On direct product factorization of homeomorphisms

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Talk outline

- Direct topological factorization: Preliminaries and definitions
- Classical motivation: Linds's work on $\mathbb{Z}$-SFTs and Perron numbers.
- A general existence result for factorization
- A $\mathbb{Z}^d$-SFT primeness example: 3-colored chessboard
- A non-sofic $\mathbb{Z}$-shift primeness example: Dyck shifts
- Questions and Problems

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- (X, T) is **topologically prime** if it has only trivial direct factors.

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Expansive systems

Recall that a $\mathbb{Z}^d$-action (X, T) is *expansive* if there exists $\epsilon > 0$ so that if $x, y \in X$ satisfy $d(T_n x, T_n y) < \epsilon$ for all $n \in \mathbb{Z}^d$ then $x = y$.

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(expansive $\mathbb{Z}^d$ action on a zero dimensional compact set $=$ $\mathbb{Z}^d$ subshift)

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- If (X, T) is not a finite product of primes then for each $n \in \mathbb{N}$ (X, T) is isomorphic to a product of n systems, $(Y_1, S_1) \times \ldots (Y_{n-1}, S_{n-1}) \times (Z_n, R_n)$, where (Z_n, R_n) is not prime.

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- Show that $\prod_{i=1}^{\infty}(Y_i, S_i)$ is a direct factor of (X, Y).

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- Up to shift-equivalence, topological factorizations of a $\mathbb{Z}$-full shift on n-symbols correspond to integer factorizations of n.

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The 3-colored chessboard

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Height functions and cocycles for 3-colored chessboard

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- Any (Hölder) continuous $\mathbb{R}$-valued cocycle $\tilde{c}$ on c_3 is a multiple of the above c, upto a homomorphism in $\mathrm{Hom}(\mathbb{Z}^d, \mathbb{R})$ and a coboundary:

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- By density of periodic points, it follows that either Y or Z are trivial.

Dyck shifts

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- $D_2 \subset \{(,[,),]\}^{\mathbb{Z}}$, a subshift consisting of all bi-infinite strings with two-types of brackets, such that corresponding opening and closing brackets are of matching types.

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- We have $h_{\mu_+}(x) = \log(k+1) + \frac{j}{n}\log(k)$ where n is the period of x and j is the number of extra β's in a period

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Primeness for Dyck shifts -continued

- Now $h_{\mu_\pm}(y,z) = h_\mu(y) + h_{\nu_\pm}(z)$.
- It follows that $h_\mu(y) = h$ is constant for all $y \in Per(y)$.

Primeness for Dyck shifts -continued

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