Characterizing possible asymptotic behaviours of cellular automata

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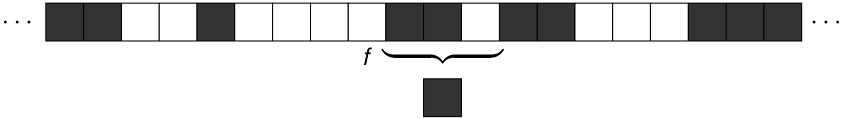
Definitions

- $\mathcal{A}, \mathcal{B}$ finite **alphabets**;
- $\mathcal{A}^*$ the (finite) **words**;
- $\mathcal{A}^{\mathbb{Z}}$ the **configurations**;
- σ the **shift action** $\sigma(a)_i = a_{i-1}$;

A **cellular automaton** is an action $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by a **local rule** $f : \mathcal{A}^{\mathbb{U}} \to \mathcal{A}$ on some neighbourhood $\mathbb{U}$.

For $\mathcal{A} = \{\blacksquare, \square\}$ and $\mathbb{U} = \{-1, 0, 1\}$:



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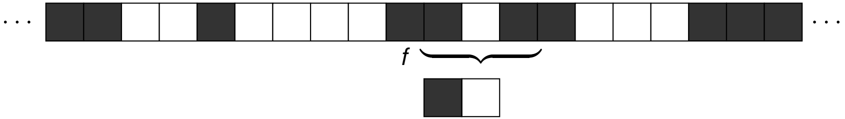

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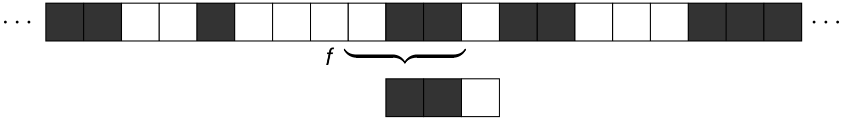

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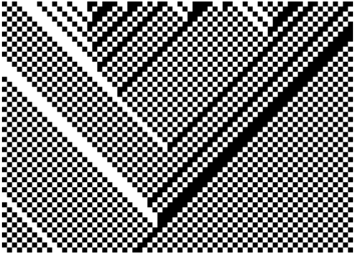
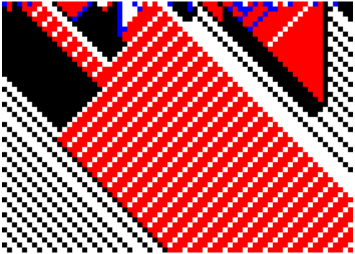
Simulations and typical asymptotic behaviour

Traffic automaton



Captive automaton



3-state cyclic automaton

Additive automaton

Measure space

$\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ the **σ-invariant probability measures** on $\mathcal{A}^\mathbb{Z}$.

$\mu([u])$ the probability that a word $u \in \mathcal{A}^*$ appears, for $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$.

Examples

Bernoulli (i.i.d) measures Let $(\lambda_a)_{a\in\mathcal{A}}$ such that $\sum \lambda_a = 1$.

$$\forall u \in \mathcal{A}^*, \mu([u]) = \prod_{i=0}^{|u|-1} \lambda_{u_i}.$$

Measures supported by a periodic orbit For a finite word w,

$$\widehat{\delta_w} = \frac{1}{|w|} \sum_{i=0}^{|w|-1} \delta_{\sigma^i(^\infty w^\infty)}.$$

Markov measures with finite memory.

Action of an automaton on an initial measure

- F extends to an action $F_* : \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$:

$$\boxed{F_*\mu(U) = \mu(F^{-1}U)}$$

for any borelian U.

- For an initial measure μ, $F_*^t\mu$ describes the repartition at time t;
- Typical asymptotic behaviour is well described by the limit(s) of $(F_*^t\mu)_{t\in\mathbb{N}}$ in the **weak-* topology**:

$$\boxed{F_*^t\mu \underset{t\to\infty}{\longrightarrow} \nu \quad \Leftrightarrow \quad \forall u \in \mathcal{A}^*, F_*^t\mu([u]) \to \nu([u]).}$$

Examples of asymptotic behaviour

Examples of asymptotic behaviour

Proposition

Let μ be the uniform Bernoulli measure on $\{0, 1, 2\}$ and F the 3-state cyclic automaton.

$$\boxed{F_*^t\mu \to \frac{1}{3}\widehat{\delta_0} + \frac{1}{3}\widehat{\delta_1} + \frac{1}{3}\widehat{\delta_2}.}$$

Main question

Question

Which measures ν are reachable as the limit of the sequence $(F_*^t\mu)_{t\in\mathbb{N}}$ for some cellular automaton F and initial measure μ?

