Lecture 1: The Domino problem on groups, part I.
CANT 2016, CIRM (Marseille)

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Objectives of this talk...

- Define the Domino problem (DP).
- Show the two main techniques to prove undecidability of DP on $\mathbb{Z}^2$
Outline of the talk.

1. Definitions

2. Undecidability of DP on $\mathbb{Z}^2$, proof I

3. Undecidability of DP on $\mathbb{Z}^2$, proof II
Configurations and Subshifts (I)

- Let $A$ be a finite alphabet, $G$ be a finitely generated group.
- Colorings $x : G \to A$ are called configurations.
- Endowed with the prodiscrete topology $A^G$ is a compact and metrizable set.
- Cylinders form a clopen basis

$$[a]_g = \{x \in A^G \mid x_g = a\}.$$ 

- A pattern is a finite intersection of cylinders, or equivalently a finite configuration $p : S \to A$
- A metric for the cylinder topology is

$$d(x, y) = 2^{-\inf\{|g| \mid g \in G : x_g \neq y_g\}},$$

where $|g|$ is the length of the shortest path from $1_G$ to $g$ in $\Gamma(G, S)$.  

Definitions

Undecidability of DP on $\mathbb{Z}^2$, proof I

Undecidability of DP on $\mathbb{Z}^2$, proof II

Configurations and Subshifts (II)

The **shift** action $\sigma : G \times A^G \rightarrow A^G$ is given by

$$(\sigma_g(x))_h = x_{g^{-1}h}.$$ 

The dynamical system $(A^G, \sigma)$ is called the **$G$-fullshift over $A$.**

**Definition**

A **$G$-subshift** is a closed and $\sigma$-invariant subset $X \subset A^G$. 
Configurations and Subshifts (II)

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The dynamical system $(A^G, \sigma)$ is called the **$G$-fullshift over $A$**.

**Definition**

A **$G$-subshift** is a closed and $\sigma$-invariant subset $X \subset A^G$.

A pattern $p \in A^S$ **appears** in a configuration $x \in A^G$ if $(\sigma_g(x))_S = p$ for some $g \in G$.

**Proposition**

$X$ is a $G$-subshift iff there exists a set $\mathcal{F}$ of forbidden patterns s.t.

$$X = X_{\mathcal{F}} := \{x \in A^G \mid \text{no pattern of } \mathcal{F} \text{ appears in } x\}.$$
Subshifts of finite type

A $G$-subshift $X$ is **of finite type** ($G$-SFT) if there exists a finite set of forbidden patterns $\mathcal{F}$ that defines it: $X = X_\mathcal{F}$.

Example:
SFTs and Wang tiles

Fix $G$ a f.g. group and $S$ a generating set for $G$. Wang tiles $\approx$ polygons with colored $2|S|$ edges.

Neighbourhood rule
SFTs and Wang tiles

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Neighbourhood rule

$X_\tau$ set of valid tilings by $\tau$
SFTs and Wang tiles

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Neighbourhood rule

$X_\tau$ set of valid tilings by $\tau$

SFT $\approx X_\tau$
The Domino problem on groups

Fix $G$ a f.g. group and $S$ a generating set for $G$.

**Domino problem on $G$**

**Input:** A finite set of Wang tiles $\tau$ on $S$

**Output:** *Yes* if there exists a valid tiling by $\tau$, *No* otherwise.
The Domino problem on groups

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**Question**

Which f.g. groups have decidable Domino Problem?
The Domino problem on groups

Fix $G$ a f.g. group and $S$ a generating set for $G$.

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**Input:** A finite set of Wang tiles $\tau$ on $S$

**Output:** Yes if there exists a valid tiling by $\tau$, No otherwise.

**Question**

Which f.g. groups have decidable Domino Problem?

→ group property, quasi-isometry invariant.
Outline of the talk.

1. Definitions
2. Undecidability of DP on $\mathbb{Z}^2$, proof I
3. Undecidability of DP on $\mathbb{Z}^2$, proof II
Sketch of the proof

Idea: encode Turing machines inside Wang tiles.

