Measures of maximal entropy of SFT on lattices and trees

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Motivation: understanding the combinatorics of multi-dimensional SFT, being able to generate patterns “uniformly”.
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Example: two-dimensional Fibonacci SFT

Set of configurations without two consecutive black squares, vertically or horizontally.
Outline of the talk

1. One-dimensional SFT and the Parry measure
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2. SFT defined on trees and $d$-Parry measures
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1. One-dimensional SFT and the Parry measure
2. SFT defined on trees and $d$-Parry measures
3. SFT and probabilistic cellular automata
Outline

1. One-dimensional SFT and the Parry measure
2. SFT defined on trees and $d$-Parry measures
3. SFT and probabilistic cellular automata
One-dimensional subshift of finite type

Let $\mathcal{A}$ be an alphabet with $n$ letters, and let $A \in M_n(\{0, 1\})$. 
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**Subshift of finite type**

The **subshift of finite type** associated to $A$ is the set $\Sigma_A$ of words $w \in \mathcal{A}^\mathbb{Z}$ such that if $A_{i,j} = 0$, $w$ does not contain the pattern $ij$.

\[ A_{i,j} = \begin{cases} 1 & \text{if } ij \text{ is an allowed pattern}, \\ 0 & \text{if } ij \text{ is a forbidden pattern}. \end{cases} \]

\[ \Sigma_A = \{ w \in \mathcal{A}^\mathbb{Z}; \forall k \in \mathbb{Z}, A_{w_k,w_{k+1}} = 1 \}. \]

In what follows, we assume that the matrix $A$ irreducible and aperiodic.
Let $\Sigma_A$ be a SFT, and let $W_k$ be the set of allowed words of length $k$. 
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Questions:

1. What is the growth rate of $|\mathcal{W}_k|$?
   Precisely, we would like to be able to compute the topological entropy of the SFT:

   $$h(\Sigma_A) = \lim_{k \to \infty} \frac{\log |\mathcal{W}_k|}{k}.$$
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$$h(\Sigma_A) = \lim_{k \to \infty} \frac{\log |\mathcal{W}_k|}{k}.$$ 

2. What do “typical” configurations look like? How to generate “uniformly” patterns of $\Sigma_A$?
Topological entropy

From Perron-Frobenius theorem, the matrix $A$ has an eigenvalue $\lambda > 0$ such that $|\mu| \leq \lambda$ for any other eigenvalue $\mu$.

**Proposition**

$$h(\Sigma_A) = \lim_{k \to \infty} \frac{\log |\mathcal{W}_k|}{k} = \log \lambda.$$
Furthermore, there is a unique choice of $r_1, \ldots, r_n \geq 0$ such that

$$\sum_{i=1}^{n} r_i = 1$$

and

$$A \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = \lambda \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}.$$
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\[
A \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = \lambda \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}.
\]

**Definition of the Parry measure**

The **Parry measure** is the Markov measure \( \pi \) of transition matrix \( P \) defined, for any \( i, j \in A \), by
\[
P_{i,j} = A_{i,j} \frac{r_j}{\lambda r_i}.
\]
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**Remark.**

\[
P_{i,j} = A_{i,j} \frac{r_j}{\sum_{k \in \mathcal{A}} A_{i,k} r_k} = A_{i,j} \frac{r_j}{\sum_{k \in S(i)} r_k},
\]

where \( S(i) = \{ k \in \mathcal{A}; A_{i,k} = 1 \} \).
Markov-uniform property of the Parry measure

