Measures of maximal entropy of SFT on lattices and trees

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Motivation: understanding the combinatorics of multi-dimensional SFT, being able to generate patterns "uniformly".

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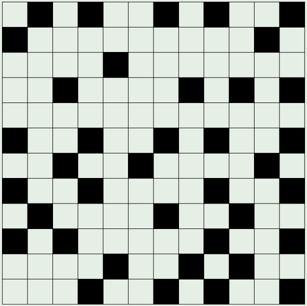
Example: two-dimensional Fibonacci SFT

Set of configurations without two consecutive black squares, vertically or horizontally.



Outline of the talk

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1. One-dimensional SFT and the Parry measure
2. SFT defined on trees and d-Parry measures

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2. SFT defined on trees and *d*-Parry measures
3. SFT and probabilistic cellular automata

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Subshift of finite type

The **subshift of finite type** associated to A is the set Σ_A of words $w \in \mathcal{A}^{\mathbb{Z}}$ such that if $A_{i,j} = 0$, w does not contain the pattern ij.

$$A_{i,j} = \begin{cases} 1 \text{ if } ij \text{ is an allowed pattern,} \\ 0 \text{ if } ij \text{ is a forbidden pattern.} \end{cases}$$

$$\Sigma_A = \{w \in \mathcal{A}^{\mathbb{Z}}; \forall k \in \mathbb{Z}, A_{w_k, w_{k+1}} = 1\}.$$

In what follows, we assume that the matrix A irreducible and aperiodic.

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1. What is the growth rate of $|\mathcal{W}_k|$?
   Precisely, we would like to be able to compute the topological entropy of the SFT:

$$h(\Sigma_A) = \lim_{k \to \infty} \frac{\log |\mathcal{W}_k|}{k}.$$

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2. What do "typical" configurations look like?
   How to generate "uniformly" patterns of Σ_A?

Topological entropy

From Perron-Frobenius theorem, the matrix A has an eigenvalue $\lambda > 0$ such that $|\mu| \leq \lambda$ for any other eigenvalue μ.

Proposition

$$h(\Sigma_A) = \lim_{k\to\infty} \frac{\log|\mathcal{W}_k|}{k} = \log\lambda.$$

Furthermore, there is a unique choice of $r_1, \ldots, r_n \geq 0$ such that $\sum_{i=1}^{n} r_i = 1$ and

$$A \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = \lambda \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}.$$

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Definition of the Parry measure

The **Parry measure** is the Markov measure π of transition matrix P defined, for any $i, j \in \mathcal{A}$, by

$$P_{i,j} = A_{i,j} \frac{r_j}{\lambda r_i}.$$

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Remark.

$$P_{i,j} = A_{i,j} \frac{r_j}{\sum_{k \in \mathcal{A}} A_{i,k} r_k} = A_{i,j} \frac{r_j}{\sum_{k \in \mathcal{S}(i)} r_k},$$

where $\mathcal{S}(i) = \{k \in \mathcal{A}; A_{i,k} = 1\}$.

Markov-uniform property of the Parry measure

Proposition

The Parry measure is **Markov-uniform**: for given $k \geq 1$ and $a, b \in \mathcal{A}$, the value

$$\pi(awb)$$

does not depend on the word $w \in \mathcal{A}^k$ such that awb is allowed.

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Proof. By definition, $P_{i,j} = A_{i,j} \frac{r_j}{\lambda r_i}$. If $awb \in \mathcal{W}_{k+2}$, then:

$$\pi(awb) = \pi_a P_{a,w_1} P_{w_1,w_2} \dots P_{w_{k-1},w_k} P_{w_k,b}$$

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$$= \frac{\pi_a r_b}{\lambda^{k+1} r_a}.$$

Measure of maximal entropy

Theorem

Let $\mathcal{M}_{\Sigma_A}$ be the set of translation invariant measures on the SFT Σ_A, and let $\pi \in \mathcal{M}_{\Sigma_A}$. The following properties are equivalent.

(i) π is the **Parry measure** associated to Σ_A,

(ii) π is a **Markov-uniform** measure on Σ_A,

(iii) π is the measure of **maximal entropy** of Σ_A,

(iv) the entropy of π is equal to the **topological entropy** $h(\Sigma_A)$.

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(iv) the entropy of π is equal to the **topological entropy** $h(\Sigma_A)$.

