Data processing and networks optimization

Part III: Subdifferential and conjugate

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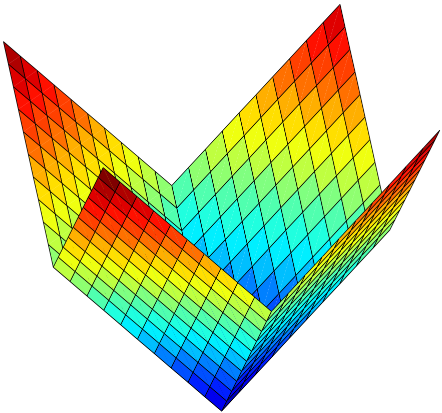
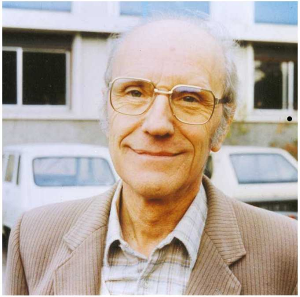
Non-smooth convex optimization



A pioneer



Jean-Jacques Moreau
(1923–2014)

Subdifferential of a convex function: definition

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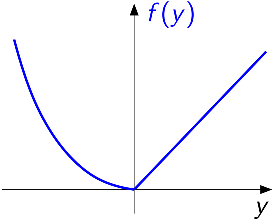

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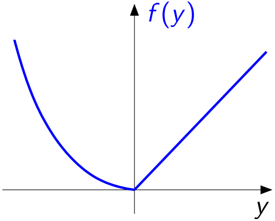

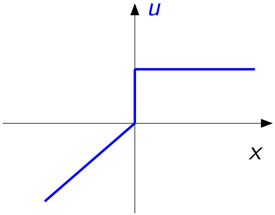

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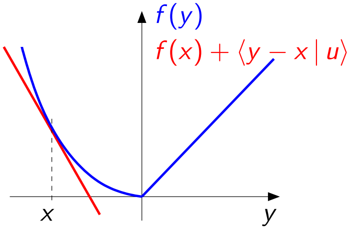

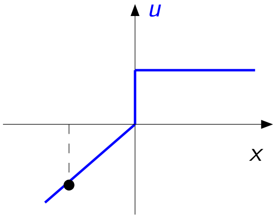

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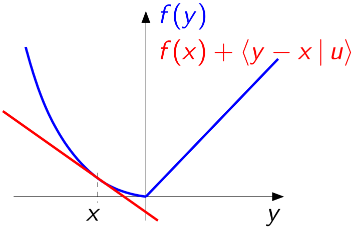

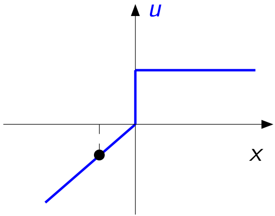

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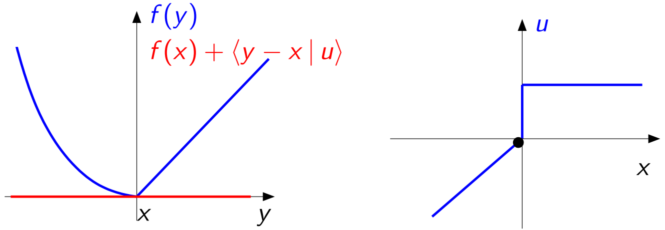

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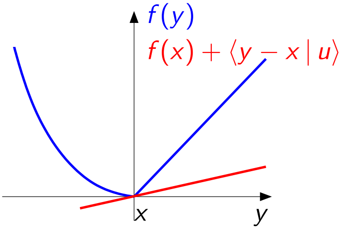

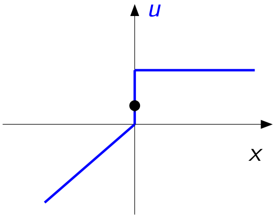

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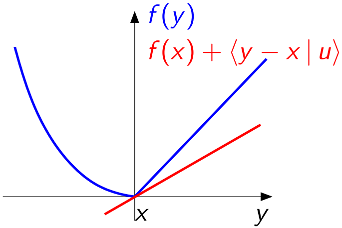