**Proposition**

The Parry measure is **Markov-uniform**: for given $k \geq 1$ and $a, b \in \mathcal{A}$, the value

$$\pi(awb)$$

does not depend on the word $w \in \mathcal{A}^k$ such that $awb$ is allowed.
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**Proof.** By definition, $P_{i,j} = A_{i,j} \frac{r_j}{\lambda r_i}$. If $awb \in \mathcal{W}_{k+2}$, then:

$$\pi(awb) = \pi_a P_{a,w_1} P_{w_1,w_2} \ldots P_{w_{k-1},w_k} P_{w_k,b}$$
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$$= \pi_a \frac{r_{w_1}}{\lambda r_a} \frac{r_{w_2}}{\lambda r_{w_1}} \ldots \frac{r_{w_k}}{\lambda r_{w_{k-1}}} \frac{r_b}{\lambda r_{w_k}}$$
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\[
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\]

\[
= \frac{\pi_a r_b}{\lambda^{k+1} r_a}.
\]
Theorem

Let $\mathcal{M}_{\Sigma_A}$ be the set of translation invariant measures on the SFT $\Sigma_A$, and let $\pi \in \mathcal{M}_{\Sigma_A}$. The following properties are equivalent.

(i) $\pi$ is the Parry measure associated to $\Sigma_A$,
(ii) $\pi$ is a Markov-uniform measure on $\Sigma_A$,
(iii) $\pi$ is the measure of maximal entropy of $\Sigma_A$,
(iv) the entropy of $\pi$ is equal to the topological entropy $h(\Sigma_A)$.

Remark. On $\mathbb{Z}^d$, $d \geq 2$, there can be several measures of maximal entropy. But the equivalence between (ii), (iii), and (iv) can be extended to some multi-dimensional SFT.
Measure of maximal entropy

**Theorem**

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**Remark.** On $\mathbb{Z}^d$, $d \geq 2$, there can be several measures of maximal entropy. But the equivalence between (ii), (iii), and (iv) can be extended to some multi-dimensional SFT.
Example: one-dimensional Fibonacci SFT

Let $A = \{0, 1\}$. The one-dimensional Fibonacci SFT is the set of words that do not contain two consecutive 1’s. It is given by:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$  

Its topological entropy is equal to $\log \varphi$, where $\varphi = \frac{1+\sqrt{5}}{2}$.

The Parry measure is the Markov measure given by

$$\pi_0 = \frac{\varphi^2}{1+\varphi^2} \quad \text{and} \quad \pi_1 = \frac{1}{1+\varphi^2}.$$
First way to generate the Parry measure

The Parry measure of the Fibonacci SFT can be generated by:
- choosing independently to write a 0 with probability \( r_0 = \frac{1}{\varphi} \)
  and a 1 with probability \( r_1 = \frac{1}{\varphi^2} \),
- rejecting the 1's creating forbidden patterns.
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Lemma

For any SFT, the Parry measure can be generated by independent draws of letters with probability \( (r_i)_{i \in A} \), with reject of a letter if it creates a forbidden pattern.
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Lemma

For any SFT, the Parry measure can be generated by independent draws of letters with probability \( (r_i)_{i \in A} \), with reject of a letter if it creates a forbidden pattern.

Proof.

\[
P_{i,j} = A_{i,j} \frac{r_j}{\sum_{k \in S(i)} r_k}.
\]
Second way to generate the Parry measure

The Parry measure of the Fibonacci SFT can be generated by:

- choosing independently to write a 0 with probability \( \tilde{r}_0 = \frac{1}{\varphi^2} \)
- and a 1 with probability \( \tilde{r}_1 = \frac{1}{\varphi} \),
- deleting pairs of consecutive 1’s.
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Confluent SFT

A SFT is **confluent** if for any $i, j, k \in \mathcal{A}$ such that both $ij$ and $jk$ are forbidden, then $i = k$. 

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Second way to generate the Parry measure

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A SFT is **confluent** if for any $i, j, k \in \mathcal{A}$ such that both $ij$ and $jk$ are forbidden, then $i = k$.

Proposition

For **confluent** SFT, the Parry measure can be generated by independent draws of letters and deletion of forbidden patterns.
Outline

1. One-dimensional SFT and the Parry measure
2. SFT defined on trees and $d$-Parry measures
3. SFT and probabilistic cellular automata
Let $A$ be a (symmetric) matrix defining allowed and forbidden patterns, and consider the corresponding SFT $\Sigma^d_A$ on the infinite regular tree of degree $d + 1$. 