- Undecidability of the Halting problem of Turing machines.
- Reduction from the Halting problem of Turing machines.
## Turing machines

### Table

<table>
<thead>
<tr>
<th>( \delta(q, x) )</th>
<th>Symbol ( x )</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>( a )</td>
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<tr>
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<td>( q_{b++} )</td>
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<tr>
<td>( q_{|} )</td>
<td>( (q_{a+}, a, \rightarrow) )</td>
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</tbody>
</table>

### Diagram

```
# # # # # # # # # #
```

State: \( q_0 \)
### Turing machines

<table>
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<tr>
<td>$q_{\parallel}$</td>
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</tbody>
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Diagram: 

```
# # a # # # # # # #
```

State $q_{b^+}$
## Turing machines

<table>
<thead>
<tr>
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<tbody>
<tr>
<td></td>
<td>$a$</td>
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<tr>
<td>$q_0$</td>
<td>⊥</td>
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<td>$q_{a^+}$</td>
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<tr>
<td>$q_{</td>
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<td>}$</td>
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</tbody>
</table>

Diagram:

```
|   | # | # | a | b | # | # | # | # |
```

$q_{||}$
Turing machines

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<td>$q_{</td>
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Turing machine transitions:

$q_{||} \downarrow$

Input tape:

```
# # a b || # # # # # #
```
Turing machines

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<td></td>
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The table represents the transition function $\delta(q, x)$ for a Turing machine, where $q_0$ is the initial state, and $q_{||}$ is a special state. The transition function shows how the machine moves from one state to another based on the input symbol $x$. The table includes states $q_0$, $q_{a^+}$, $q_{b^+}$, $q_{b^{++}}$, and $q_{||}$, and transitions for symbols $a$, $b$, $\parallel$, and $\#$.
Turing machines

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<td>$b$</td>
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<td>$b$</td>
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# Turing machines

## Definitions

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## Undecidability of DP on $\mathbb{Z}^2$, proof I

**Turing machines**

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# # a b || # # # # # #
```
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<tr>
<td>q₀</td>
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<tr>
<td>qa⁺</td>
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<td>qb⁺</td>
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<td>qb⁺⁺</td>
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<tr>
<td>q∥</td>
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**Diagram:**

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### Turing machines

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<tr>
<td></td>
<td>$\parallel$</td>
<td>$q_{\parallel} (q_{\parallel}, \parallel, \leftarrow)$</td>
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\[
\begin{array}{cccccc}
    # & # & a & a & b & # & # & # & # & # \\
\end{array}
\]
## Turing machines

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Example Turing machine state diagram:

```
# # a a b b # # # #
```

State $q_{\parallel}$
### Turing machines

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<td>( \perp )</td>
</tr>
<tr>
<td>( q_{</td>
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</tbody>
</table>

Diagram:

```
#  #  a  a  b  b  ||  #  #  #  #
```

Transition function \( \delta(q, x) \) for a Turing machine.
Turing machines

<table>
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<tr>
<td>$q_0$</td>
<td>$\perp$</td>
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<td></td>
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<td></td>
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<td>$\perp$</td>
<td></td>
</tr>
<tr>
<td>$q_{|}$</td>
<td>$(q_{a^+}, a, \rightarrow)$</td>
<td>$(q_{|}, b, \leftarrow)$</td>
<td>$(q_{|}, |, \leftarrow)$</td>
<td>$(q_{|}, |, \cdot)$</td>
<td></td>
</tr>
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</table>

**Theorem (Turing, 1936)**

The Halting problem (to know whether a Turing machine $M$ halts on input $w$ or not) is undecidable.

**Theorem**

The Blank tape Halting problem (to know whether a Turing machine $M$ halts on the empty input) is undecidable.
Turing machines and Wang tiles

Encode Turing machine computations inside Wang tiles:

- no computation head
- initial configuration \((\infty\#\infty, q_0)\)
- \(\delta(q, a) = (q', a', .)\)
- \(\delta(r, a) = (r', a', \rightarrow)\)
- \(\delta(s, a) = (s', a', \leftarrow)\)
Turing machines and Wang tiles

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We want: \(\tau\) admits a tiling iff \(M\) does not halt on the empty input.
Which tilings?

We *forbid* tiles with an halting state $q_f$. 

Which tilings?

We forbid tiles with an halting state $q_f$.

If $\mathcal{M}$ does not halt on the empty input, we have a tiling.
Which tilings?

We **forbid** tiles with an halting state $q_f$.

If $\mathcal{M}$ does not halt on the empty input, we have a tiling. But...
The Origin Constrained Domino problem

What we have not proven:

Not-Yet-Theorem

The Domino problem is undecidable on $\mathbb{Z}^2$. 
The Origin Constrained Domino problem

What we have not proven:

**Not-Yet-Theorem**

The Domino problem is undecidable on $\mathbb{Z}^2$.

What we have proven:

**Theorem (Kahr, Moore & Wang 1962, Büchi 1962)**

The Origin Constrained Domino problem is undecidable on $\mathbb{Z}^2$.

where

**Origin Constrained Domino problem**

**Input:** A finite set of Wang tiles $\tau$, a tile $t \in \tau$

**Output:** *Yes* if there exists a valid tiling by $\tau$ with $t$ at the origin, *No* otherwise.
How to initialize computations?

Build one infinite in time and space computation zone?

- **Compactness** $\implies$ we cannot force one given tile to appear exactly once in every valid tiling
How to initialize computations?

Build one infinite in time and space computation zone?

> **Compactness** ⇒ we cannot force one given tile to appear exactly once in every valid tiling

Build arbitrarily big computation zones?

> **Compactness** ⇒ if we have arbitrarily big *rectangles* in our tilings, then we also have a tiling with no rectangle.
How to initialize computations?

Build one infinite in time and space computation zone?

▶ **Compactness** ⇒ we cannot force one given tile to appear exactly once in every valid tiling

Build arbitrarily big computation zones?

▶ **Compactness** ⇒ if we have arbitrarily big rectangles in our tilings, then we also have a tiling with no rectangle.

One solution: hierarchy of computation zones (thus arbitrarily big zones) that intersect a lot.
Robinson tileset

The Robinson tileset, where tiles can be rotated and reflected.
Robinson tileset

The Robinson tileset, where tiles can be rotated and reflected.
Existence of a valid tiling

**Proposition**

Robinson’s tileset admits at least one valid tiling.
Existence of a valid tiling

Proposition

Robinson’s tileset admits at least one valid tiling.

Proof:

- We can build arbitrarily large patterns (called macro-tiles) with the same structure.
- We thus conclude by compactness.
Macro-tiles of level 1.
Macro-tiles of level 1

Macro-tiles of level 1.

They behave like large 🕐.
From macro-tiles of level 1 to macro-tiles of level 2
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From macro-tiles of level 1 to macro-tiles of level 2
From macro-tiles of level 1 to macro-tiles of level 2
From macro-tiles of level $n$ to macro-tiles of level $n + 1$
About Robinson’s tiling structure

Hierarchy of squares: squares of level $n$ are gathered by 4 to form a square of level $n + 1$
About Robinson’s tiling structure

Hierarchy of squares: squares of level \( n \) are gathered by 4 to form a square of level \( n + 1 \)

Proposition

The only valid tilings by the Robinson tileset form a hierarchy of squares.
Valid tilings (I)