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Answer

All (take $F = Id$ and $\mu = \nu$).

Main question

Better question

Which measures ν are reachable as the limit of the sequence $(F_*^t\mu)_{t\in\mathbb{N}}$ for some cellular automaton F and **simple** initial measure μ (e.g. the uniform Bernoulli measure)?

In a sense, this would correspond to the "physically relevant" measure for F.

Section 2

Necessary conditions: computability obstructions

Topological obstructions

Topological obstruction

The accumulation points of $(F_*^t\mu)_{t\in\mathbb{N}}$ form a nonempty and **compact** set.

Measures and computability

$f : \mathbb{N} \to \mathbb{N}$ is **computable** if there exists a Turing machine that, on any input $n \in \mathbb{N}$, stops and outputs $f(n)$ (up to encoding).

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A probability measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is:

computable if $\boxed{u \to \mu([u]) \text{ is computable}}$,

i.e. if there exists $f : \mathcal{A}^* \times \mathbb{N} \to \mathbb{Q}$ computable such that

$$\boxed{|\mu([u]) - f(u,n)| < 2^{-n}}.$$

($\Leftrightarrow$ can be **simulated** by a probabilistic Turing machine)

Examples of computable measures

- Any periodic orbit measure;
- Any Bernoulli or Markov measure with computable parameters.

Measures and computability

$f : \mathbb{N} \to \mathbb{N}$ is **computable** if there exists a Turing machine that, on any input $n \in \mathbb{N}$, stops and outputs $f(n)$ (up to encoding).

A probability measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is:

semi-computable if there exists a computable function $f : \mathcal{A}^* \times \mathbb{N} \to \mathbb{Q}$ such that

$$\boxed{|\mu([u]) - f(u,n)| \underset{n\to\infty}{\longrightarrow} 0.}$$

($\Leftrightarrow$ **limit** of a computable sequence of measures)

Examples of computable measures

- Any periodic orbit measure;
- Any Bernoulli or Markov measure with computable parameters.

Computability obstruction

Action of an automaton on a computable measure

- If μ is computable, then $F_*^t\mu$ is **computable**;
- If μ is computable, and $F_*^t\mu \underset{t\to\infty}{\longrightarrow} \nu$, then ν is **semi-computable**.

Section 3

Sufficient conditions: construction of limit measures

State of the art

Theorem [Boyet, Poupet, Theyssier 06]

There is an automaton F such that the language of words u satisfying

$$F_*^t\mu([u]) \not\to 0$$

in **not computable** for any nondegenerate Bernoulli measure μ.

Theorem [Boyer, Delacourt, Sablik 10]

Let μ be the uniform Bernoulli measure.
For a large class of subshifts $U \subset \mathcal{A}^{\mathbb{Z}}$ (under computability conditions), there is an automaton F such that

$$\boxed{U = \overline{\bigcup_{\nu \in \mathcal{V}(F_*^t\mu)} supp(\nu)}.}$$

Main result

Action of an automaton on a computable measure

If μ is computable, and $F_*^t\mu \underset{t\to\infty}{\longrightarrow} \nu$, then ν is **semi-computable**.

Motto:

> *"The only obstruction is the computability obstruction"*

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Action of an automaton on a computable measure

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Motto:

"The only obstruction is the computability obstruction"

Theorem

Let ν be a **semi-computable** measure. There exists:

- an alpabet $\mathcal{B} \supset \mathcal{A}$
- a cellular automaton $F : \mathcal{B} \to \mathcal{B}$

such that, for any **ergodic** and **full-support** measure $\mu \in \mathcal{M}_\sigma(\mathcal{B}^\mathbb{Z})$,

$$\boxed{F_*^t\mu \underset{t\to\infty}{\longrightarrow} \nu}$$

Approximation by periodic orbits

Proposition

Measures supported by periodic orbits are dense in $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$.

Example: Uniform Bernoulli measure

$w_0 = 01$
$w_1 = 0011$
$w_2 = 00010111$
$w_3 = 0000110100101111$

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$w_0 = 01$
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Proposition

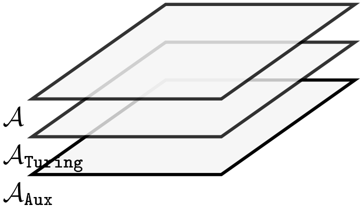
If $\nu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is semi-computable, there is a **computable** sequence of words $(w_n)_{n\in\mathbb{N}}$ such that $\widehat{\delta_{w_n}} \to \nu$.