Remark. On $\mathbb{Z}^d, d \geq 2$, there can be several measures of maximal entropy. But the equivalence between *(ii)*, *(iii)*, and *(iv)* can be extended to some multi-dimensional SFT.

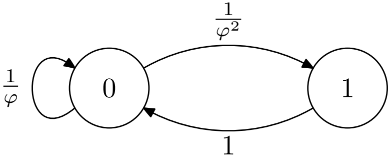
Example: one-dimensional Fibonacci SFT

Let $\mathcal{A} = \{0, 1\}$. The one-dimensional Fibonacci SFT is the set of words that do not contain two consecutive 1's. It is given by:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its topological entropy is equal to $\log \varphi$, where $\varphi = \frac{1+\sqrt{5}}{2}$.

The Parry measure is the Markov measure given by



with $\pi_0 = \frac{\varphi^2}{1+\varphi^2}$ and $\pi_1 = \frac{1}{1+\varphi^2}$.

First way to generate the Parry measure

The Parry measure of the Fibonacci SFT can be generated by:

- choosing independently to write a 0 with probability $r_0 = \frac{1}{\varphi}$ and a 1 with probability $r_1 = \frac{1}{\varphi^2}$,
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For any SFT, the Parry measure can be generated by independent draws of letters with probability $(r_i)_{i \in \mathcal{A}}$, with reject of a letter if it creates a forbidden pattern.

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Proof.

$$P_{i,j} = A_{i,j}\frac{r_j}{\sum_{k\in\mathcal{S}(i)} r_k}.$$

Second way to generate the Parry measure

The Parry measure of the Fibonacci SFT can be generated by:

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- deleting pairs of consecutive 1's.

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Confluent SFT

A SFT is **confluent** if for any $i, j, k \in \mathcal{A}$ such that both ij and jk are forbidden, then $i = k$.

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Proposition

For **confluent** SFT, the Parry measure can be generated by independent draws of letters and deletion of forbidden patterns.

Outline

Let A be a (symmetric) matrix defining allowed and forbidden patterns, and consider the corresponding SFT Σ_A^d on the infinite regular tree of degree $d+1$.

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Question: how to construct Markov-uniform measures on Σ_A^d?

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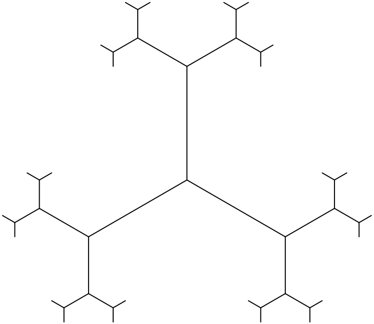

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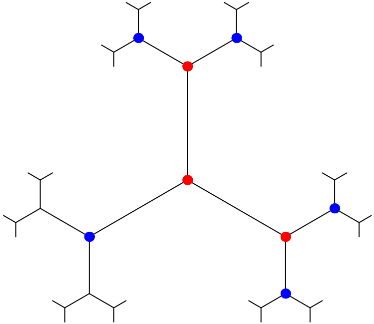

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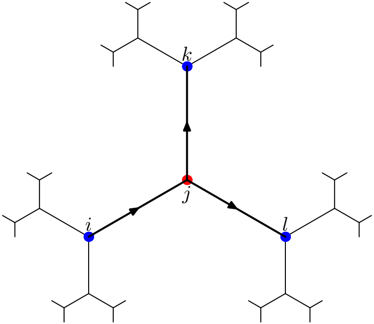

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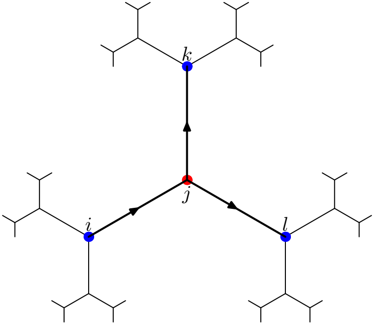

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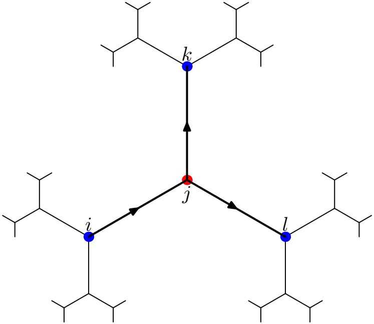

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For given i, k, l, we want this value to be independent of the letter j such that the pattern is allowed.