The two forms in Robinson tileset, cross (bumpy corners) and arms (dented corners).

![Cross and arms](image_url)
Valid tilings (I)

The two forms in Robinson tileset, cross (bumpy corners) and arms (dented corners).

![Crosses and arms diagrams]

Obviously, two crosses cannot be in contact (neither through an edge nor a vertex) thus a cross must be surrounded by eight arms.
Valid tilings (II)

You cannot have things like

![Diagrams of invalid tilings]
Valid tilings (II)

You cannot have things like

The only possibilities are thus
Valid tilings (II)

You cannot have things like

The only possibilities are thus
Valid tilings (III)

So each $\square$ is part of a macro tile of level 1.

that behaves like a big $\square$, and so on...
Undecidability of the Domino Problem (II)

**Solution**

Embed Turing machine computations inside the hierarchy of squares given by Robinson’s tiling.
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Embed Turing machine computations inside the hierarchy of squares given by Robinson’s tiling.

Theorem (Berger 1966, Robinson 1971)

The Domino Problem is undecidable on $\mathbb{Z}^2$. 
Outline of the talk.

1. Definitions
2. Undecidability of DP on $\mathbb{Z}^2$, proof I
3. Undecidability of DP on $\mathbb{Z}^2$, proof II
Sketch of the proof

Idea: encode **piecewise affine maps** inside Wang tiles.

- Undecidability of the Mortality problem of Turing machines.
- Undecidability of the Mortality problem of piecewise affine maps.
- Reduction from the Mortality problem of piecewise affine maps.
Mortality problem of Turing machines

Take $\mathcal{M}$ a deterministic Turing machine with an halting state $q_f$.

!! configurations of $\mathcal{M}$ do not have finite support !!

A configuration $(x, q)$ is a **non-halting configuration** if it never evolves into the halting state.
Mortality problem of Turing machines

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!! configurations of $\mathcal{M}$ do not have finite support !!

A configuration $(x, q)$ is a non-halting configuration if it never evolves into the halting state.

**Theorem (Hooper, 1966)**

The Mortality problem of Turing machines is undecidable.

**Proof:** very technical, uses Minsky 2-counters machines.
Rational piecewise affine maps in $\mathbb{R}^2$

Take $f_i : U_i \to \mathbb{R}^2$ for $i \in [1; n]$ some rational affine maps, with $U_1, U_2, \ldots, U_n$ disjoint unit squares with integer corners.

Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ with domain $U = \bigcup_{i=1}^{n} U_i$ by

$$\vec{x} \mapsto f_i(\vec{x}) \text{ if } \vec{x} \in U_i.$$

A point $\vec{x} \in \mathbb{R}^2$ is an immortal starting point for $(f_i)_{i=1}^{n}$ if for every $n \in \mathbb{N}$, the point $f^n(\vec{x})$ lies inside the domain $U$.

Mortality problem of piecewise affine maps

**Input:** a system of rational affine maps $f_1, f_2, \ldots, f_n$ with disjoint unit squares $U_1, U_2, \ldots, U_n$ with integer corners.

**Output:** Yes the system has an immortal starting point, No otherwise.
Rational piecewise affine maps and Turing machines (I)

We use the **moving tape** Turing machines model.

Assume that $\mathcal{M}$ has alphabet $A = \{0, 1, \ldots, a - 1\}$ and states $Q = \{0, 1, \ldots, b - 1\}$.

Given $\mathcal{M}$ a Turing machine, we construct a system $f_1, f_2, \ldots, f_n$ of piecewise affine maps s.t.

- A configuration of $\mathcal{M}$ is coded by two real numbers.
- A transition of $\mathcal{M}$ is coded by one $f_i$.
- $f_1, f_2, \ldots, f_n$ has an immortal starting point if and only if $\mathcal{M}$ has an immortal configuration.
Rational piecewise affine maps and Turing machines (II)

Configuration \((x, q)\) is coded by \((\ell, r) \in \mathbb{R}^2\) where

\[
\ell = \sum_{i=-1}^{-\infty} M^i x_i,
\]

and

\[
r = Mq + \sum_{i=0}^{\infty} M^{-i} x_i,
\]

where \(M\) is an integer s.t. \(M > a\) and \(M > b\).
Rational piecewise affine maps and Turing machines (II)

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where \(M\) is an integer s.t. \(M > a\) and \(M > b\).

The transition \(\delta(q, a) = (q', a', \rightarrow)\) is coded by the affine transformation

\[
\begin{pmatrix} \ell \\ r \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{M} & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \ell \\ r \end{pmatrix} + \begin{pmatrix} a' \\ M(q' - a - Mq) \end{pmatrix}
\]

with domain \([0, 1] \times [Mq, Mq + 1]\).
Rational piecewise affine maps and Turing machines (II)

- A Turing machine $\mathcal{M}$ is transformed into a system $f_1, \ldots, f_n$ of rational piecewise affine maps.
