A few words on percolation properties of freezing monotone CA

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Percolation problems

Freezing alone

Definition

Let S be a finite set. A cellular automaton $f : S^{\mathbb{Z}^2} \to S^{\mathbb{Z}^2}$ is a map defined by a finite *neighborhood* $N \subset \mathbb{Z}^2$ and a local rule $F : S^N \to S$ such that $f(x)_{\vec{n}} = F(x_{\vec{n}+N})$ for all $x \in S^{\mathbb{Z}^2}$.

Definition

Let P be a finite poset. A cellular automaton f on $P^{\mathbb{Z}^2}$ is *freezing*, if $x_{\vec{n}} \leq f(x)_{\vec{n}}$. It is *monotone*, if $x \leq y$ implies $f(x) \leq f(y)$ in the cellwise ordering.

Lemma

A cellular automaton f on $\{0,1\}^{\mathbb{Z}^2}$ (with poset structure $0 < 1$) is freezing and monotone if and only if there exists a finite family E of finite triggering subsets of $\mathbb{Z}^2 \setminus \{\vec{0}\}$ such that

$$f(x)_{\vec{0}} = 1 \iff x_{\vec{0}} = 1 \vee \exists N \in E : x_N = 1^N.$$

Definition

Let μ be a measure on $S^{\mathbb{Z}^2}$. The *μ-limit set* of a cellular automaton f on $S^{\mathbb{Z}^2}$ is

$$\Omega_f^\mu = \overline{\bigcup_{\nu \in M} \mathrm{Supp}(\nu)},$$

where M is the set of limit points of the sequence $(f^n(\mu))_{n\in\mathbb{N}}$.

More concretely, Ω_f^μ is the set of configurations x such that no pattern w occurring in x satisfies $\lim_n \mu(f^{-n}([w]_{\vec{0}})) = 0$.

Definition

Let μ be a measure on $S^{\mathbb{Z}^2}$ and let f be a cellular automaton on $S^{\mathbb{Z}^2}$. We say f *trivializes* μ, if $|\Omega_f^\mu| = 1$.

Lemma

Let μ be a measure on $S^{\mathbb{Z}^2}$ and let f be a cellular automaton on $S^{\mathbb{Z}^2}$. The following conditions are equivalent:

- *f trivializes μ*
- $\lim_n f^n(\mu) = \delta_x$ *for some (unary)* $x \in S^{\mathbb{Z}^2}$
- *for some $s \in S$ and μ-almost every x, we have $f^n(x_{\vec{0}}) = s$ for all large enough n.*

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- Usually the goal is to find the exact speed of convergence for specific threshold rules.
- The freezing monotone binary CA generalize most bootstrap percolation models; all threshold models are special cases
- This class is also discussed in 'Neighborhood Family Percolation' by Bollobas, Smith and Uzzell

Theorem (Bollobas, Smith, Uzzell, Balister, Przykucki)

It is decidable whether a freezing monotone binary CA trivializes every nontrivial Bernoulli measure.

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Theorem (us)

It is decidable whether a freezing monotone binary CA trivializes some nontrivial Bernoulli measure.

Definition

For a freezing monotone cellular automaton f on $\{0,1\}^{\mathbb{Z}^2}$, we write $E(f)$ for its triggering sets E. We denote

$$F(f) = \{N \in E(f) \mid \vec{0} \notin \mathrm{CHull}(N)\},$$

and $G(f) = E(f) \setminus F(f)$. For a finite family E of incomparable subsets of $\mathbb{Z}^2 \setminus \{\vec{0}\}$, we denote by f_E the cellular automaton defined by $E(f_E) = E$.

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Theorem

Let f be a freezing monotone cellular automaton on $\{0,1\}^{\mathbb{Z}^2}$. Then f trivializes a nontrivial Bernoulli measure if and only if $F(f)$ is nonempty.

Lemma

Let $g : \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}$ *be the CA where*

$$g(x)_{\vec{v}} = 1 \iff x_{\vec{v}} = 1 \vee (x_{\vec{v}+(0,1)} = 1 \wedge x_{\vec{v}+(1,1)} = 1).$$

Then g *trivializes some Bernoulli measure* μ_p*.*

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Theorem

Let f *be a freezing monotone cellular automaton on* $\{0,1\}^{\mathbb{Z}^2}$*. Then* f *trivializes a nontrivial Bernoulli measure if ...* $F(f)$ *is nonempty.*

If $F(f)$ contains N, then $\vec{0} \notin \mathrm{CHull}(N)$, and we show that even $f_{\{N\}}$ trivializes a nontrivial Bernoulli measure.

Theorem

Let f be a freezing monotone cellular automaton on $\{0,1\}^{\mathbb{Z}^2}$. Then f trivializes a nontrivial Bernoulli measure ... only if $F(f)$ is nonempty.

If $F(f)$ is empty, then $\vec{0} \in \mathrm{CHull}(N)$ for all $N \in E(f)$. If we take a nice enough polygon B (or, say, a big enough ball), then a configuration with 0 in B and 1 outside it is fixed.

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If $F(f)$ is empty, then $\vec{0} \in \mathrm{CHull}(N)$ for all $N \in E(f)$. If we take a nice enough polygon B (or, say, a big enough ball), then a configuration with 0 in B and 1 outside it is fixed. Then f is actually subcritical, in the sense that there exists a finite obstruction.

Conjecture

For freezing monotone CA on a general poset alphabet, it is decidable whether some nontrivial Bernoulli measure is trivialized.

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Theorem

There exists a freezing CA f on $\{0,1\}^{\mathbb{Z}^2}$ and two nontrivial Bernoulli measures μ_1 and μ_2 such that $\mu_1(1) < \mu_2(1)$, and f trivializes μ_1 but not μ_2.

Proof.

$F = G \circ H$ where

- H flips an $n \times n$ square with 0s in the corners to all 1s if the density is roughly $\mu_1(1)$.
- G applies the CA g to $n \times n$ blocks of all 1s.
- $F^n = G^n \circ H$.

□

Thank you for listening!