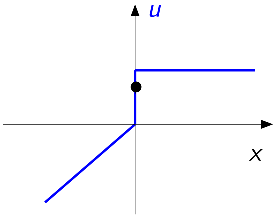

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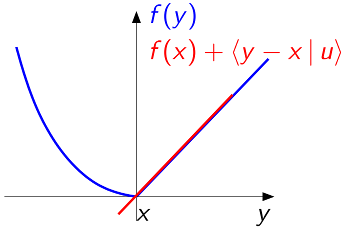

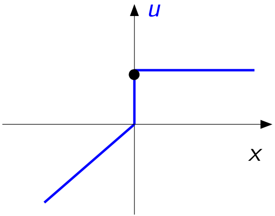

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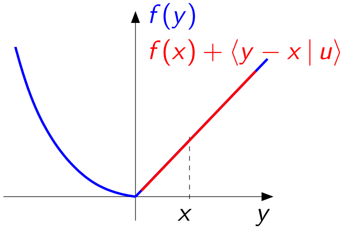

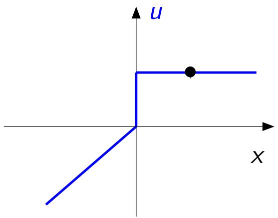

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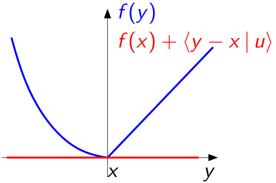

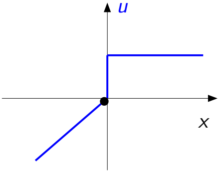

Fermat rule: $0 \in \partial f(x) \Leftrightarrow x \in \mathrm{Argmin} f$

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- If $x \notin \operatorname{dom} f$, then $\partial f(x) = \varnothing$.
- For every $x \in \operatorname{dom} f$, $\partial f(x)$ is a closed and convex set.

Subdifferential of a convex function: properties

If $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ is convex and it is Gâteaux differentiable at x, then

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$$(\forall y \in \mathcal{H}) \qquad \langle \nabla f(x) \mid y \rangle = \lim_{\substack{\alpha \to 0 \\ \alpha \neq 0}} \frac{f(x + \alpha y) - f(x)}{\alpha}.$$

Proof:
For every $\alpha \in [0, 1]$ and $y \in \mathcal{H}$,

$$\begin{aligned} & f\big(x + \alpha(y - x)\big) \le (1 - \alpha) f(x) + \alpha f(y) \\ \Rightarrow \quad & \langle \nabla f(x) \mid y - x \rangle = \lim_{\substack{\alpha \to 0 \\ \alpha \neq 0}} \frac{f\big(x + \alpha(y - x)\big) - f(x)}{\alpha} \le f(y) - f(x) \end{aligned}$$

Then $\nabla f(x) \in \partial f(x)$.

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<u>Proof</u>:
Inversely, if $u \in \partial f(x)$, then, for every $\alpha \in \left[0,+\infty\right[$ and $y \in \mathcal{H}$,

$$\begin{aligned} & f(x+\alpha y) \geq f(x) + \langle u \mid x+\alpha y - x\rangle \\ \Rightarrow \quad & \langle \nabla f(x) \mid y\rangle = \lim_{\substack{\alpha\to 0\\ \alpha\neq 0}} \frac{f(x+\alpha y)-f(x)}{\alpha} \geq \langle u \mid y\rangle \end{aligned}$$

By selecting $y = u - \nabla f(x)$, it results that $\|u-\nabla f(x)\|^2 \leq 0$ and then $u = \nabla f(x)$.

Subdifferential of a convex function: properties

Let $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ be Gâteaux differentiable on $\operatorname{dom} f$.
Then, f is convex if and only if

$$(\forall (x,y) \in (\operatorname{dom} f)^2) \quad f(y) \geq f(x) + \langle \nabla f(x) \mid y - x \rangle \,.$$

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Proof:

We have already seen that the gradient inequality holds when f is convex and differentiable at $x \in \mathcal{H}$.