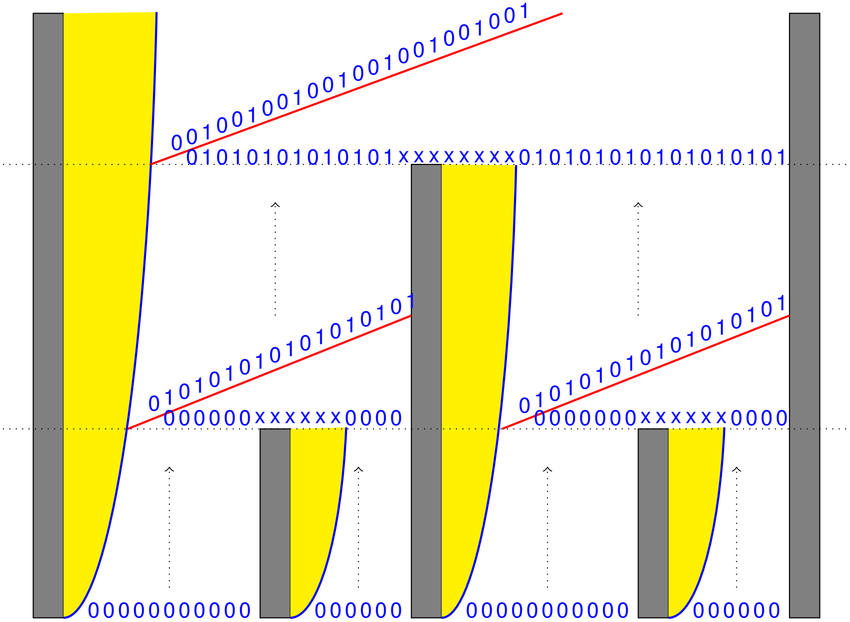
Our construction will compute each w_n and approach the measure $\widehat{\delta_{w_n}}$ by writing concatenated copies of w_n on all the configuration.

Idea of the construction

We expand the alphabet with additional layers to perform computation.

$\mathcal{B} =$




001001001001001001001001
01010101010101xxxxxxxx010101010101010101
010101010101010101
000000xxxxxx0000
010101010101010101
0000000xxxxxx0000
00000000000
000000
00000000000
000000

Section 4

Extensions and related results

Extensions and related results

Questions

1. Implementation of the construction? No (but for good reasons)

Implementation

- Non-trivial Turing machines satisfying space constraints;
- Large number of states;
  (for $|\mathcal{B}| = 2$, at least 2244 times more than the corresponding Turing machine)
- Speed of convergence $O\left(\frac{1}{\log t}\right)$ in the best case.

Extensions and related results

Questions

1. Implementation of the construction? No (but for good reasons)
2. No auxiliary states? Yes, if the target measure is not full-support

Theorem

Let ν be a **non full-support**, **semi-computable** measure.
Then there exists an automaton $F : \mathcal{A} \to \mathcal{A}$ such that, for any measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ **σ-mixing** and **full-support**,

$$\boxed{F_*^t \mu \underset{t\to\infty}{\longrightarrow} \nu.}$$

Extensions and related results

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Idea: use forbidden words to encode auxiliary states.

Remark

If $F_*^t\mu \to \nu$ where ν is a full support measure, then F is a **surjective** automaton and **the uniform Bernoulli measure is invariant**.

Extensions and related results

Questions

1. Implementation of the construction? No (but for good reasons)
2. No auxiliary states? Yes, if the target measure is not full-support
3. Sets of accumulation points? Yes, with a computability condition on compact sets

Theorem

Let $\mathcal{V}$ be a nonempty, compact, **connected**, Σ_2-**computable** set of measures.
Then there exists an automaton $F : \mathcal{A} \to \mathcal{A}$ such that, for any measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ $\boldsymbol{\sigma}$**-mixing** and **full-support**,

The set of accumulation points of $(F_*^t\mu)_{t\in\mathbb{N}}$ is $\mathcal{V}$.

Computability of compact sets

There are different non-equivalent ways of defining computability on compact sets. We use its distance function $d_{\mathcal{V}} : \mu \to \min_{\nu \in \mathcal{V}} d_{\mathcal{M}}(\mu, \nu)$, where:

$$d_{\mathcal{M}}(\mu_1, \mu_2) = \sum_{n=0}^{\infty} \frac{1}{2^n} \max_{u \in \mathcal{A}^n} |\mu_1([u]) - \mu_2([u])|$$

Computable compact set

$\mathcal{V}$ computable $\Leftrightarrow$ $d_{\mathcal{V}} : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \mapsto \mathbb{R}$ computable