Question: how to construct Markov-uniform measures on $\Sigma^d_A$?
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**Question:** how to construct Markov-uniform measures on \( \Sigma^d_A \)?
Let $A$ be a (symmetric) matrix defining allowed and forbidden patterns, and consider the corresponding SFT $\Sigma^d_A$ on the infinite regular tree of degree $d + 1$.

**Question:** how to construct Markov-uniform measures on $\Sigma^d_A$?
**Idea 1:** consider a (reversible) Markov chain $P$ on the alphabet $\mathcal{A}$, of stationary distribution $\pi$.
Choose the letter at one given vertex according to $\pi$ and then label the other vertices using $P$. 
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Example:

Probability of this pattern: $\pi(i)P_{i,j}P_{j,k}P_{j,l}$
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**Example:**

![Diagram of a tree with labels $i$, $j$, $k$, and $l$.]

Probability of this pattern:

$$\pi(i)P_{i,j}P_{j,k}P_{j,l} = \pi(j)P_{j,i}P_{j,k}P_{j,l} = \ldots$$
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Example:

For given $i, k, l$, we want this value to be independent of the letter $j$ such that the pattern is allowed.

Probability of this pattern:

$$\pi(i) P_{i,j} P_{j,k} P_{j,l} = \pi(j) P_{j,i} P_{j,k} P_{j,l} = \ldots$$
**Idea 2:** like for the Parry measure, choose $P$ under the form:

$$P_{i,j} = A_{i,j} \frac{r_j}{\sum_{s \in \mathcal{A}} A_{i,s} r_s} = A_{i,j} \frac{r_j}{\sum_{s \in S(i)} r_s},$$

for some probability vector $(r_i)_{i \in \mathcal{A}}$. 
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Then,

$$\pi(i) P_{ij} P_{jk} P_{kl} = \pi(i) \frac{r_j}{\sum_{s \in A} A_{i,s} r_s} \frac{r_k}{\sum_{s \in A} A_{j,s} r_s} \frac{r_l}{\sum_{s \in A} A_{j,s} r_s}.$$
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\]

Let us try to choose \((r_i)_{i \in \mathcal{A}}\) such that:

\[
\sum_{s \in \mathcal{A}} A_{j,s} r_s = \lambda r_j^{1/2},
\]

for any \( j \in \mathcal{A} \)!
For a tree of degree $d + 1$, the problem is to find a probability distribution $(r_i)_{i \in A}$ such that for some $\lambda > 0$,

$$A \begin{pmatrix} r_1 \\
... \\
\vdots \\
r_n \end{pmatrix} = \lambda \begin{pmatrix} r_1^{1/d} \\
... \\
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**Proposition**

Let $A$ be an irreducible non-negative matrix, and let $d \geq 1$. There exist $\lambda > 0$ and $r_1, \ldots, r_n > 0$ satisfying $\sum_{i=1}^{n} r_i = 1$ and:

$$A \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = \lambda \begin{pmatrix} r_1^{1/d} \\ \vdots \\ r_n^{1/d} \end{pmatrix}.$$
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*Proof.* Fixed point theorem.
For a tree of degree $d + 1$, the problem is to find a probability distribution $(r_i)_{i \in A}$ such that for some $\lambda > 0$,

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**Proposition**

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**Proof.** Fixed point theorem.

