees and orbits

Lemma 11
Let $A = [n]$ and $B \subseteq A$ and $a \in A - B$.
Let N_a^B be the set of neighbors of a which are in B.

- *Let $S_B \subseteq \mathcal{S}_n$ be the subgroup of permutations σ such that $\sigma(a) = a$ for every $a \in A - B$.*
- *Let $G_B^{N_a} \subseteq S_B$ be the subgroup of S_B which maps N_a^B onto itself.*

Then $\mid Orb_{S_B}(\mathfrak{A}) \mid$ is divisible by $\binom{|B|}{|N_a^B|}$.

Proof:

$$\mid G_B^{N_a} \mid \cdot \binom{\mid B \mid}{\mid N_a^B \mid} = \mid S_B \mid = \mid Aut_{S_B}(\mathfrak{A}) \mid \cdot \mid Orb_{S_B}(\mathfrak{A}) \mid =$$

$$\mid Orb_{G_B^{N_a}}(\mathfrak{A}) \mid \cdot \mid Aut_{G_B^{N_a}}(\mathfrak{A}) \mid \cdot \binom{\mid B \mid}{\mid N_a^B \mid}$$

But we have $Aut_{G_B^{N_a}}(\mathfrak{A}) = Aut_{S_B}(\mathfrak{A})$, hence the result. Q.E.D.

Choosing special values

We fix m, the modulus, and d, the degree.

We put $c = d! \cdot m$.

Lemma 12

For every $t \in \mathbb{N}$ and $0 < d_1 \leq d$ we have that m divides $\binom{t \cdot c}{d_1}$.

Proof: Write out the definitions.

$$\binom{t \cdot c}{d_1} = \binom{t \cdot d! \cdot m}{d_1} = \frac{(t \cdot d! \cdot m)!}{(t \cdot d! \cdot m - d_1)! \cdot d_1!} =$$

$$t \cdot m \cdot d! \cdot \frac{\prod_{i=1}^{d_1-1}(t \cdot m \cdot d! - i)}{d_1!} =$$

$$\frac{t \cdot m \cdot d!}{d_1 \cdot (d_1 - 1)!} \cdot \prod_{i=1}^{d_1-1}(t \cdot m \cdot d! - i) = \frac{t \cdot m \cdot d!}{d_1} \cdot \prod_{i=1}^{d_1-1} \frac{(t \cdot m \cdot d! - i)}{i}$$

Q.E.D

Connected structures, I

Recall: m is the modulus, and d the degree. $c = d! \cdot m$.

Lemma 13
For every $t \in \mathbb{N}$ and every connected $\mathfrak{A}$ with $\mid A \mid \geq t \cdot c + 1$, and for every $B \subseteq A$ with $\mid B \mid = t \cdot c$, we have that $\mid Orb_{S_B}(\mathfrak{A}) \mid$ is divisible by m.

Proof: For t, A, B as required, there is $a \in A - B$ with at d_a neighbors in B, and $1 \leq d_a \leq d$.

By Lemma 11 $\mid Orb_{S_B}(\mathfrak{A}) \mid$ is divisible by $\binom{|B|}{d_a}$.

Using Lemma 12 with $\mid B \mid = t \cdot c$ and $1 \leq d_a \leq d$, we get,
it is also divisible by m.

Connected structures, II

Now we use Lemma 10 with $\mathcal{G} = S_B$ and $\mid B \mid = t \cdot c$.

But this means that B has to be fixed **independently** of the particular structure $\mathfrak{A}$ in $\mathcal{P}^n$.

However, as all the $\mathfrak{A} \in \mathcal{P}^n$ are connected, there is always an $a \in A - B$ which has neighbors in B.

Hence, by Lemma 13, for every $\mathfrak{A} \in \mathcal{P}^n$
$\mid Orb_{S_B}(\mathfrak{A}) \mid$ is divisible by m.

Q.E.D.

Remark: *We did not use a particular $\mathcal{P}$ of finite DU-index. We only used connectedness (which implies that the DU-index is 2).*

Disconnected structures, I

Let $\mathcal{P}$ be class of structures of bounded degree d and finite D-index α.

Let $\mathcal{D}_i, i \leq \alpha$ be the DU-equivalence classes. with respect to $\mathcal{P}$.

