Lecture 3: Domino problem and (a)periodicity.
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Outline of the talk.

1. Domino problem and periodicity
2. Block gluing SFTs on $\mathbb{Z}^2$
3. Strongly aperiodic subshifts
4. Lovász Local Lemma in Symbolic Dynamics
We can define two notions of periodic configuration:

- A configuration \( x \in A^{\mathbb{Z}^2} \) is **weakly periodic** if its stabilizer is infinite.
  
  \( \Leftrightarrow \) \( x \) admits a non-trivial direction \( \vec{u} \) of periodicity.

- A configuration \( x \in A^{\mathbb{Z}^2} \) is **strongly periodic** if its stabilizer is of finite index in \( \mathbb{Z}^2 \): \([\mathbb{Z}^2 : \text{Stab}(x)] < \infty \).

  \( \Leftrightarrow \) \( x \) admits two non-collinear directions \( \vec{u}, \vec{v} \) of periodicity.
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  \[ \Leftrightarrow \text{ } x \text{ admits two non-collinear directions } \overrightarrow{u}, \overrightarrow{v} \text{ of periodicity}. \]

**Proposition**

On \( \mathbb{Z}^2 \), if an SFT contains a weakly periodic configuration, then it contains a strongly periodic one.

**Proof:** on the blackboard.
Wang’s conjecture (1961)

If a set of Wang tiles can tile the plane, then they can always be arranged to do so periodically.
Domino problem and periodicity on $\mathbb{Z}^2$ (II)

Wang’s conjecture (1961)

A non-empty SFT contains a periodic configuration.
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A non-empty SFT contains a periodic configuration.

Suppose Wang’s conjecture is true. Then you can decide DP!

**Semi-algorithm 1:**

1. gives a finite periodic pattern, if it exists
2. loops otherwise

**Semi-algorithm 2:**

1. gives an integer \( n \) so that there is no \([1; n] \times [1; n]\) locally admissible pattern, if it exists
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Wang’s conjecture (1961)

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**Semi-algorithm 1:**
1. gives a finite periodic pattern, if it exists
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**Semi-algorithm 2:**
1. gives an integer $n$ so that there is no $[1; n] \times [1; n]$ locally admissible pattern, if it exists
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**Consequence**

The undecidability of DP implies existence of an aperiodic SFT.
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Block gluing subshifts on $\mathbb{Z}^2$ (I)

A subshift $X \subset A^{\mathbb{Z}^2}$ is **block-gluing** with gap $g \in \mathbb{N}$ if for any two finite supports $S_1, S_2 \subset \mathbb{Z}^2$ at distance at least $g$, and for any $x, y \in X$

there exists $z \in X$ s.t. $z|_{S_1} = x|_{S_1}$ and $z|_{S_2} = y|_{S_2}$.

**Remark:** this is a **uniform** mixing condition.
Proposition (Folklore, written in Pavlov & Schraudner 2015)

A non-empty block-gluing SFT has a periodic configuration.

**Proof:** on the blackboard.
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Proof: on the blackboard.

Consequence

The Domino problem is decidable for block-gluing SFTs.
Block gluing subshifts on $\mathbb{Z}^2$ (II)

Proposition (Folklore, written in Pavlov & Schraudner 2015)
A non-empty block-gluing SFT has a periodic configuration.

Proof: on the blackboard.

Consequence
The Domino problem is decidable for block-gluing SFTs.

Remark: Actually we prove something stronger: we can decide whether a locally admissible pattern is globally admissible (the language is decidable).
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**Example:** Robinson’s SFT is strongly aperiodic
Strongly aperiodic subshifts (II)

Question

Which f.g. groups admit strongly aperiodic SFTs?
Strongly aperiodic subshifts (II)

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- Generalization of Kari’s construction to some $G \times \mathbb{Z}$ (Jeandel, 2015).
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- Surface groups (Cohen & Goodman-Strauss, 2015).
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- groups $\mathbb{Z}^2 \rtimes H$ where $H$ has decidable \textbf{WP} (Barbieri & Sablik, 2016).
Strongly aperiodic subshifts (II)

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**Question (simpler)**

Does every f.g. group admit strongly aperiodic subshifts?
Strongly aperiodic subshifts (III)

Theorem (Gao, Jackson & Seward, 2009)
Every f.g. group $G$ has a strongly aperiodic subshift on alphabet $\{0, 1\}$. 
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Proof: ???
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Proof: ???

