CR05: Tilings, between dynamical systems and computability

Course 3: (A)periodicity

October 1st, 2015
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1. Wang tiles model
   - Wang tiles and SFT
   - Periodicity and the tiling problem

2. Robinson’s tiling
   - Robinson’s tiles force aperiodicity

3. Substitutive subshifts
   - Fracture lines
   - Multidimensional substitutions
   - Substitutive subshifts

4. Mozes theorem
   - Statement
   - Sketch of the proof
The tiling problem (with Wang tiles)

Wang tiles = unit squares with colored edges
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Wang tiles = unit squares with colored edges

**Tiling problem:** given $\tau$ a finite set of Wang tiles, does there exist a valid tiling of the plane by $\tau$? (a.k.a. **Domino problem**
Wang tiles and $\mathbb{Z}^2$-SFT

Wang tiles $\rightarrow \mathbb{Z}^2$-SFT

Let $\tau$ be a finite set of Wang tiles.
Wang tiles and $\mathbb{Z}^2$-SFT

**Wang tiles $\rightarrow \mathbb{Z}^2$-SFT**

Let $\tau$ be a finite set of Wang tiles.

Construct the $\mathbb{Z}^2$-SFT on the alphabet $\tau$ given by the set of $2 \times 1$ and $1 \times 2$ forbidden patterns that violate matching rules.
Wang tiles and $\mathbb{Z}^2$-SFT

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**$\mathbb{Z}^2$-SFT $\rightarrow$ Wang tiles**

forbidden patterns of size $k$
Wang tiles model

Robinson’s tiling

Substitutive subshifts

Mozes theorem

Wang tiles and $\mathbb{Z}^2$-SFT

**Wang tiles $\rightarrow \mathbb{Z}^2$-SFT**

Let $\tau$ be a finite set of Wang tiles.

![Wang tiles example](image)

Construct the $\mathbb{Z}^2$-SFT on the alphabet $\tau$ given by the set of $2 \times 1$ and $1 \times 2$ forbidden patterns that violate matching rules.

![Forbidden patterns image](image)

**$\mathbb{Z}^2$-SFT $\rightarrow$ Wang tiles**

forbidden patterns of size $2n + 1$

![Examples of forbidden patterns](image)
Wang tiles and $\mathbb{Z}^2$-SFT

**Wang tiles $\rightarrow \mathbb{Z}^2$-SFT**

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Construct the $\mathbb{Z}^2$-SFT on the alphabet $\tau$ given by the set of $2 \times 1$ and $1 \times 2$ forbidden patterns that violate matching rules.

**$\mathbb{Z}^2$-SFT $\rightarrow$ Wang tiles**

allowed patterns of size $2n + 1$

Wang tiles with colors in $A^{[0;2n] \times [0;n]} \cup A^{[0;n] \times [0;2n]}$
The tiling problem (with susbhifts)

**Tiling problem:** given $\tau$ a finite set of Wang tiles, does there exist a valid tiling of the plane by $\tau$ ? (a.k.a. **Domino problem**)

**Emptiness problem:** given a finite set of forbidden patterns $F$, could one determine whether $X_F = \emptyset$ ?
Locally and globally admissible patterns

**Tiling/Emptiness problem:** We look for an infinite configuration that respects local constraints (given by Wang tiles/forbidden patterns).

What about **finite** configurations (=patterns)?
Locally and globally admissible patterns

**Tiling/Emptiness problem:** We look for an infinite configuration that respects local constraints (given by Wang tiles/forbidden patterns).

What about **finite** configurations (=patterns) ?

- A pattern is **locally admissible** if it respects the local constraints (i.e. if Wang tiles colors match/contains no forbidden pattern).
- A pattern is **globally admissible** if it can be extended to an infinite configuration that respects local constraints.
Let $\tau$ be a finite set of Wang tiles.
Semi-algorithm for periodicity (I)

Let $\tau$ be a finite set of Wang tiles.

It is easy to generate, for every integers $n, m \in \mathbb{N}$, all locally admissible patterns of size $n \times m$. 
Semi-algorithm for periodicity (I)

Let $\tau$ be a finite set of Wang tiles.

It is easy to generate, for every integers $n, m \in \mathbb{N}$, all locally admissible patterns of size $n \times m$.

