(Projective) Subdynamics of Multidimensional Subshifts, part I.

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Outline

1 Symbolic dynamics
   • Shift spaces and subshifts
   • Classes of subshifts
   • 2D vs 1D sofic subshifts

2 Projective Subdynamics and Subactions
   • Definitions
   • Introductive examples
   • Effective subshifts as projective subdynamics
Symbolic dynamics
Projective Subdynamics and Subactions

Full-shift, shift action and subshift

- $\mathcal{A}$ a finite alphabet and $d \in \mathbb{N}$
- $x \in \mathcal{A}^\mathbb{Z}^d$ is a configuration
- $\mathcal{A}^\mathbb{Z}^d$ endowed with the prodiscrete topology is a compact metric space
- shift action $\sigma : \mathbb{Z}^d \times \mathcal{A}^\mathbb{Z}^d \to \mathcal{A}^\mathbb{Z}^d$, 
  $\left(\sigma(n_1, \ldots, n_d)(x)\right)_{(i_1, \ldots, i_d)} = x(i_1+n_1, \ldots, i_d+n_d)$
- the dynamical system $\left(\mathcal{A}^\mathbb{Z}^d, \sigma\right)$ is the $d$-dimensional full-shift on $\mathcal{A}$
**Full-shift, shift action and subshift**

- $\mathcal{A}$ a finite alphabet and $d \in \mathbb{N}$
- $x \in \mathcal{A}^{\mathbb{Z}^d}$ is a *configuration*
- $\mathcal{A}^{\mathbb{Z}^d}$ endowed with the prodiscrete topology is a compact metric space
- *shift action* $\sigma : \mathbb{Z}^d \times \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$,
  \[
  (\sigma(n_1,\ldots,n_d)(x))(i_1,\ldots,i_d) = x(i_1+n_1,\ldots,i_d+n_d)
  \]
- the dynamical system $(\mathcal{A}^{\mathbb{Z}^d}, \sigma)$ is the *$d$-dimensional full-shift on $\mathcal{A}$*

**Definition**

A *subshift* is a closed and $\sigma$-invariant subset of $\mathcal{A}^{\mathbb{Z}^d}$.

\[
\begin{align*}
\{ x \in \{0,1\}^{\mathbb{Z}^2} : x(i,j) = 1 & \iff i = j = 0 \} & \text{ not $\sigma$-invariant} \\
\{ x \in \{0,1\}^{\mathbb{Z}^2} : \text{ only one 1 appears in } x \} & \text{ not closed} \\
\{ x \in \{0,1\}^{\mathbb{Z}^2} : \text{ at most one 1 appears in } x \} & \text{ is a subshift}
\end{align*}
\]
Combinatorial point of view

- A **pattern** is a local function $p : S \rightarrow A$, where $S \subset \mathbb{Z}^d$ is finite.

- Given a pattern $u \in A^S$, it generates the **cylinder**

  $$[u] = \left\{ x \in A^{\mathbb{Z}^d} : x|_S = u \right\}.$$

- If $F$ is a set of patterns, the **subshift generated by $F$** is

  $$X_F = \left\{ x \in A^{\mathbb{Z}^d} : \text{no pattern of } F \text{ appears in } x \right\}.$$

- A subshift is thus the complement of a union of cylinders

  $$X_F = A^{\mathbb{Z}^d} \setminus \left( \bigcup_{i \in \mathbb{Z}^d, u \in F} \sigma_i([u]) \right).$$
Language of a subshift

Definition

The *language of size n* of a $\mathbb{Z}^d$-subshift $X$ is

$$\mathcal{L}_n(X) := \{ p : [-n; n]^d \to A : \exists x \in X, p \text{ appears in } x \}.$$ 

The *language* of a $\mathbb{Z}^d$-subshift $X$ is

$$\mathcal{L}(X) := \bigcup_{n \geq 0} \mathcal{L}_n(X).$$

The *complement of the language* $\mathcal{L}(X)^c$ is the biggest set of forbidden patterns.
Language of a subshift

