A few words on percolation properties of freezing monotone CA

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Definition

Let $S$ be a finite set. A cellular automaton $f : S^\mathbb{Z}^2 \to S^\mathbb{Z}^2$ is a map defined by a finite neighborhood $N \subset \mathbb{Z}^2$ and a local rule $F : S^N \to S$ such that $f(x)_\vec{n} = F(x_\vec{n} + N)$ for all $x \in S^\mathbb{Z}^2$.

Definition

Let $P$ be a finite poset. A cellular automaton $f$ on $P^\mathbb{Z}^2$ is freezing, if $x_\vec{n} \leq f(x)_\vec{n}$. It is monotone, if $x \leq y$ implies $f(x) \leq f(y)$ in the cellwise ordering.
Lemma

A cellular automaton $f$ on $\{0, 1\}^{\mathbb{Z}^2}$ (with poset structure $0 < 1$) is freezing and monotone if and only if there exists a finite family $E$ of finite triggering subsets of $\mathbb{Z}^2 \setminus \{\vec{0}\}$ such that

$$f(x)_{\vec{0}} = 1 \iff x_{\vec{0}} = 1 \lor \exists N \in E : x_N = 1^N.$$
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Percolation problems

Freezing alone

**Definition**

Let $\mu$ be a measure on $\mathbb{Z}^2$. The $\mu$-limit set of a cellular automaton $f$ on $\mathbb{Z}^2$ is

$$\Omega_f^{\mu} = \bigcup_{\nu \in M} \text{Supp}(\nu),$$

where $M$ is the set of limit points of the sequence $\left(f^n(\mu)\right)_{n \in \mathbb{N}}$.

More concretely, $\Omega_f^{\mu}$ is the set of configurations $x$ such that no pattern $w$ occurring in $x$ satisfies $\lim_{n} \mu(f^{-n}([w]_0)) = 0$.

**Definition**

Let $\mu$ be a measure on $\mathbb{Z}^2$ and let $f$ be a cellular automaton on $\mathbb{Z}^2$. We say $f$ trivializes $\mu$, if $|\Omega_f^{\mu}| = 1$. 
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**Lemma**

Let $\mu$ be a measure on $S^\mathbb{Z}^2$ and let $f$ be a cellular automaton on $S^\mathbb{Z}^2$. The following conditions are equivalent:

- $f$ trivializes $\mu$
- $\lim_n f^n(\mu) = \delta_x$ for some (unary) $x \in S^\mathbb{Z}^2$
- for some $s \in S$ and $\mu$-almost every $x$, we have $f^n(x_0) = s$ for all large enough $n$. 

Percolation problems
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The freezing monotone binary CA generalize most bootstrap percolation models; all threshold models are special cases.

This class is also discussed in ‘Neighborhood Family Percolation’ by Bollobas, Smith and Uzzell.
Theorem (Bollobas, Smith, Uzzell, Balister, Przykucki)

*It is decidable whether a freezing monotone binary CA trivializes every nontrivial Bernoulli measure.*
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Theorem (us)

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Definition

For a freezing monotone cellular automaton $f$ on $\{0, 1\}^\mathbb{Z}^2$, we write $E(f)$ for its triggering sets $E$. We denote

$$F(f) = \{ N \in E(f) \mid \vec{0} \notin \text{CHull}(N) \},$$

and $G(f) = E(f) \setminus F(f)$. For a finite family $E$ of incomparable subsets of $\mathbb{Z}^2 \setminus \{\vec{0}\}$, we denote by $f_E$ the cellular automaton defined by $E(f_E) = E$. 

Theorem

Let $f$ be a freezing monotone cellular automaton on $\{0, 1\}^\mathbb{Z}^2$. Then $f$ trivializes a nontrivial Bernoulli measure if and only if $F(f)$ is nonempty.
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**Theorem**

Let $f$ be a freezing monotone cellular automaton on $\{0, 1\}^{\mathbb{Z}^2}$. Then $f$ trivializes a nontrivial Bernoulli measure if and only if $F(f)$ is nonempty.
Lemma

Let \( g : \{0, 1\}^{\mathbb{Z}^2} \rightarrow \{0, 1\}^{\mathbb{Z}^2} \) be the CA where

\[
g(x)_{\vec{v}} = 1 \iff x_{\vec{v}} = 1 \lor (x_{\vec{v}+(0,1)} = 1 \land x_{\vec{v}+(1,1)} = 1).
\]

Then \( g \) trivializes some Bernoulli measure \( \mu_p \).
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Lemma

Let $g : \{0, 1\}^{\mathbb{Z}^2} \rightarrow \{0, 1\}^{\mathbb{Z}^2}$ be the CA where

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Then $g$ trivializes some Bernoulli measure $\mu_p$.

Theorem

Let $f$ be a freezing monotone cellular automaton on $\{0, 1\}^{\mathbb{Z}^2}$. Then $f$ trivializes a nontrivial Bernoulli measure if

$\ldots$ $F(f)$ is nonempty.

If $F(f)$ contains $N$, then $0 \notin \text{CHull}(N)$, and we show that even $f_{\{N\}}$ trivializes a nontrivial Bernoulli measure.
Theorem

Let $f$ be a freezing monotone cellular automaton on $\{0, 1\}^{\mathbb{Z}^2}$. Then $f$ trivializes a nontrivial Bernoulli measure ... only if $F(f)$ is nonempty.

If $F(f)$ is empty, then $\vec{0} \in \text{CHull}(N)$ for all $N \in E(f)$. If we take a nice enough polygon $B$ (or, say, a big enough ball), then a configuration with 0 in $B$ and 1 outside it is fixed.
**Theorem**

Let $f$ be a freezing monotone cellular automaton on $\{0, 1\} \times \mathbb{Z}^2$. Then $f$ trivializes a nontrivial Bernoulli measure ... only if $F(f)$ is nonempty.

If $F(f)$ is empty, then $\vec{0} \in \text{CHull}(N)$ for all $N \in E(f)$. If we take a nice enough polygon $B$ (or, say, a big enough ball), then a configuration with 0 in $B$ and 1 outside it is fixed. Then $f$ is actually subcritical, in the sense that there exists a finite obstruction.
Conjecture

For freezing monotone CA on a general poset alphabet, it is decidable whether some nontrivial Bernoulli measure is trivialized.
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For binary monotone CA, it is decidable whether some nontrivial Bernoulli measure is trivialized.
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Theorem

There exists a freezing CA $f$ on $\{0, 1\}^{\mathbb{Z}^2}$ and two nontrivial Bernoulli measures $\mu_1$ and $\mu_2$ such that $\mu_1(1) < \mu_2(1)$, and $f$ trivializes $\mu_1$ but not $\mu_2$.

Proof.

$F = G \circ H$ where

- $H$ flips an $n \times n$ square with 0s in the corners to all 1s if the density is roughly $\mu_1(1)$.
- $G$ applies the CA $g$ to $n \times n$ blocks of all 1s.
- $F^n = G^n \circ H$. 

□
Thank you for listening!