If you find a locally admissible pattern with matching edges, then $\tau$ tiles the plane periodically.
Semi-algorithm for periodicity (II)

**Semi-algorithm:**

1. gives a pattern that tiles the plane periodically if it exists
2. loops otherwise
Semi-algorithm for periodicity (II)

Semi-algorithm:
1. gives a pattern that tiles the plane periodically if it exists
2. loops otherwise

Questions:
1. Can you check whether the locally admissible patterns are globally admissible?
2. Is it true that if \( \tau \) admits no periodic pattern, then \( \tau \) does not tile the plane?
Wang’s conjecture and the tiling problem

Wang’s conjecture (1961)

If a set of Wang tiles can tile the plane, then they can always be arranged to do so periodically.
Wang’s conjecture and the tiling problem

Wang’s conjecture (1961)
If a set of Wang tiles can tile the plane, then they can always be arranged to do so periodically.

Suppose Wang’s conjecture is true. Then you can decide the tiling problem!

Semi-algorithm 1:
1. gives a pattern that tiles the plane periodically if it exists
2. loops otherwise

Semi-algorithm 2:
1. gives an integer $n$ so that $[1; n] \times [1; n]$ cannot be tiled if it exists
2. loops otherwise
Back to Wang’s conjecture

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If a set of Wang tiles can tile the plane, then they can always be arranged to do so periodically.
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- and even 11 Wang tiles (Jeandel and Rao, 2015)!
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Remark

More than that, all these constructions actually show the undecidability of the tiling problem (from which you deduce the existence of an aperiodic tileset). ⇒ see next Course
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   - Multidimensional substitutions
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5. Mozes theorem
   - Statement
   - Sketch of the proof
Robinson tileset

The Robinson tileset, where tiles can be rotated.
The Robinson tileset, where tiles can be rotated and reflected.
Robinson tileset

The Robinson tileset, where tiles can be rotated and reflected.
The Robinson tileset, where tiles can be rotated and reflected.

http://perso.ens-lyon.fr/nathalie.aubrun/robinson.html

See you in 15 minutes!!
Existence of a valid tiling

**Proposition**

Robinson’s tileset admits at least one valid tiling.
Existence of a valid tiling

Proposition

Robinson’s tileset admits at least one valid tiling.

Proof:

- We can build arbitrarily large patterns (called macro-tiles) with the same structure.
- We thus conclude by compactness.
Macro-tiles of level 1.
Macro-tiles of level 1.

They behave like large □.
From macro-tiles of level 1 to macro-tiles of level 2
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From macro-tiles of level 1 to macro-tiles of level 2
From macro-tiles of level $n$ to macro-tiles of level $n + 1$
This valid tiling is aperiodic

**Proposition**

The valid tiling $x$ obtained by compactness is aperiodic.

**Proof:**

- Centers of macro-tiles of level $n$ are located on the lattice $2^{n+1}\mathbb{Z} \times 2^{n+1}\mathbb{Z}$.
- Suppose $x$ admits a direction of periodicity $\overrightarrow{u}$.
- Then there exists an integer $n$ s.t. $2^{n+1} > \|\overrightarrow{u}\|$.
- Thus a macro-tile of level $n$ overlaps with its translation.
- $\Rightarrow$ contradiction.
All valid tilings are aperiodic (I)

The two forms in Robinson tileset, cross (bumpy corners) and arms (dented corners).
All valid tilings are aperiodic (I)

The two forms in Robinson tileset, cross (bumpy corners) and arms (dented corners).

Obviously, two crosses cannot be in contact (neither through an edge nor a vertex) thus a cross must be surrounded by eight arms.
All valid tilings are aperiodic (II)

You cannot have things like

\[ \begin{align*}
&\quad \text{Diagram 1} \\
&\quad \text{Diagram 2} \\
&\quad \text{Diagram 3}
\end{align*} \]
All valid tilings are aperiodic (II)

You cannot have things like

![Diagram of invalid tilings]

The only possibilities are thus

![Diagram of valid tilings]
All valid tilings are aperiodic (II)

You cannot have things like

The only possibilities are thus
All valid tilings are aperiodic (III)

So each \[
\square
\]

is part of a macro tile of level 1

that behaves like a big \[
\square
\]

and so on…
About Robinson’s tiling structure

Hierarchy of squares: squares of level $n$ are gathered by 4 to form a square of level $n + 1$. 