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The *language of size* $n$ of a $\mathbb{Z}^d$-subshift $X$ is

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Proposition

The topological and combinatorial definitions coincide.
Subshifts of finite type

The subshift $X\{\begin{array}{c}
\text{red}, \\
\text{blue},
\end{array}\}$ contains the following configurations

![Configurations](image-url)
Subshifts of finite type

The subshift \( X \{ \begin{array}{ccc}
\text{\cellcolor{red}} & \text{\cellcolor{blue}} & \text{\cellcolor{blue}} \\
\text{\cellcolor{blue}} & \text{\cellcolor{red}} & \text{\cellcolor{blue}} \\
\text{\cellcolor{blue}} & \text{\cellcolor{blue}} & \text{\cellcolor{blue}} 
\end{array} \} \) contains the following configurations

---

Definition

A subshift is \textit{of finite type (SFT)} if it can be defined by a finite set of forbidden patterns. It is of \textit{rank k} if these finite patterns may be chosen of size \( k \).

- simplest class for the combinatorial definition
- 2D-SFT \( \equiv \) tilings by Wang tiles
- closely related to cellular automata theory
A **sofic subshift** is the image of a SFT under a continuous and $\sigma$-commuting map.

A continuous and $\sigma$-commuting map $\Phi : A^{Z^d} \rightarrow B^{Z^d}$ given by the local function $\phi$:

$$\phi(x) \in A^{Z^2}$$

$$\Phi(\phi) \in B^{Z^2}$$

On $\mathbb{Z}$, sofic subshifts are exactly those recognized by finite automata. In higher dimension, no characterization is known.
Sofic subshifts

Definition

A *sofic subshift* is the image of a SFT under a continuous and $\sigma$-commuting map.

Continuous and $\sigma$-commuting map $\Rightarrow$ Sliding block map (cellular automaton)  

\[ \Phi : \mathcal{A}^{\mathbb{Z}^d} \rightarrow \mathcal{B}^{\mathbb{Z}^d} \]

given by the local function $\phi$

- SFT on which information can be erased.
- On $\mathbb{Z}$, sofic subshifts are exactly those recognized by finite automata.
- In higher dimension, no characterization is known.
An example of purely sofic subshift

Let $X_{\leq 1} = \{x \in \{0, 1\}^{\mathbb{Z}^2} : \text{at most one 1 appears in } x\}$.

- Suppose that $X_{\leq 1}$ is a rank $k$ SFT.
- Then a configuration that contains two 1’s at distance $2k + 1$ cannot be rejected.

$\Rightarrow X_{\leq 1}$ is not an SFT!
An example of purely sofic subshift

Let $X_{\leq 1} = \{ x \in \{0, 1\}^{\mathbb{Z}^2} : \text{at most one } 1 \text{ appears in } x \}$. 
An example of purely sofic subshift

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An example of purely sofic subshift

Let $X_{\leq 1} = \{ x \in \{0, 1\}^{\mathbb{Z}^2} : \text{at most one 1 appears in } x \}$.
An example of non-sofic subshift

The *mirror subshift* is defined on alphabet \{□, □, □} by

\[ X_{\text{mirror}} = \{□, □\}^\mathbb{Z}^2 \cup \{\ldots\} \]
An example of non-sofic subshift

The *mirror subshift* is defined on alphabet \{\[,\[\[,\[\} by

\[X_{\text{mirror}} = \{\[,\[\}[\mathbb{Z}^2 \cup \{\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array},\ldots\}\]

Suppose \(X_{\text{mirror}}\) is sofic.
Then \(\exists \Sigma \subset A_{\mathbb{Z}^2} \) a \(k\)-SFT and \(\Pi\) a block map of order \(r\), such that

\[\Pi : \Sigma \rightarrow X_{\text{mirror}}\] is onto.
An example of non-sofic subshift

The *mirror subshift* is defined on alphabet \{□, ■, ★\} by

\[ X_{\text{mirror}} = \{□, ■\}^{\mathbb{Z}^2} \cup \left\{ \begin{array}{c}
\begin{array}{c}
\text{patterns}
\end{array}
\end{array}, \begin{array}{c}
\text{patterns}
\end{array}, \ldots \right\} \]