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<u>Proof</u>:
Conversely, if the gradient inequality is satisfied, we have, for every $(x,y) \in (\operatorname{dom} f)^2$ and $\alpha \in [0,1]$, $\alpha x + (1-\alpha) y \in \operatorname{dom} f$, and

$$\begin{aligned} f(x) &\geq f(\alpha x + (1-\alpha)y) + (1-\alpha) \langle \nabla f(\alpha x + (1-\alpha)y) \mid x - y \rangle \\ f(y) &\geq f(\alpha x + (1-\alpha)y) + \alpha \langle \nabla f(\alpha x + (1-\alpha)y) \mid y - x \rangle \,. \end{aligned}$$

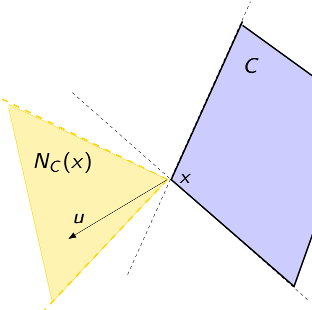
By multiplying the first inequality by α and the second one by $1-\alpha$ and summing them, we get

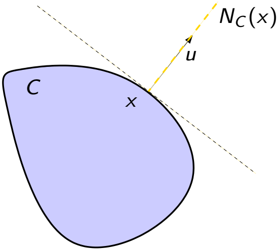
$$\alpha f(x) + (1-\alpha) f(y) \geq f(\alpha x + (1-\alpha) y).$$

Subdifferential of a convex function: example

For every $x \in \mathcal{H}$, $\partial \iota_C(x)$ is the **normal cone** to C at x defined by

$$N_C(x) = \begin{cases} \big\{ u \in \mathcal{H} \;\big|\; (\forall y \in C) \;\; \langle u \mid y - x \rangle \leq 0 \big\} & \text{if } x \in C \\ \varnothing & \text{otherwise.} \end{cases}$$





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- Si $x \in \operatorname{int} C$ alors $N_C(x) = \{0\}$.
- If C is a vector space, then for every $x \in C$, $N_C(x) = C^{\perp}$.
- Let $c \in \mathcal{H}$, $\rho \in \,]0,+\infty[$ and $C = \overline{\mathcal{B}}(c,\rho) = \big\{y \in \mathcal{H} \;\big|\; \|y - c\| \leq \rho\big\}$.
  For every $x \in C$,

$$N_C(x) = \begin{cases} \big\{\alpha(x - c) \;\big|\; \alpha \in [0,+\infty[\big\} & \text{if } \|x - c\| = \rho \\ \{0\} & \text{si } \|x - c\| < \rho. \end{cases}$$

Subdifferential calculus

Let $\mathcal{H}$ and $\mathcal{G}$ be real Hilbert spaces.

- Let $f\colon \mathcal{H} \to]-\infty, +\infty]$ be proper, then for every $\lambda \in]0, +\infty[$ $\partial(\lambda f) = \lambda \partial f$.
- Let $f\colon \mathcal{H} \to]-\infty, +\infty]$ and $g\colon \mathcal{G} \to]-\infty, +\infty]$ be two proper functions, and let $L \in \mathcal{B}(\mathcal{G}, \mathcal{H})$. Then, $\partial f + L^* \partial g L \subset \partial(f + g \circ L)$. In addition, if $f \in \Gamma_0(\mathcal{H})$, $f \in \Gamma_0(\mathcal{G})$, and

$$\operatorname{dom} g \cap \operatorname{int}\big(L(\operatorname{dom} f)\big) \neq \varnothing \text{ or } \operatorname{int}(\operatorname{dom} g) \cap L(\operatorname{dom} f) \neq \varnothing,$$

  then $\partial f + L^* \partial g L = \partial(f + g \circ L)$.
  Particular case:
  If $f \in \Gamma_0(\mathcal{H})$, $f \in \Gamma_0(\mathcal{G})$, $L \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ and f is finite valued, then $\partial f + \partial g = \partial(f + g)$.

Exercise 1: Huber function

Let $\rho > 0$ and set

$$f : \mathbb{R} \to \mathbb{R} : \mapsto \begin{cases} \frac{x^2}{2}, & \text{if } |x| \leq \rho \\ \rho|x| - \frac{\rho^2}{2}, & \text{otherwise} \end{cases}$$

1. What is the domain of f ?
2. Is f differentiable ? twice-differentiable ?
3. Prove that f is convex.

Exercise 2

Let $\mathcal{H}$ be a Hilbert space. Let $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ and let $C \subset \mathcal{H}$ such that $\operatorname{dom} f \cap C \neq \varnothing$.