Idea 2: like for the Parry measure, choose P under the form:

$$P_{i,j} = A_{i,j} \frac{r_j}{\sum_{s \in \mathcal{A}} A_{i,s} r_s} = A_{i,j} \frac{r_j}{\sum_{s \in \mathcal{S}(i)} r_s},$$

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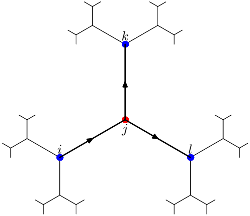

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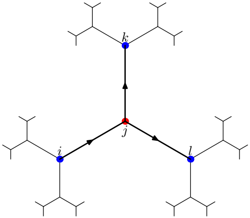
Let us try to choose $(r_i)_{i \in \mathcal{A}}$ such that:

$$\sum_{s \in \mathcal{A}} A_{j,s} r_s = \lambda {r_j}^{1/2},$$

for any $j \in \mathcal{A}$!



For a tree of degree $d+1$, the problem is to find a probability distribution $(r_i)_{i \in \mathcal{A}}$ such that for some $\lambda > 0$,

$$A \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = \lambda \begin{pmatrix} r_1^{1/d} \\ \vdots \\ r_n^{1/d} \end{pmatrix}.$$

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Proposition

Let A be an irreducible non-negative matrix, and let $d \geq 1$. There exist $\lambda > 0$ and $r_1, \ldots, r_n > 0$ satisfying $\sum_{i=1}^{n} r_i = 1$ and:

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Proof. Fixed point theorem.

Remark. λ and $(r_i)_{i\in\mathcal{A}}$ may not be unique.

Proposition

If the distribution of probability $(r_i)_{i\in\mathcal{A}}$ satisfies

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for some $\lambda > 0$, then the Markov chain given by:

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Remark. The reversible invariant measure is given by $\pi_i = \gamma r_i^{1+1/d}$.

Example: Fibonacci SFT on trees

We search $P = \begin{pmatrix} \alpha & 1-\alpha \\ 1 & 0 \end{pmatrix}$, such that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} = \lambda \begin{pmatrix} \alpha^{1/d} \\ (1-\alpha)^{1/d} \end{pmatrix}.$$

For any $d \geq 1$, there exists a unique d-Parry measure, which is given by $r_0 = \alpha$ and $r_1 = 1 - \alpha$, where α is the unique positive solution of the equation

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For $d = 1$, we recover $r_0 = \frac{1}{\varphi}$ and $r_1 = \frac{1}{\varphi^2}$.
But we also have examples of SFT having several d-Parry measures...

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$$d\sum_{i=1}^{n}\pi(i)\log(\pi(i)) - \frac{d+1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\pi(i)P_{i,j}\log\pi(i)P_{i,j}$$

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Markov measures maximise the f-invariant.

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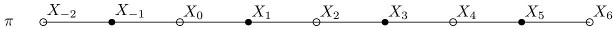
Questions:

- If there are several d-Parry measures, how to guess which ones maximise the f-invariant?
- Is there a relation between all this and some growth rate of the number of allowed patterns?

Outline

Example: one-dimensional Fibonacci SFT

Consider a configuration distributed according to the Parry measure π of the Fibonacci SFT.



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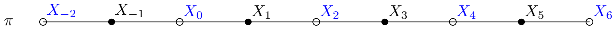

For all $i \in \mathbb{Z}$, if $X_{2i} = X_{2i+2} = 0$, we flip the value of X_{2i+1} with probability $1/2$.

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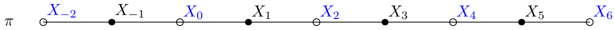

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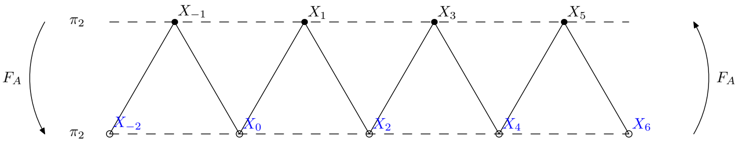