Suppose \( X_{\text{mirror}} \) is sofic.
Then \( \exists \Sigma \subset A^{\mathbb{Z}^2} \) a \( k \)-SFT and \( \Pi \) a block map of order \( r \), such that

\[ \Pi : \Sigma \rightarrow X_{\text{mirror}} \text{ is onto.} \]
An example of non-sofic subshift

The *mirror subshift* is defined on alphabet \{ □, ■, ▢ \} by

\[ X_{\text{mirror}} = \{ □, ■ \} \mathbb{Z}^2 \cup \{ \text{patterns} \}, \ldots \]

Suppose \( X_{\text{mirror}} \) is sofic.

Then \( \exists \Sigma \subset A^{\mathbb{Z}^2} \) a \( k \)-SFT and \( \Pi \) a block map of order \( r \), such that

\[ \Pi : \Sigma \to X_{\text{mirror}} \text{ is onto.} \]

\[ |A|^{4nr+8nk+4r^2} < 2^{n^2} \]
An example of non-sofic subshift

The *mirror subshift* is defined on alphabet \{ , , \} by

\[ X_{\text{mirror}} = \{ , \} \mathbb{Z}^2 \cup \left\{ \begin{array}{c}
\text{\tiny \{ , \}} \\
\text{\tiny , } \\
\text{\tiny , } \\
\end{array} \right\} \]

Suppose \( X_{\text{mirror}} \) is sofic.
Then \( \exists \Sigma \subset A^{\mathbb{Z}^2} \) a \( k \)-SFT and \( \Pi \) a block map of order \( r \), such that

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\[ |A|^{4nr+8nk+4r^2} < 2^{n^2} \]
An example of non-sofic subshift

The *mirror subshift* is defined on alphabet \{ [square], [black square], [red square] \} by

\[ X_{\text{mirror}} = \{ [square], [black square] \} \mathbb{Z}^2 \cup \{ \text{patterns} \} \]

Suppose \( X_{\text{mirror}} \) is sofic. Then \( \exists \Sigma \subset A^{\mathbb{Z}^2} \) a \( k \)-SFT and \( \Pi \) a block map of order \( r \), such that

\[ \Pi : \Sigma \rightarrow X_{\text{mirror}} \text{ is onto.} \]

\[ |A|^{4nr+8nk+4r^2} < 2^{n^2} \]
Effectively closed subshifts

\[ \text{SFT} \subset \text{Sofic susbhifts} \subset \text{Effectively closed} \]

**Definition**

A subshift is **effectively closed** (or **effective**) if its complement is a computable union of cylinders.

**Property**

\( X \) is effectively closed if and only one of the followings holds

(i) \( X = X_\mathcal{F} \) for some recursively enumerable set \( \mathcal{F} \) of forbidden patterns

(ii) \( X = X_\mathcal{F} \) for some recursive set \( \mathcal{F} \) of forbidden patterns

Remark:

There exist non effectively closed subshifts (countability argument).
Effectively closed subshifts

SFT ⊆ Sofic suhshifts ⊆ Effectively closed

**Definition**
A subshift is *effectively closed* (or *effective*) if its complement is a computable union of cylinders.

**Property**
$X$ is effectively closed if and only one of the followings holds
(i) $X = X_F$ for some recursively enumerable set $F$ of forbidden patterns
(ii) $X = X_F$ for some recursive set $F$ of forbidden patterns

**Remark:** There exist non effectively closed subshifts (countability argument).
A *Turing machine* is a tuple $\mathcal{M} = (Q, \Gamma, \#, q_0, \delta, Q_F)$ where:
- $Q$ is a finite set of states, $q_0 \in Q$ is the initial state;
- $\Gamma$ is a finite alphabet;
- $\# \notin \Gamma$ blank symbol
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\leftarrow, \downarrow, \rightarrow\}$ transition function;
- $F \subset Q_F$ finite set of final states.

