Small aperiodic tilings and entropy

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Source of aperiodicity

Consider a bi-infinite sequence x_n such that :
$x_{n+1} = x_n \times 2$ or $x_n \times \frac{1}{3}$ pour tout n.

Such a sequence is aperiodic or null.

Assume $x_n = x_m \neq 0$ then there are a and b such that $3^a = 2^b$ and $a + b = n - m$.

Tilings

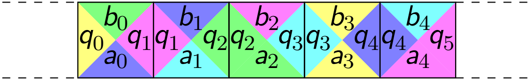
We can see tilings as sequences of horizontal lines, link each color to an integer, then the average of each line is a real.

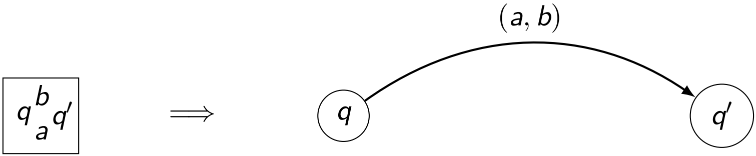
Lemma

Every tiling such that the average each line is not zero and wether twice or a third of the previous one is aperiodic.

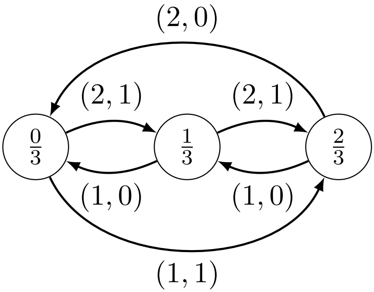
Transductors

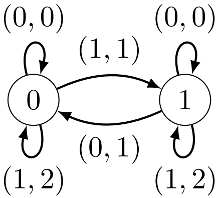
One can also see tilings as transductors between the lines.



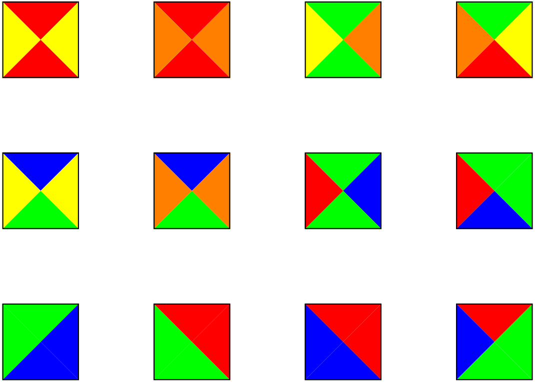


Multiplication transductors





The tileset



Existence of a tiling

- Mechanical words of average between $1/3$ and 2 are recognized by at least one of the transductors.
- For each mechanical word we can construc an other one with average twice or $1/3$, such that the pair is accepted by one transductor.
- A bi-infinite sequence of such words is a tiling of the plane.

Aperiodicity

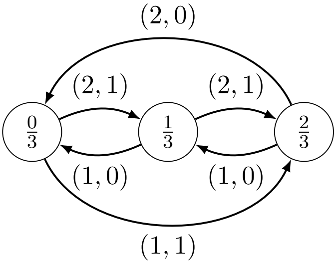
Lemma

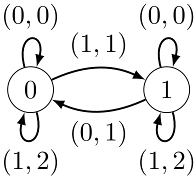
All the lines have an average.

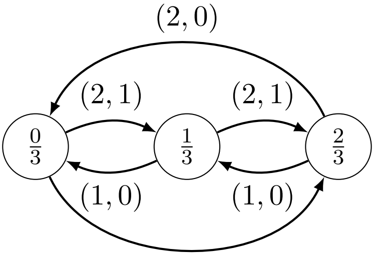
Corollary

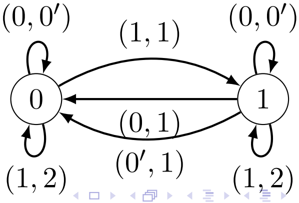
Each tiling with this tileset is wether aperiodic or null.