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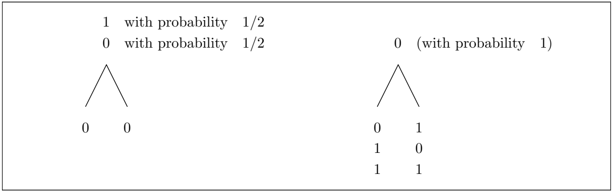
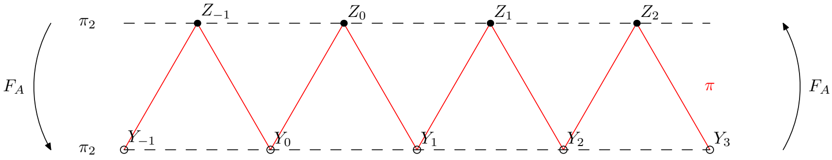
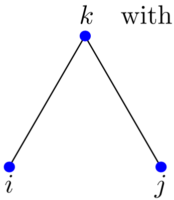




The projection π_2 of the Parry measure on odd (resp. even) sites is an invariant measure of the probabilistic cellular automaton.

General one-dimensional SFT

For a general one-dimensional SFT Σ_A, let us consider the PCA F_A defined by:



$$k \quad \text{with proba} \quad 1/|\{s \in \mathcal{A}\,;\; isj \in \mathcal{W}_3\}| \quad \text{if } ikj \in \mathcal{W}_3$$
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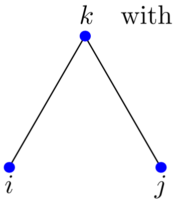

k with proba $1/|\{s \in \mathcal{A}\,;\ isj \in \mathcal{W}_3\}|$ if $ikj \in \mathcal{W}_3$
(and with proba 0 otherwise)

Proposition

The projection π_2 of the Parry measure on odd (resp. even) sites is an invariant measure of the PCA F_A.
From π_2, we recover π by one application of the PCA.

The same result holds on $\mathbb{Z}^d, d \geq 2$ and on infinite trees (bipartite graphs).

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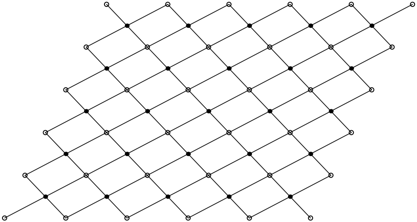
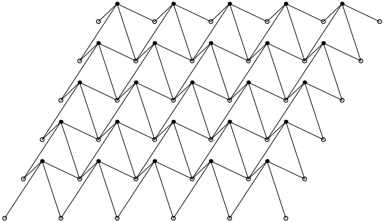
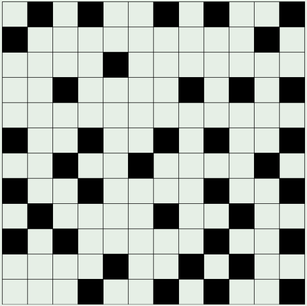


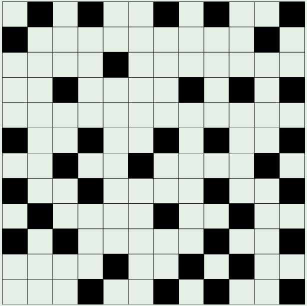


Two-dimensional case.

Back to the two-dimensional Fibonacci SFT

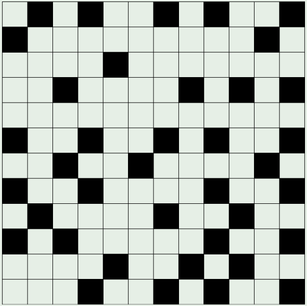


Back to the two-dimensional Fibonacci SFT



- Using PCA, new ideas for **perfect sampling** of patterns according to the measure of maximal entropy of the SFT.

Back to the two-dimensional Fibonacci SFT



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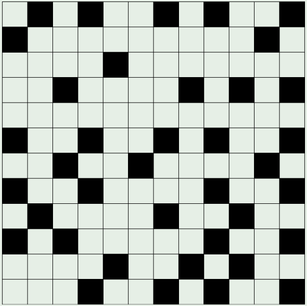
Back to the two-dimensional Fibonacci SFT



- Using PCA, new ideas for **perfect sampling** of patterns according to the measure of maximal entropy of the SFT.
- And an interesting detour through SFT defined on trees...
- Extension to sofic SFT?