Theorem (A. Barbieri & Thomassé, 2015)
Every f.g. group $G$ has a strongly aperiodic subshift on alphabet $\{0, 1\}$.
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Lovász Local Lemma

(see Anton Chaplygin’s talk yesterday)

\((A_i)_{i=1}^{n} \text{ mutually independent}\)

Each \(A_i\) can be avoided \[\implies A_1, \ldots, A_n\] can be avoided.

**Proposition**

If events \(A_1, \ldots, A_n\) are mutually independent, then

\[
Pr \left( \bigcap_{i=1}^{n} \bar{A}_i \right) = \prod_{i=1}^{n} (1 - Pr(A_i))
\]

What about the dependent case?
Lovász Local Lemma

(see Anton Chaplygin’s talk yesterday)

\( (A_i)_{i=1}^{n} \text{ not very dependent} \)
Each \( A_i \) can be avoided \( \Rightarrow A_1, \ldots, A_n \) can be avoided.

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**Lovász Local Lemma (1975)**

Let \( \mathcal{A} = \{A_1, A_2, \ldots, A_n\} \). For \( A_i \in \mathcal{A} \), let \( \Gamma(A_i) \) be the subset of \( \mathcal{A} \) such that \( A_i \) is independent of the collection \( \mathcal{A} \setminus (\{A_i\} \cup \Gamma(A_i)) \). Suppose there are \( x_1, \ldots, x_n \) such that \( 0 \leq x_i < 1 \) and:

\[
\forall A_i \in \mathcal{A} : Pr(A_i) \leq x_i \prod_{A_j \in \Gamma(A)} (1 - x_j)
\]

then the probability of avoiding \( A_1, A_2, \ldots, A_n \) is positive.
How to use LLL in Symbolic Dynamics?

Suppose you want to prove that the subshift $X$ is non-empty.

- Uniform Bernoulli measure on configurations space.
- Bad events $\approx$ forbidden patterns.
- Compactness $+$ LLL (if applicable) show the non-emptiness of the subshift.
Lovász Local Lemma in Symbolic Dynamics (II)

Let $G$ be a f.g. group, $A$ a finite alphabet and $\mu$ the uniform Bernoulli probability measure on $A^G$.

A sufficient condition for being non-empty

Let $X \subset A^G$ be a subshift defined by $\mathcal{F} = \bigcup_{n \geq 1} \mathcal{F}_n$, where $\mathcal{F}_n \subset A^{B_n}$. Suppose that there exists a function $x : \mathbb{N} \times G \rightarrow (0, 1)$ such that:

$$\forall n \in \mathbb{N}, g \in G, \mu(A_{n,g}) \leq x(n, g) \prod_{gS_n \cap hS_k \neq \emptyset} (1 - x(k, h)),$$

where $A_{n,g} = \{ x \in A^G : x|_{gS_n} \in \mathcal{F}_n \}$. Then the subshift $X$ is non-empty.
Strong aperiodicity vs. the distinct neighborhood property

A subshift $X \subset A^G$ is **strongly aperiodic** if all its configurations have trivial stabilizer

$$\forall x \in X, \forall g \in G, \sigma^g(x) = x \Rightarrow g = 1_G.$$  

Fix $A = \{0, 1\}$.

A configuration $x \in \{0, 1\}^G$ has the **distinct neighborhood property** if for every $h \in G \setminus \{1_G\}$, there exists a finite $T \subset G$ s.t.

$$\forall g \in G, x|_{ghT} \neq x|_{gT}.$$
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Proposition

If $x \in \{0, 1\}^G$ has the distinct neighborhood property, then the subshift $\text{Orb}_\sigma(x)$ is strongly aperiodic.