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Computable compact set

$$\mathcal{V} \text{ computable} \quad \Leftrightarrow \quad d_{\mathcal{V}} : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \mapsto \mathbb{R} \text{ computable}$$

$$\Leftrightarrow \quad \exists f : \mathcal{A}^* \times \mathbb{N} \mapsto \mathbb{Q} \text{ computable,}$$

$$|d_{\mathcal{V}}(\widehat{\delta_w}) - f(w,n)| \leq \frac{1}{2^n}$$

and $\exists b : \mathbb{N} \mapsto \mathbb{Q}$ computable,

$$d_{\mathcal{M}}(\mu_1, \mu_2) < b(m) \Rightarrow |f(\mu_1, n) - f(\mu_2, n)| \leq \frac{1}{2^m}$$

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Σ_2-computable compact set ($\Leftrightarrow$ $\emptyset'$-lower-semi-computable compact set)

$\mathcal{V}$ Σ_2-computable $\quad\Leftrightarrow\quad$ $d_{\mathcal{V}} = \liminf d_i$

where d_i is a sequence of elements $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \mapsto \mathbb{R}$ with $\exists f : \mathbb{N} \times \mathcal{A}^* \times \mathbb{N} \mapsto \mathbb{Q}$ computable,

$$|d_i(\widehat{\delta_w}) - f(i, w, n)| \leq \frac{1}{2^n}$$

and $\exists b : \mathbb{N} \mapsto \mathbb{Q}$ computable,

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Extensions and related results

Questions

3. Sets of accumulation points? Yes, with a computability condition on compact sets
4. Cesaro mean convergence? Yes

Theorem

Let $\mathcal{V}' \subset \mathcal{V}$ be two compact, **connected**, Σ_2-**computable** sets of measures.
Then there exists an automaton $F : \mathcal{A} \to \mathcal{A}$ such that, for any measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ $\boldsymbol{\sigma}$**-mixing** and **full-support**,

The set of accumulation points of $(F_*^t \mu)_{t\in\mathbb{N}}$ is $\mathcal{V}$.

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7. Using the initial measure as an argument or an oracle? Some simple cases

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7. Using the initial measure as an argument or an oracle? Some simple cases
8. Higher dimensions? Work in progress with Martin Delacourt

Computation in the space of measures

Let us consider the operator

$$\mu \mapsto \text{accumulation points of } (F_*^t \mu)_{t \in \mathbb{R}}$$

The previous construction gave us operators that were essentially **constant** (on a large domain).

Question

Which operators $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ (ou $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}) \to \mathcal{P}(\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}))$) can be realized in this way?

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Theorem

Let $\nu : \mathbb{R} \to \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ be a **semi-computable** operator. There is:

- an alphabet $\mathcal{B} \supset \mathcal{A}$,
- an automaton $F : \mathcal{B}^\mathbb{Z} \to \mathcal{B}^\mathbb{Z}$

such that, for any **full-support** and **exponentially σ-mixing** measure μ,

$$\boxed{F_*^t\mu \underset{t\to\infty}{\longrightarrow} \nu\left(\mu\left(\boxed{\mathtt{I}}\right)\right)}.$$

Some examples

Let $M \subset \mathcal{M}_\sigma(\mathcal{B}^{\mathbb{Z}})$ be the set of **full-support**, **exponentially $\boldsymbol{\sigma}$-mixing** measures.

Example 1: Density classification

There exists an automaton $F : \mathcal{B}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ realizing the operator:

$$
\begin{aligned}
M &\to \mathcal{M}_\sigma(\{0,1\}^{\mathbb{Z}}) \\
\mu &\mapsto \begin{cases} \widehat{\delta_0} & \text{if } \mu(\boxed{1}) < \frac{1}{2} \\ \widehat{\delta_1} & \text{otherwise.} \end{cases}
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Example 2: A simple oracle

There exists an automaton $F : \mathcal{B}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ realizing the operator:

$$M \to \mathcal{M}_\sigma(\{0,1\}^{\mathbb{Z}})$$
$$\mu \mapsto Ber(\mu(\boxed{\text{I}}))$$

Implementation of a simple case

Fibonacci word

Consider the morphism :

$$\begin{array}{rccc} \varphi : & \mathcal{A}^* & \to & \mathcal{A}^* \\ & 0 & \mapsto & 01 \\ & 1 & \mapsto & 0 \end{array}$$

Then the sequence $\varphi^n(0)$ converges to an infinite word called **Fibonacci word**:

$$\varphi^\infty(0) = 0100101001001010010101\ldots$$

and it is **uniquely ergodic**.

An open question

Open question

For $d > 1$, the periodic measures

$$\{\widehat{\delta_u} \mid u \in \mathcal{A}^{n^d}\}$$

are dense in the set of σ-invariant measures $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}^d})$. Is the rate of convergence

$$\max_{\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}^d})} \min_{u \in \mathcal{A}^{n^d}} d_{\mathcal{M}}(\mu, \widehat{\delta_u})$$

a computable sequence, or does it have a computable upper bound?