- All the structures of degree bigger than d are in one class, say $\mathcal{D}_0$.
- $\mathcal{P}$ is also one of the classes.
  (Allowing $\mathfrak{B}$ to be empty)

m and d are still fixed and $c = m \cdot d!$.
We look now at structures with universe $[n]$.

Let $t \in \mathbb{N}$ and $B = [t \cdot c]$.

- Let $\mathcal{D}_i^0$ be those structures in $\mathcal{D}_i$ for which there is $a \in A - B$ which is connected to some $b \in B$, and
- $\mathcal{D}_i^1$ be those structures in which no such a exists.

Disconnected structures, II

We want to compute $d_i(n) = d_{\mathcal{D}_i}$ and $d_i^j(n) = d_{\mathcal{D}_i^j}$ modulo m
for each $i \leq \alpha$ and $j = 0, 1$.

Clearly, $d_i(n) = d_i^0(n) + d_i^1(n)$.

For $\mathfrak{A} \in \mathcal{D}_i^0$ we apply Lemma 11. Let d_a be the number of neighbors of a in B.

Hence, $Orb_{S_B}(\mathfrak{A})$ is divisible by $\binom{t \cdot c}{d_a}$.

By Lemma 12 $Orb_{S_B}(\mathfrak{A})$ is divisible by m,
hence $Orb(\mathfrak{A})$ and $d_i^0(n)$ are divisible by m.

Conclusion:

$$d_i(n) = d_i^0(n) + d_i^1(n) = d_i^1(n) \pmod{m}$$

Disconnected structures, III

For $\mathfrak{A} \in \mathcal{D}_i^1$ we note that $\mathfrak{A}$ can be uniquely written as

$$\mathfrak{A}_1 \sqcup \mathfrak{A}_2$$

with universes $A_1 = [t \cdot c]$ and $A_2 = \{t \cdot c + 1, \ldots, n\}$.

Fact: The equivalence class $[\mathfrak{A}_1 \sqcup \mathfrak{A}_2]$ is uniquely determined by the equivalence classes $[\mathfrak{A}_1]$ and $[\mathfrak{A}_2]$.

Now we put $t(n) = \lfloor \frac{n-1}{c} \rfloor$ and $\widehat{n} = n \pmod{c}$.

Conclusion: We get, summing over all possibilities

$$d_i(n) = \sum_{j=1}^{\alpha} \mu_{i,j,m,\widehat{n}} d_j(t(n)) \pmod{m}$$

Disconnected structures, IV

This gives us $(\alpha \times \alpha)$-matrices

$$M(m, \widehat{n}) = (\mu_{i,j,m,\widehat{n}})_{i,j}$$

Let $\bar{d}(n) = (d_1(n), \ldots, d_\alpha(n))^{tr}$.

For each $\widehat{n} \in \mathbb{Z}_c$ we get now the relationship

$$\bar{d}(n) = M(m, \widehat{n}) \cdot \bar{d}(t(n)) \pmod{m}$$

Using the characteristic polynomials $p_{m,\widehat{n}}(\lambda)$ of all the matrices $M(m, \widehat{n})$, we can now compute the required linear recurrence modulo m. Q.E.D.

Graphs of unbounded degree, I

The case of graphs of unbounded degree has several complications.

- Finite DU-index of $\mathcal{P}$ does not suffice.
  We have to make a stronger assumtion: **Finite Specker index**.
- The restriction to relations of arity at most 2 is essential.
- The proof, although similar, is considerably more complicated.

Specker index, survey

- To define the Specker index of a graph property $\mathcal{P}$ one defines a binary operation $subst(H, a, G)$ where in the pointed graph $H = (V_H, E_H, a)$ the vertex a is substituted by $G = (V_G, E_G)$.

- $DU(\mathcal{P})$-equivalence is now replaced by $subst(\mathcal{P})$-equivalence:
$G_1 \sim_{\mathcal{P}} G_2$ iff for all H and $a \in V_H$ we have

$$subst(H, a, G_1) \in \mathcal{P} \text{ iff } subst(H, a, G_2) \in \mathcal{P}$$

- The Specker index of $\mathcal{P}$ is the number of $subst(\mathcal{P})$-equivalence classes.

Observation:

- The DU-index of $\mathcal{P}$ is always smaller or equal The Specker-index of $\mathcal{P}$.

- $MSOL$-definability of $\mathcal{P}$ implies finite Specker index.