Tiling problems on Baumslag-Solitar groups.

MCU 2013

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Outline

1 Tilings on groups
   - Definition
   - Finitely presented groups

2 Classical problems
   - Existence of aperiodic tile sets
   - Domino problem

3 Baumslag-Solitar groups
   - Why are they interesting ?
   - A weakly aperiodic tile set on BS(2, 3)
Group presentations

- generators: $a, b$
- relations: $a^{-1}b^{-1}ab = \varepsilon$ (or $ab = ba$)
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- $aba = a^2b = ba^2 = b^{-1}a^2b^2 = \ldots$
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- $\langle a, b \mid ab = ba \rangle$
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- \( aba = a^2 b = ba^2 = b^{-1} a^2 b^2 = \ldots \)
- \( \langle a, b \mid ab = ba \rangle \cong \mathbb{Z}^2 \)
Group presentations

- generators: $a, b$
- relations: $a^{-1}b^{-1}ab = e$ (or $ab = ba$)
- elements of the group: words on the alphabet \{a, b, a^{-1}, b^{-1}\}
- $aba = a^2b = ba^2 = b^{-1}a^2b^2 = \ldots$
- $\langle a, b \mid ab = ba \rangle \approx \mathbb{Z}^2 \approx \langle a, b, c \mid ab = ba, ab = c, ac = ca, bc = cb \rangle$
Cayley graph

Representation of a group with an undirected graph:
- vertices are elements of the group
- edges are labelled by the generators $g_i$
- an edge labelled by $g_i$ between $h$ and $h.g_i$
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$$\langle a, b/a^4 = b^2 = \varepsilon, b.a = a^3.b \rangle$$
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Tilings on groups

On $\mathbb{Z}^2$: Wang tiles

A tile = pattern with one colour for each generator and each inverse; finite tile set $\tau$; a configuration (or tiling) $\in \tau^G = \text{colouring of the Cayley graph that respects the neighbourhood rule.}$
Tilings on groups

On $\mathbb{Z}^2$: Wang tiles

Generalization to a group $G$:

- a *tile* = pattern with one colour for each generator and each inverse;
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- a *configuration* (or *tiling*) $\in \tau^G =$ colouring of the Cayley graph that respects the neighbourhood rule.
Finitely presented groups

A group is finitely presented if it possesses a presentation having
- a finite number of generators;
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- a finite number of generators;
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Interest:
- structure with a finite representation...
- which may nevertheless be complex:

Theorem (Novikov, 1955 & Boone, 1957)

There are finitely presented groups with an undecidable word problem.
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A tiling $x \in A^G$ is \textit{m-periodic} with $m \in G$ non-trivial if

$$\forall g \in G, x_g = x_{m \cdot g}.$$ 

The set of periods of a tiling $x$, denoted by $\text{Per}(x)$, is thus a sub-group of $G$. 
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- $x$ is weakly periodic if $Per(x)$ contains an infinite cyclic subgroup
- $x$ is strongly non-periodic if it is not weakly periodic
- $\tau$ is strongly aperiodic if a valid tiling exists and if it admits only strongly non-periodic tilings.

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Classical problems: aperiodic tile sets

Remarks:

- Strong aperiodicity implies weak aperiodicity.
- On $\mathbb{Z}^2$ the two notions coincide (but not on $\mathbb{Z}^3$...).
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Aperiodicity

- On free groups, every tile set has a strongly periodic configuration (compactness argument).
- There exist strongly aperiodic tile sets on \( \mathbb{Z}^2 \) [Ber66, Rob71].
Classical problems: domino problem

**Question:** Let $G$ be a group generated by $G$. Is it possible to find an algorithm that takes as input a finite set of Wang tiles $\tau$ on $G$, and outputs *Yes* if and only if there exists a valid tiling by $\tau$?