Forbid the zero-tiling

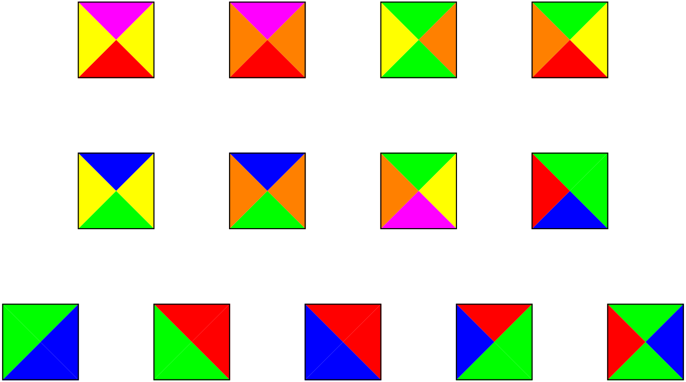




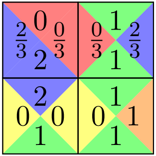


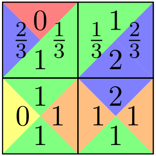


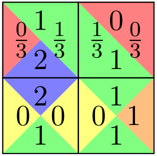
The final tileset

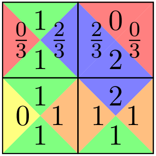


Two substitutive pairs

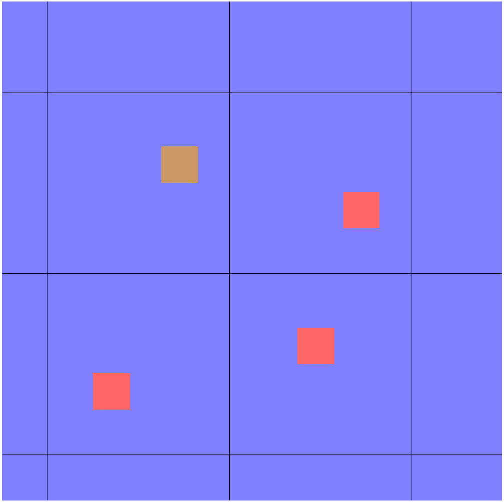
$A =$ 

$A' =$ 

$B =$ 

$B' =$ 

Density of the pairs

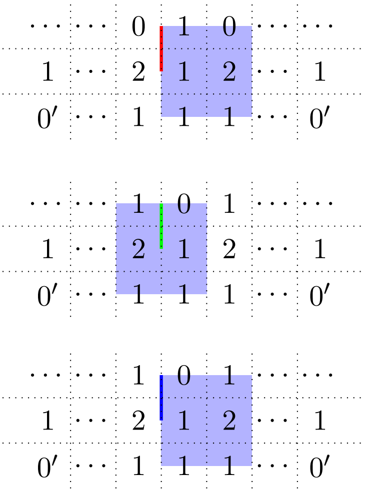


Origins of the pairs

Lemma

One element of thos paris appear above each pattern of the form $01^\alpha 0$ *with* $\alpha > 3$.

Middle case



Leftmost case

1	0	1	0	1	$\cdots$	$\cdots$
1	1	2	2	2	$\cdots$	1
$0'$	1	1	1	1	$\cdots$	$0'$

0	1	0	1	1	$\cdots$	$\cdots$
1	1	2	2	2	$\cdots$	1
$0'$	1	1	1	1	$\cdots$	$0'$

0	0	1	1	0	$\cdots$	$\cdots$
1	1	2	2	2	$\cdots$	1
$0'$	1	1	1	1	$\cdots$	$0'$

Rightmost case

···	···	···	0	1	0	1
1	···	2	2	2	1	1
$0'$	···	1	1	1	1	$0'$

···	···	···	1	1	0	0
1	···	2	2	2	1	1
$0'$	···	1	1	1	1	$0'$

···	···	···	1	0	1	0
1	···	2	2	2	1	1
$0'$	···	1	1	1	1	$0'$

Horizontal density of those patterns

Lemma

There is a constant k such that in any pattern of size k of any line of average between $4/5$ *and* $9/10$*, one pattern of the form* $01^{\alpha}0$ *with* $\alpha > 3$*.*

Vertical density of line with wanted average

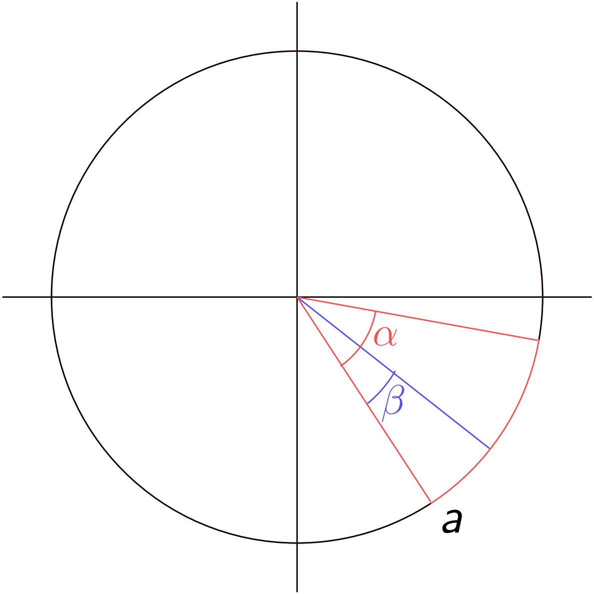
$f : [\frac{1}{3}; 2] \mapsto [\frac{1}{3}; 2]$ such that

$$f(x) = \begin{cases} 2x & \text{if } x \in [\frac{1}{3}; 1] \\ \frac{1}{3}x & \text{if } x \in [1; 2] \end{cases}$$

Lemma

Given any interval the maximal number of iterations of f between two occurences in this interval is bounded.

Remark : This function can be mapped to an irrationnal rotation of the circle.


α
β
a

Putting all together

- Consider *K* lconsecutive lines, at least one have average between $4/5$ and $9/10$.
- In any pattern of size *K* of any line of average between $4/5$ and $9/10$, one pattern of the form $01^{\alpha}0$ with $\alpha > 3$.
- One element of one substitutive pair appear each time such a pattern appear.
- This tileset have positive entropy.

Open problems

- Is one of those pairs dense alone in a given tiling ?
- Forbid one pattern of each substitutive pair. Has the new tileset positive entropy ?
- Is it possible to better characterize the Kari-words ?