![Diagram of Robinson's tiling structure](image)
About Robinson’s tiling structure

Hierarchy of squares: squares of level $n$ are gathered by 4 to form a square of level $n + 1$

![Diagram of Robinson's tiling]

**Proposition**

The are uncountably many different valid tilings by the Robinson tileset.
Fracture lines

Some sequences of choices (ultimately constant sequences) leads to
Fracture lines

Some sequences of choices (ultimately constant sequences) leads to

But it is possible to enrich the tiles to get rid of fracture lines! (idea: synchronize squares of same level)
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Let $A$ be a finite alphabet. A $(k, \ell)$-substitution is an application
\[ s : A \to A^{[1;k] \times [1;\ell]} . \]

Example:

\[ s : \]

\[ \begin{array}{ccc}
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } \\
\end{array} \quad \begin{array}{ccc}
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\end{array} \]

\[ s : \]

\[ \begin{array}{ccc}
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\text{ } & \text{ } & \text{ } \\
\end{array} \]
Iterate a substitution on a pattern

We can produce arbitrarily big rectangular patterns, called the $s$-patterns. $s$-patterns are patterns that can be written as $s^n(a)$ for some letter $a \in A$ and $n \in \mathbb{N}$.

Example:
Two notions of substitutive subshifts (I)

The sub-pattern substitutive subshift $X_s$ is given by

$$X_s = \left\{ x \in A^\mathbb{Z}^d : \forall p \sqsubset x, \exists a \in A, \exists n \in \mathbb{N}, p \sqsubset s^n(a) \right\}$$

The only allowed patterns are $s$-patterns.
The sub-pattern substitutive subshift $X_s$ is given by

$$X_s = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \forall p \sqsubset x, \exists a \in \mathcal{A}, \exists n \in \mathbb{N}, p \sqsubset s^n(a) \right\}$$

The only allowed patterns are $s$-patterns.

The limit substitutive subshift $X'_s$ is given by

$$X'_s = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \forall n \in \mathbb{N}, \exists y \in \mathcal{A}^{\mathbb{Z}^d}, \exists i \in \mathbb{Z}^d, s^n(y) = \sigma^i(x) \right\}$$

It is possible to find a pre-image of any order under $s$. 
Two notions of substitutive subshifts (II)

Example:

\[ \circ \mapsto \circ \circ \quad \text{and} \quad \bullet \mapsto \bullet \circ \]
Two notions of substitutive subshifts (II)

Example:

\[ \circ \rightarrow \circ \circ \quad \text{and} \quad \bullet \rightarrow \bullet \circ \]

\[
X_s = \{ \circ^{\mathbb{Z}^2} \} \quad \text{and} \quad X'_s = \{ \circ^{\mathbb{Z}^2} \} \cup \{ \sigma^i(x_\bullet), i \in \mathbb{Z}^2 \}
\]

where the configuration \( x_\bullet \) is such that \( x(i,j) = \bullet \) if and only if \( (i,j) = (0,0) \).
Two notions of substitutive subshifts (II)

**Example:**

\[
\begin{align*}
\circ & \mapsto \circ \circ \quad \text{and} \quad \bullet & \mapsto \bullet \circ \\
X_s &= \left\{ \circ \mathbb{Z}^2 \right\} \quad \text{and} \quad X'_s = \left\{ \circ \mathbb{Z}^2 \right\} \cup \left\{ \sigma^i(x_{\bullet}), \, i \in \mathbb{Z}^2 \right\}
\end{align*}
\]

where the configuration \( x_{\bullet} \) is such that \( x(i,j) = \bullet \) if and only if \((i,j) = (0,0)\).

**Proposition**

Let \( s \) be a rectangular substitution, then \( X_s \subset X'_s \).
A substitution \( s \) has \textbf{unique derivation} if for all \( x \in X'_s \) there exists a unique \( y \in X'_s \) and a unique \( i \in \mathbb{Z}^d \) s.t. \( x = \sigma^i(s(y)) \).

**Proposition**

If a substitution \( s \) has unique derivation, then every configuration \( x \in X'_s \) is aperiodic.
A substitution $s$ has **unique derivation** if for all $x \in X'_s$ there exists a unique $y \in X'_s$ and a unique $i \in \mathbb{Z}^d$ s.t. $x = \sigma^i(s(y))$.

**Proposition**

If a substitution $s$ has unique derivation, then every configuration $x \in X'_s$ is aperiodic.

**Proof:** If a configuration $x \in X'_s$ is periodic with periodicity vectors $(m, 0)$ and $(0, n)$, then the pre-image of $x$ under $s$ is also periodic with periodicity vectors $(m', 0)$ and $(0, n')$.