**Remark:** The problem does not depend on the set of generators chosen for $G$. 
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**Domino problem**

- Decidable on free groups.
- Undecidable on $\mathbb{Z}^2$ [Ber66, Rob71]
- Undecidable on the hyperbolic plane [Kar07, Mar08].
- Decidable when $G$ is virtually free [MS85] (= has a free sub-group of finite index).
Domino problem on a group

$G$ has finite tree-width

Word problem algebraic on $G$

$G$ is virtually free

Domino problem decidable on $G$
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**Aim:** Necessary condition on $G$ to make the domino problem decidable?
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Definition

Baumslag-Solitar group: $BS(m, n) = \langle a, b | a^m b = b a^n \rangle$
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Theorem (Magnus, 1932)
Every finitely presented group defined by a single relation has a decidable word problem.
### Definition

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Definition

Baumslag-Solitar group: $BS(m, n) = \langle a, b | a^m b = ba^n \rangle$

Theorem (Magnus, 1932)

Every finitely presented group defined by a single relation has a decidable word problem.

Theorem (Baumslag-Solitar, 1962)

The groups $BS(m, n)$ are not virtually free.

In the sequel: $BS(2, 3) = \langle a, b | a^2 b = ba^3 \rangle$
Structure

\[ b^{-1} \quad bab^{-2} \]
\[ \varepsilon \quad bab^{-1} \]
\[ b \quad ba \]
Techniques known

How to build aperiodic tile sets?

- give ad-hoc *local rules*
  - strongly aperiodic tile set on $\mathbb{Z}^2$ [Rob71], $\mathbb{H}^2$ [GS10]
- use *substitutions* [Oll08] or *fixpoint theorem* [DRS09]
  - gives self-similar tilings, hence strongly aperiodic tile set, but only for $\mathbb{Z}^d$
    (or amenable groups)
- simulate an *aperiodic dynamical system*
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How to prove the undecidability of the domino problem?

- reduction from the Halting problem
- reduction from the immortality problem for piecewise affine maps
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Remark: On \( \mathbb{Z}^d \) the undecidability of DP implies the existence of a strongly aperiodic tile set!
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Which technique on BS groups?

- Infinitely many layers that merge infinitely often.
- Each layer is isomorphic to a tessellation of $\mathbb{H}^2$.
- But we cannot directly use the tileset of $[\text{Kar07}] \sim$ synchronization problems!!
An aperiodic tile set: sketch of the construction

Let $T : [\frac{2}{3}; 2] \rightarrow [\frac{2}{3}; 2]$ be the piecewise linear map defined by

$$T : x \mapsto \begin{cases} 
2x & \text{if } x \in [\frac{2}{3}; 1] \\
\frac{2}{3}x & \text{if } x \in ]1; 2] 
\end{cases}$$

Properties

- The dynamical system $T$ is aperiodic.
- Following [Kar07], we construct a finite tile set $\tau$.
- There does not exist a strongly periodic valid tiling by $\tau$.
- There exists a weakly periodic valid tiling by $\tau$ (period $\omega = bab^{-1}a^2ba^{-1}b^{-1}a^{-2}$).
The tile set $\tau$

$c \in \{0, \frac{1}{3}, \frac{2}{3}\}$

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Example of tiling by $\tau$

Tiling by $\tau$ corresponding to the orbit $(\ldots, \frac{5}{4}, \frac{5}{6}, \frac{5}{3}, \ldots)$ in $T$. 
Theorem (A.& Kari)

There exist weakly aperiodic tile sets on BS\((m, n)\) for every \(m, n > 0\).

Theorem (A.& Kari)

The domino problem is undecidable on BS\((m, n)\).

**Proof:** Reduction from the undecidability of the mortality problem for piecewise affine maps.
Conclusion

⇒ A class of groups with undecidable domino problem...
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Use different characterizations of virtually free groups.
Conclusion

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- but no progress about the reciprocal statement of [MS85].
- More interesting: what happens on $< a, b | ab^m = ba^n >$?
- Use different characterizations of virtually free groups.

Thank you for your attention!
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