Rational piecewise affine maps and Turing machines (II)

- A Turing machine $\mathcal{M}$ is transformed into a system $f_1, \ldots, f_n$ of rational piecewise affine maps.

- $\mathcal{M}$ has an immortal starting point iff $f_1, \ldots, f_n$ has.
Rational piecewise affine maps and Turing machines (II)

- A Turing machine $\mathcal{M}$ is transformed into a system $f_1, \ldots, f_n$ of rational piecewise affine maps.

- $\mathcal{M}$ has an immortal starting point iff $f_1, \ldots, f_n$ has.

Theorem

The Mortality problem of piecewise affine maps is undecidable.
Rational affine maps inside Wang tiles (I)

Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ a rational affine map as before. The tile

\[
\begin{array}{c}
\vec{n} \\
\vec{w} \\
\vec{s} \\
\vec{e}
\end{array}
\]

is said to **compute** the function $f$ if

\[
f(\vec{n}) + \vec{w} = \vec{s} + \vec{e}.
\]
Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ a rational affine map as before. The tile

$$
\begin{array}{|c|c|}
\hline
\vec{n} & \vec{e} \\
\hline
\vec{w} & \vec{s} \\
\hline
\end{array}
$$

is said to \textbf{compute} the function $f$ if

$$f(\vec{n}) + \vec{w} = \vec{s} + \vec{e}.$$

And on a row:

$$
\begin{array}{c}
\vec{w} = \vec{w}_1 \\
\vec{s}_1 & \vec{s}_2 \\
\vec{n}_1 & \vec{n}_2 \\
\vdots
\end{array} 
\begin{array}{c}
\vec{s}_{k-1} & \vec{s}_k \\
\vec{n}_{k-1} & \vec{n}_k \\
\vec{e}_k = \vec{e}
\end{array}
$$

$$f \left( \frac{\vec{n}_1 + \cdots + \vec{n}_k}{k} \right) + \frac{1}{k} \vec{w} = \frac{\vec{s}_1 + \cdots + \vec{s}_k}{k} + \frac{1}{k} \vec{e}.$$
Rational affine maps inside Wang tiles (II)

For $x \in \mathbb{R}$, a **representation of** $x$ is a sequence of integers $(x_k)_{k \in \mathbb{Z}}$ s.t.

- $\forall k \in \mathbb{Z}, x_k \in \lfloor x \rfloor, \lfloor x \rfloor + 1$;
- $\forall k \in \mathbb{Z}, \lim_{n \to \infty} \frac{x_{k-n} + \cdots + x_{k+n}}{2n + 1} = x$. 
Rational affine maps inside Wang tiles (II)

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- $\forall k \in \mathbb{Z}$,

$$
\lim_{n \to \infty} \frac{x_k-n + \cdots + x_k+n}{2n+1} = x.
$$

Define $B_k(x) = \lfloor kx \rfloor - \lfloor (k-1)x \rfloor$ for every $k \in \mathbb{Z}$. Then $B(x) = (B_k(x))_{k \in \mathbb{Z}}$ is the **balanced representation of** $x$. 
Rational affine maps inside Wang tiles (II)

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$$B(x) = (B_k(x))_{k \in \mathbb{Z}}$$

is the **balanced representation of** $x$.

For $\vec{x} \in \mathbb{R}^2$ and $k \in \mathbb{Z}$, define $B_k(\vec{x})$ coordinate by coordinate.

If $\vec{x}$ is in $U_i = [n, n + 1] \times [m, m + 1]$, then

$B_k(\vec{x}) \in \{(n, m), (n, m + 1), (n + 1, m), (n + 1, m + 1)\}$ for every $k \in \mathbb{Z}$.
The tile set corresponding to $f_i(\vec{x}) = M\vec{x} + \vec{b}$ consists of tiles

$$B_k(\vec{x})$$

$$f_i(A_{k-1}(\vec{x})) - A_{k-1}(f_i(\vec{x}))$$

$$+ (k - 1)\vec{b}$$

$$f_i(A_k(\vec{x})) - A_k(f_i(\vec{x}))$$

$$+ k\vec{b}$$

$$B_k(f_i(\vec{x}))$$

for every $k \in \mathbb{Z}$ and $\vec{x} \in U_i$. 
Rational affine maps inside Wang tiles (III)

The tile set corresponding to $f_i(\vec{x}) = M\vec{x} + \vec{b}$ consists of tiles

$$B_k(\vec{x})$$

$$f_i(A_{k-1}(\vec{x})) - A_{k-1}(f_i(\vec{x}))$$

$$+(k-1)\vec{b}$$

$$B_k(f_i(\vec{x}))$$

for every $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

Since $U_i$ is bounded and $f_i$ rational, there are finitely many tiles!
A system of rational affine maps \( f_1, f_2, \ldots, f_n \) defined on \( U_1, U_2, \ldots, U_n \) with integer corners.

- Each \( f_i \) maps to a finite set of tiles \( T_i \)
- Set of tiles \( T = \bigcup T_i \) with additional markings (every row tiled by a single \( T_i \))
- \( T \) admits a tiling of the plane iff \( f_1, f_2, \ldots, f_n \) has an immortal point.
Rational affine maps inside Wang tiles (IV)

- A system of rational affine maps \( f_1, f_2, \ldots, f_n \) defined on \( U_1, U_2, \ldots, U_n \) with integer corners.
- Each \( f_i \mapsto \) a finite set of tiles \( T_i \)
- Set of tiles \( T = \bigcup T_i \) with additional markings (every row tiled by a single \( T_i \))
- \( T \) admits a tiling of the plane iff \( f_1, f_2, \ldots, f_n \) has an immortal point.

**Theorem (Kari, 2007)**
The Domino problem is undecidable on \( \mathbb{Z}^2 \).
Conclusion

- Two proofs of the undecidability of **DP** on $\mathbb{Z}^2$.
- Encode a small computational model inside Wang tiles.
- What about f.g. groups?
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- Encode a small computational model inside Wang tiles.
- What about f.g. groups?

Thank you for your attention!!