The rule $\delta(q_1, x) = (q_2, y, \leftarrow)$ will be encoded by the pattern

```
  z ← q_2  y  z'  
  z  x ← q_1 z'  
```
Turing machines and SFT (II)

\[ M \] Turing machine \( \rightsquigarrow \) finite set of patterns \( F_M \) \( \rightsquigarrow \) SFT \( X_{F_M} \)
Turing machines and SFT (II)

$\mathcal{M}$ Turing machine $\leadsto$ finite set of patterns $F_\mathcal{M} \leadsto$ SFT $X_{F_\mathcal{M}}$
Turing machines and SFT (II)

$\mathcal{M}$ Turing machine $\leadsto$ finite set of patterns $F_\mathcal{M} \leadsto$ SFT $X_{F_\mathcal{M}}$
$M$ Turing machine $\leadsto$ finite set of patterns $F_M \leadsto$ SFT $X_{F_M}$
2D vs 1D sofic subshifts

1D sofic subshifts

- $X_F = \emptyset$? is decidable
- Entropy is computable (nonnegative rational multiples of log of Perron numbers)
- Representation by finite automata/matrix
- Every SFT has a periodic configuration
- Soficness $\iff$ finite number of followers set

2D sofic subshifts

- $X_F = \emptyset$? is undecidable
- Entropy is not computable (right recursively enumerable numbers)
- Representation by Wang tiles, textile systems
- $\exists$ aperiodic SFT
Necessary conditions for soficness in 2D

- If $X$ is a minimal subshift with positive entropy, then $X$ is not sofic.
  
  [Desai, 2006]
Necessary conditions for soficness in 2D

- If $X$ is a minimal subshift with positive entropy, then $X$ is not sofic. [Desai, 2006]

- If $X$ is effective and if the *Kolmogorov complexity* of every $p \in \mathcal{L}_n(X)$ is greater than $\mathcal{O}(n)$, then $X$ is not sofic. [Durand, Romaschenko & Shen, 2008]
Necessary conditions for soficness in 2D

- If $X$ is a minimal subshift with positive entropy, then $X$ is not sofic. [Desai, 2006]

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- Too many extender sets implies non-soficness. [Kass & Madden 2013] and [Pavlov, 2013]
Outline

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   - 2D vs 1D sofic subshifts

2. Projective Subdynamics and Subactions
   - Definitions
   - Introductive examples
   - Effective subshifts as projective subdynamics
Initially introduced by Johnson, Kass and Madden in 2007.

**Definition**

Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a $\mathbb{Z}^d$ subshift and $L \subseteq \mathbb{Z}^d$ a $k$-dimensional sublattice ($1 \leq k < d$). The *$L$-projective subdynamics of $X$* is

$$P_L(X) := \{x|_L : x \in L\} \subseteq \mathcal{A}^L.$$

- $(P_L(X), \sigma_{L \times P_L(X)})$ is a $\mathbb{Z}^k$-subshift.
- $P_L(X)$: globally admissible configurations of shape $L$ in $X$.
- Loss of information about the original subshift.
Projective Subdynamics

Initially introduced by Johnson, Kass and Madden in 2007.

**Definition**

Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a $\mathbb{Z}^d$ subshift and $L \subseteq \mathbb{Z}^d$ a $k$-dimensional sublattice $(1 \leq k < d)$. The *$L$-projective subdynamics of $X$* is

$$P_L(X) := \{x|_L : x \in L\} \subseteq \mathcal{A}^L.$$ 

- $(P_L(X), \sigma_L \times P_L(X))$ is a $\mathbb{Z}^k$-subshift.
- $P_L(X)$: globally admissible configurations of shape $L$ in $X$.
- Loss of information about the original subshift.