**Remark.** $\lambda$ and $(r_i)_{i \in A}$ may not be unique.
Proposition

If the distribution of probability \((r_i)_{i \in A}\) satisfies

\[
A \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = \lambda \begin{pmatrix} r_1^{1/d} \\ \vdots \\ r_n^{1/d} \end{pmatrix}
\]

for some \(\lambda > 0\), then the Markov chain given by:

\[
P_{i,j} = A_{i,j} \frac{r_j}{\sum_{s \in A} A_{i,s} r_s} = A_{i,j} \frac{r_j}{\lambda r_i^{1/d}},
\]

defines a **Markov-uniform** measure on the SFT \(\Sigma_A\).
Proposition

If the distribution of probability \((r_i)_{i \in A}\) satisfies

\[
\begin{pmatrix}
    r_1 \\
    \vdots \\
    r_n
\end{pmatrix} = \lambda
\begin{pmatrix}
    r_1^{1/d} \\
    \vdots \\
    r_n^{1/d}
\end{pmatrix}
\]

for some \(\lambda > 0\), then the Markov chain given by:

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P_{i,j} = A_{i,j} \frac{r_j}{\sum_{s \in A} A_{i,s} r_s} = A_{i,j} \frac{r_j}{\lambda r_i^{1/d}},
\]

defines a **Markov-uniform** measure on the SFT \(\Sigma_A\). We call such a measure a **d-Parry measure**.
Proposition

If the distribution of probability \((r_i)_{i \in A}\) satisfies

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A \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = \lambda \begin{pmatrix} r_1^{1/d} \\ \vdots \\ r_n^{1/d} \end{pmatrix}
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for some \(\lambda > 0\), then the Markov chain given by:

\[
P_{i,j} \propto A_{i,j} \frac{r_j}{\sum_{s \in A} A_{i,s} r_s} = A_{i,j} \frac{r_j}{\lambda r_i^{1/d}},
\]

defines a **Markov-uniform** measure on the SFT \(\Sigma_A\).
We call such a measure a \(d\)-**Parry measure**.

**Remark.** The reversible invariant measure is given by \(\pi_i = \gamma r_i^{1+1/d}\).
Example: Fibonacci SFT on trees

We search $P = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 & 0 \end{pmatrix}$, such that

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ 1 - \alpha \end{pmatrix} = \lambda \begin{pmatrix} \alpha^{1/d} \\ (1 - \alpha)^{1/d} \end{pmatrix}$.

For any $d \geq 1$, there exists a unique $d$-Parry measure, which is given by $r_0 = \alpha$ and $r_1 = 1 - \alpha$, where $\alpha$ is the unique positive solution of the equation

$\alpha^{d+1} = 1 - \alpha$.

For $d = 1$, we recover $r_0 = \frac{1}{\phi}$ and $r_1 = \frac{1}{\phi^2}$.
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For any \( d \geq 1 \), there exists a unique \( d \)-Parry measure, which is given by \( r_0 = \alpha \) and \( r_1 = 1 - \alpha \), where \( \alpha \) is the unique positive solution of the equation

\[
\alpha^{d+1} = 1 - \alpha.
\]

For \( d = 1 \), we recover \( r_0 = \frac{1}{\varphi} \) and \( r_1 = \frac{1}{\varphi^2} \).
But we also have examples of SFT having several \( d \)-Parry measures...
**Question:** is there a relation between these $d$-Parry measures and some analogue of entropy on trees?
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**\( f \)-invariant of Bowen**

The \textbf{\( f \)-invariant} is a measure-conjugacy invariant, introduced by L. Bowen to generalize the theory of entropy to free group actions.

For a Markov measure, its value is given by:

\[
d_n \sum_{i=1}^{\pi(i)} \log(\pi(i)) - \frac{d+1}{2} n \sum_{i=1}^{\pi(i)} \sum_{j=1}^{\pi(i)} \pi(i) P_{i,j} \log(\pi(i)) P_{i,j} = d - 1 \frac{1}{2} n \sum_{i=1}^{\pi(i)} \log(\pi(i)) - \frac{d+1}{2} n \sum_{i=1}^{\pi(i)} \sum_{j=1}^{\pi(i)} \pi(i) P_{i,j} \log(\pi(i)) P_{i,j}.
\]

Markov measures maximize the \( f \)-invariant.
Question: is there a relation between these $d$-Parry measures and some analogue of entropy on trees?