Proof: on the blackboard.
Distinct neighborhood property with LL

**Proposition**

Every infinite f.g. group $G$ has a configuration $x \in \{0, 1\}^G$ with the distinct neighborhood property.

**Proof:**

- Take $(s_i)_{i \in \mathbb{N}}$ an enumeration of $G$ with $s_0 = 1_G$.
- Choose $(T_i)_{i \in \mathbb{N}}$ a sequence of finite sets of $G$ s.t.
  $$T_i \cap s_i T_i = \emptyset$$
  and $|T_i| = C i$ for some constant $C$.

- $A_{n,g} = \{ x \in \{0, 1\}^G \mid x|_{gT_n} = x|_{gs_n T_n} \}$.
- $x(n, g) = 2^{-\frac{Cn}{2}}$. 

Theorem

Every f.g. group $G$ has a strongly aperiodic subshift on alphabet $\{0, 1\}$. 

The Lovász Local Lemma in Symbolic Dynamics
Proposition

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- Choose $(T_i)_{i \in \mathbb{N}}$ a sequence of finite sets of $G$ s.t.
  \[ T_i \cap s_i T_i = \emptyset \text{ and } |T_i| = Ci \text{ for some constant } C. \]
- Let $A_{n,g} = \{ x \in \{0, 1\}^G \mid x|_{gT_n} = x|_{gs_n T_n} \}$.
- Let $x(n,g) = 2^{\frac{-Cn}{2}}$.

Theorem

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\[ x(n,g) = 2^{\frac{-Cn}{2}}. \]
An effectively closed strongly aperiodic subshift (I)

A subshift is $G$-effectively closed if it can be defined by a set of forbidden patterns recognizable by a Turing machine with oracle $\text{WP}(G)$. 
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**Theorem (Alon, Grytczuk, Haluszczak & Riordan, 2002)**

Every finite graph with degree $\leq \Delta$ has a square-free coloring with $2e^{16} \Delta^2$ colors.
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**Theorem (Alon, Grytczuk, Haluszczak & Riordan, 2002)**

Every finite graph with degree $\leq \Delta$ has a square-free coloring with $2e^{16}\Delta^2$ colors.

**Proposition**

Let $G$ a f.g. group and $S$ a generating set. Then $\Gamma(G, S)$ has a square-free coloring with $2^{19}|S|^2$ colors.
An effectively closed strongly aperiodic subshift (II)

**Theorem (A. Barbieri & Thomassé, 2015)**

Every f.g. group $G$ has a $G$-effectively closed strongly aperiodic subshift.

**Sketch of the proof:**

- Fix $S$ and take $X \subset A^G$ be the subshift such that every square in $\Gamma(G, S)$ is forbidden.
- Let $g \in G$ such that $\sigma^g(x) = x$ for some $x \in X$.
- Factorize $g$ as $uwv$ with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1_G$. 
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- Factorize $g$ as $uvw$ with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1_G$.
- If not, let $w = w_1 \ldots w_n$ and consider the odd length walk $\pi = v_0 v_1 \ldots v_{2n-1}$ on $\Gamma(G, S)$ defined by:

$$v_i = \begin{cases} 1_G & \text{if } i = 0 \\ w_1 \ldots w_i & \text{if } i \in \{1, \ldots, n\} \\ ww_1 \ldots w_{i-n} & \text{if } i \in \{n+1, \ldots, 2n-1\} \end{cases}$$
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- $\pi$ is a path, and $x_{v_i} = x_{v_{i+n}} \Rightarrow g = 1_G$. 


Conclusion

- Every one-ended f.g. group with decidable WP has strongly aperiodic SFTs?
- Does there exist $G$ with decidable DP and strongly aperiodic SFTs?
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Thank you for your attention!!