In the sequel, we will concentrate on $P_{\vec{e}_1 \mathbb{Z}}(X)$ (PS along the horizontal direction).
Proposition (Johnson, Kass & Madden, 2007)

\[ h_{\text{top}}(P_{\bar{e}_1\mathbb{Z}}(X)) \geq h_{\text{top}}(X). \]

Proof:

\[
h_{\text{top}}(X) = \lim_{n \to \infty} \frac{1}{n^2} \log (|\mathcal{L}_n(X)|)
= \lim_{n \to \infty} \frac{1}{n^2} \log (|\mathcal{L}_n(P_{\bar{e}_1\mathbb{Z}}(X))|^n)
\leq \lim_{n \to \infty} \frac{1}{n} \log (|\mathcal{L}_n(P_{\bar{e}_1\mathbb{Z}}(X))|)
= h_{\text{top}}(P_{\bar{e}_1\mathbb{Z}}(X))
\]
Subdynamics

**Definition**

Let \( X \subseteq \mathcal{A}^{\mathbb{Z}^d} \) be a \( \mathbb{Z}^d \) subshift and \( Y \subseteq \mathcal{B}^{\mathbb{Z}^k} \) a \( \mathbb{Z}^k \)-subshift \((1 \leq k < d)\). Then \( Y \) is a **subaction of** \( X \) if the dynamical systems \((X, \sigma |_{\mathbb{Z}^k})\) and \((Y, \sigma |_{\mathbb{Z}^k})\) are isomorphic.

- Much stronger than projective subdynamics
- The subshift \( Y \) is defined on a possibly non-finite alphabet
- No loss of information
Questions

- What are projective subdynamics of 2D sofic subshifts
- What are projective subdynamics of 2D SFT?
- What are subactions of sofic subshifts?
- What are subactions of 2D SFT?
Questions

- What are projective subdynamics of 2D sofic subshifts = effective subshifts
- What are projective subdynamics of 2D SFT?
- What are 1D subactions of 3D sofic subshifts = effective dynamical systems
- What are subactions of 2D SFT?
What can be PS of sofic subshifts? (0)

▶ Trivially, every 1D sofic subshift...

\[ \Sigma \subset B^\mathbb{Z} \text{ SFT, } \Pi : \Sigma \to X \text{ block map} \]
What can be PS of sofic subshifts? (0)

- Trivially, every 1D sofic subshift...

\[
\begin{align*}
\Pi(x_5) &\in X \\
\Pi(x_6) &\in X \\
\Pi(x_7) &\in X \\
\Pi(x_8) &\in X \\
\Pi(x_9) &\in X \\
\Pi(x_{10}) &\in X \\
\Pi(x_{11}) &\in X \\
\Pi(x_{12}) &\in X \\
\Pi(x_{13}) &\in X \\
\Pi(x_{14}) &\in X \\
\Pi(x_{15}) &\in X \\
\Pi(x_{16}) &\in X \\
\Pi(x_{17}) &\in X \\
\Pi(x_{18}) &\in X \\
\Pi(x_{19}) &\in X
\end{align*}
\]

\[
X \subset A^\mathbb{Z} \text{ sofic}
\]
\[
\Sigma \subset B^\mathbb{Z} \text{ SFT, } \Pi : \Sigma \to X \text{ block map}
\]
What can be PS of sofic subshifts? (0)

- Trivially, every 1D sofic subshift...

\[ \Pi(x_{19}) \in X \]
\[ \Pi(x_{18}) \in X \]
\[ \Pi(x_{17}) \in X \]
\[ \Pi(x_{16}) \in X \]
\[ \Pi(x_{15}) \in X \]
\[ \Pi(x_{14}) \in X \]
\[ \Pi(x_{13}) \in X \]
\[ \Pi(x_{12}) \in X \]
\[ \Pi(x_{11}) \in X \]
\[ \Pi(x_{10}) \in X \]
\[ \Pi(x_{9}) \in X \]
\[ \Pi(x_{8}) \in X \]
\[ \Pi(x_{7}) \in X \]
\[ \Pi(x_{6}) \in X \]
\[ \Pi(x_{5}) \in X \]

**SFT** \( \Sigma \)

\[ X \subset A^\mathbb{Z} \) sofic

\[ \Sigma \subset B^\mathbb{Z} \) SFT, \( \Pi : \Sigma \rightarrow X \) block map

**Conjecture (Jeandel)**

\( X \) is sofic \( \iff \) \( X^\mathbb{Z} \) is sofic.
What can be PS of sofic subshifts? (I)

The 1D subshift $X_{a^n b^n}$.

cccaaaabbbbbccccccaaaabbbbc
What can be PS of sofic subshifts? (I)

- The 1D subshift $X_{a^n b^n}$. 
What can be PS of sofic subshifts? (I)

- The 1D subshift $X_{a^n b^n}$. 
What can be PS of sofic subshifts? (I)

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What can be PS of sofic subshifts? (I)

- The 1D subshift $X_{a^n b^n}$. 