Since one has $|m'| < |m|$ and $|n'| < |n|$, iterating this process leads to a contradiction.
Other properties of substitutive subshifts

Let $s$ be a deterministic substitution. Then the subshifts $X_s$ and $X'_s$ satisfy the following properties:

- They are **effective** subshifts.

- They have **zero entropy** (find an upper bound on their (factor) complexity function). **This is no longer true for a set $S$ of substitutions !!**

- If $s$ is primitive, all configurations in $X_s$ are **quasi-periodic.**
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Theorem (Mozes, 1989)

Let $s$ be a non-degenerate deterministic multidimensional substitution – all letters are mapped to a pattern of size at least 2 in all directions. Then the subshift $X_s$ is sofic.
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**Addendum**

Let $s$ be a non-degenerate deterministic multidimensional substitution – all letters are mapped to a pattern of size at least 2 in all directions. Then the subshift $X'_s$ is sofic.
Global idea

Principle for a $2 \times 2$ deterministic substitution
Global idea

Principle for a $2 \times 2$ deterministic substitution
Global idea

Principle for a $2 \times 2$ deterministic substitution
Sketch of the proof (I)

**Theorem (Mozes, 1989)**

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Sketch of the proof (I)

**Theorem (Mozes, 1989)**

Let $s$ be a non-degenerate deterministic multidimensional substitution – all letters are mapped to a pattern of size at least 2 in all directions. Then the subshift $X_s$ is sofic.

We construct a set of Wang tiles $\tau$ that contains

- **letter tiles**, on which letters from $A$ are written
- **arrows, junction and lines tiles**, that transmit and gather information between letter tiles

and such that $\pi(X_\tau) = X_s$. 
Sketch of the proof (II)

**Letter tiles:** They contain

1. a letter \( a \in A \)
2. a substitution rule \( b \rightarrow p \) s.t. \( a \) appears in pattern \( p \)
3. the position \((i, j)\) of \( a \) inside the pattern \( p \)

\[
\begin{array}{c|c|c|c}
& b & a & b \\
\hline
b & b & a & b \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
& a \\
\hline
(2, 1) & b & a & b \\
\hline
b & b & a & b \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
& a \\
\hline
(2, 2) & b & a & b \\
\hline
b & b & a & b \\
\end{array}
\]
Sketch of the proof (II)

**Letter tiles:** They contain

1. A letter $a \in A$
2. A substitution rule $b \rightarrow p$ s.t. $a$ appears in pattern $p$
3. The position $(i, j)$ of $a$ inside the pattern $p$

**Parity tiles:** They ensure that letter tiles only appear on the lattice $2\mathbb{Z} \times 2\mathbb{Z}$.
Sketch of the proof (III)

**Arrows, Junction and Lines tiles:** These tiles carry information and ensure the global structure of the tiling.
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Let $s$ be a non-degenerate deterministic multidimensional substitution – all letters are mapped to a pattern of size at least 2 in all directions. Then the subshift $X_s$ is sofic.
Wang tiles model

Robinson’s tiling

Substitutive subshifts

Mozes theorem

Sketch of the proof (V)

Theorem (Mozes, 1989)

Let $s$ be a non-degenerate deterministic multidimensional substitution – all letters are mapped to a pattern of size at least 2 in all directions. Then the subshift $X_s$ is sofic.

Proof:

- $X_s \subseteq \pi(X_{\tau})$: encode the history (all pre-images) into a tiling.
- $\pi(X_{\tau}) \subseteq X_s$: decompose any tiling into blocks and check that the pre-image is also a valid tiling.
How to use Mozes theorem?

Mozes result can be used every time you want to construct sofic subshifts with an hierarchical structure given by a substitution (most of the time to perform Turing machine calculations):

- prove the undecidability of the Domino Problem
- prove that $\mathbb{Z}$-effective subshifts can be embedded inside $\mathbb{Z}^2$-sofic subshifts
How to use Mozes theorem?

Mozes result can be used every time you want to construct sofic subshifts with an hierarchical structure given by a substitution (most of the time to perform Turing machine calculations):

- prove the undecidability of the Domino Problem
- prove that $\mathbb{Z}$-effective subshifts can be embedded inside $\mathbb{Z}^2$-sofic subshifts

Of course the result has mostly theoretical interest, since the number of tiles needed is really huge!
Important ideas in this course

- 1D to 2D $\Rightarrow$ SFTs and sofic subshifts are fundamentally different
- Link between aperiodicity and a decision problem
- Aperiodicity can be forced by local rules in dimension 2
- Substitutive systems can be enforced by local rules.