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$$d \sum_{i=1}^{n} \pi(i) \log(\pi(i)) - \frac{d + 1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi(i)P_{i,j} \log P_{i,j}$$

$$= \frac{d - 1}{2} \sum_{i=1}^{n} \pi(i) \log(\pi(i)) - \frac{d + 1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi(i)P_{i,j} \log P_{i,j}.$$
**Question:** is there a relation between these $d$-Parry measures and some analogue of entropy on trees?

### $f$-invariant of Bowen

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\begin{align*}
& d \sum_{i=1}^{n} \pi(i) \log(\pi(i)) - \frac{d + 1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi(i) P_{i,j} \log \pi(i) P_{i,j} \\
= & \frac{d - 1}{2} \sum_{i=1}^{n} \pi(i) \log(\pi(i)) - \frac{d + 1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi(i) P_{i,j} \log P_{i,j}.
\end{align*}
$$

Markov measures maximise the $f$-invariant.
Proposition

If a measure on $\Sigma_A^d$ maximises the $f$-invariant, then it is a $d$-Parry measure.
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Questions:

- If there are several $d$-Parry measures, how to guess which ones maximise the $f$-invariant?
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Questions:

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- Is there a relation between all this and some growth rate of the number of allowed patterns?
Outline

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2. SFT defined on trees and $d$-Parry measures
3. SFT and probabilistic cellular automata
Example: one-dimensional Fibonacci SFT

Consider a configuration distributed according to the Parry measure $\pi$ of the Fibonacci SFT.

$$X_{-2} \rightarrow X_{-1} \rightarrow X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \rightarrow X_5 \rightarrow X_6$$
Example: one-dimensional Fibonacci SFT

Consider a configuration distributed according to the Parry measure $\pi$ of the Fibonacci SFT.

For all $i \in \mathbb{Z}$, if $X_{2i} = X_{2i+2} = 0$, we flip the value of $X_{2i+1}$ with probability $1/2$.
By the Markov-uniform property, the new sequence is still distributed according to $\pi$. 
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\[
\begin{array}{c|c|c}
1 & \text{with probability} & 1/2 \\
0 & \text{with probability} & 1/2 \\
0 & \text{(with probability} & 1 \\
\hline
0 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

\[ \begin{array}{c}
Y - 1 \\
Z - 1 \\
Y_0 \\
Z_0 \\
Y_1 \\
Z_1 \\
Y_2 \\
Z_2 \\
Y_3 \\
\end{array} \]

The projection \( \pi_2 \) of the Parry measure on odd (resp. even) sites is an invariant measure of the probabilistic cellular automaton.
General one-dimensional SFT

For a general one-dimensional SFT $\Sigma_A$, let us consider the PCA $F_A$ defined by:

\[
k \text{ with proba } \frac{1}{|\{s \in A; isj \in W_3\}|} \text{ if } ikj \in W_3
\]

(and with proba 0 otherwise)

Proposition

The projection $\pi_2$ of the Parry measure on odd (resp. even) sites is an invariant measure of the PCA $F_A$.

From $\pi_2$, we recover $\pi$ by one application of the PCA.
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The same result holds on $\mathbb{Z}^d$, $d \geq 2$ and on infinite trees (bipartite graphs).
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Two-dimensional case.
Back to the two-dimensional Fibonacci SFT
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- Using PCA, new ideas for **perfect sampling** of patterns according to the measure of maximal entropy of the SFT.
Back to the two-dimensional Fibonacci SFT

- Using PCA, new ideas for **perfect sampling** of patterns according to the measure of maximal entropy of the SFT.
- And an interesting detour through SFT defined on trees...
Back to the two-dimensional Fibonacci SFT

- Using PCA, new ideas for **perfect sampling** of patterns according to the measure of maximal entropy of the SFT.
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- Extension to sofic SFT?