![Diagram of a 1D subshift $X_{a^n b^n}$]
What can be PS of sofic subshifts? (I)

▶ The 1D subshift $X_{a^n b^n}$. And even a subaction!
What can be PS of sofic subshifts? (II)

- The 1D subshift $X_{a^n b^n c^n}$ (neither sofic nor algebraic).
What can be PS of sofic subshifts? (II)

- The 1D subshift $X_{a^n b^n c^n}$. And even a subaction!
What can be PS of sofic subshifts? (III)

▶ Any effective subshift $X$ that contains a uniform configuration.
What can be PS of sofic subshifts? (III)

- Any effective subshift $X$ that contains a uniform configuration.

$X_{\{ab: a, b \in A\}}$ (SFT)
What can be PS of sofic subshifts? (III)

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$X\{a, b \in A\}$ (SFT)

$\tilde{X}_{\leq 1}$ (sofic)
What can be PS of sofic subshifts? (III)

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$\tilde{X}_{\leq 1}$ (sofic)

$X_M$ (SFT)

TM with 3 tapes:
- one tape of calculation
- two tapes to check the validity of $x$
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- one tape of calculation
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Any effective subshift $X$ that contains a uniform configuration.

On the third layer:
- The Turing Machine $M$ works on the first tape and enumerates forbidden patterns for $X$ (initialization thanks to the second layer).
- Each time a forbidden patterns is produced, it is copied out on the two other tapes.
- Patterns written on the second (resp. third) tape are shifted to the left (resp. right) at each step of computation.
What can be PS of sofic subshifts? (III)

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On the third layer:

- The Turing Machine $M$ works on the first tape and enumerates forbidden patterns for $X$ (initialization thanks to the second layer).
- Each time a forbidden pattern is produced, it is copied out on the two other tapes.
- Patterns written on the second (resp. third) tape are shifted to the left (resp. right) at each step of computation.
- If a pattern written on the two last tapes matches with the corresponding pattern in $x$, then the configuration is forbidden (intercation by local rules with the first layer).
- If a forbidden pattern for $X$ appears in $x$, it will eventually be detected and the configuration is rejected.
- If $x \in X$, the configuration is accepted.
Any effective subshift $X$ that contains a uniform configuration.

$X \{a, b \in A\}$ (SFT)

$\tilde{X}_{\leq 1}$ (sofic)

$X_M$ (SFT)

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What can be PS of sofic subshifts? (III)

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What can be PS of sofic subshifts? (III)

- Any effective subshift $X$ that contains a uniform configuration.

$\Rightarrow X$ is a PS of a sofic subshift
Hochman’s result

**Theorem (Hochman 2008)**

- Any effective $\mathbb{Z}^d$-subshift may be obtained as the subaction of a $\mathbb{Z}^{d+2}$ sofic subshift.
- Any effective $\mathbb{Z}^d$ dynamical system may be obtained as the subaction of a $\mathbb{Z}^{d+2}$ sofic subshift.

The proof is based on

- the use of *Turing machines as SFT*,
- *substitutive tilings* to construct computation zones in 3D.
Conclusion of Part I

- Challenging question: characterize soficness in higher dimension.

- Projective subdynamics and subaction: decrease dimension to better understand 2D subshifts.

- Complete characterization of PS/subactions of sofic subshifts (Hochman)

- Coming soon:
  - Sketch of Hochman’s proof...
  - that can be improved to dimension $d + 1$!
  - Some other results about PS/subactions of SFT
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Thank you for your attention!