1. Give a sufficient condition for $x \in \mathcal{H}$ to be a global minimizer of $f + \iota_C$.
2. When $f \in \Gamma_0(\mathcal{H})$, C is a nonempty closed convex set, and f is Gâteaux-differentiable on $\mathcal{H}$, give a necessary and sufficient condition for $x \in \mathcal{H}$ to be a minimizer of $f + \iota_C$.

Conjugate

Adrien-Marie Legendre
(1752–1833)

Werner Fenchel
(1905–1988)

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Conjugate: definition

Let $\mathcal{H}$ be a Hilbert space and $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$.
The conjugate of f is the function $f^*\colon \mathcal{H} \to [-\infty, +\infty]$ such that

$$(\forall u \in \mathcal{H}) \qquad f^*(u) = \sup_{x \in \mathcal{H}} \big(\langle x \mid u \rangle - f(x) \big) .$$

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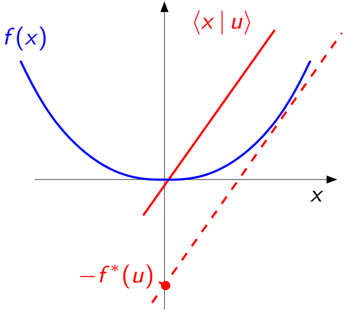

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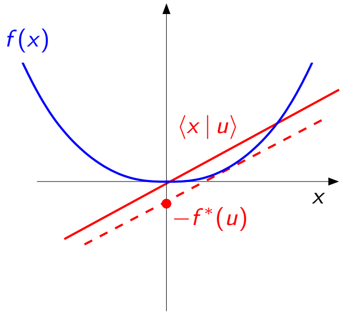

Conjugate: definition

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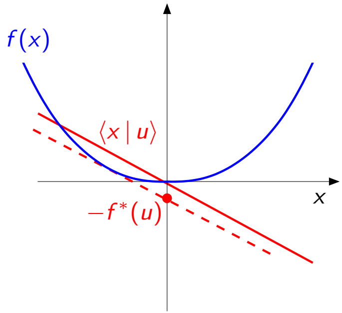

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Examples :

- $f = \frac{1}{2}\| \cdot \|^2 \Rightarrow f^* = \frac{1}{2}\| \cdot \|^2$

  Proof : For every $(x, u) \in \mathcal{H}^2$, $\langle x \mid u \rangle - \frac{1}{2}\|x\|^2 = \frac{1}{2}\|u\|^2 - \frac{1}{2}\|u - x\|^2$ is maximum in $x = u$.
  Consequently, $f^*(u) = \frac{1}{2}\|u\|^2$.

Conjugate: definition

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- $f = \frac{1}{2}\|\cdot\|^2 \Rightarrow f^* = \frac{1}{2}\|\cdot\|^2$.
- Let $\phi\colon \mathbb{R} \to \left]-\infty, +\infty\right]$ be a even function. $(\phi \circ \|\cdot\|)^* = \phi^* \circ \|\cdot\|$.
- $(\forall x \in \mathbb{R}^N)\ f(x) = \frac{1}{q}\|x\|_q^q$ with $q \in \left]1, +\infty\right[$
  $\Rightarrow (\forall u \in \mathbb{R}^N)\ f^*(u) = \frac{1}{q^*}\|u\|_{q^*}^{q^*}$ with $\frac{1}{q} + \frac{1}{q^*} = 1$

Conjugate: definition

Let $\mathcal{H}$ be a Hilbert space and $f\colon \mathcal{H} \to \,]-\infty,+\infty]$.
The **conjugate** of f is the function $f^*\colon \mathcal{H} \to [-\infty,+\infty]$ such that

$$(\forall u \in \mathcal{H}) \qquad \boxed{f^*(u) = \sup_{x\in \operatorname{dom} f} \big(\langle x \mid u\rangle - f(x)\big)}$$

- If f is even then f^* is even.

Proof :

$$\begin{aligned}(\forall u \in \mathcal{H}) \quad f^*(-u) &= \sup_{x\in\mathcal{H}} \big(\langle x \mid -u\rangle - f(x)\big)\\ &= \sup_{x\in\mathcal{H}} \big(\langle x \mid u\rangle - f(-x)\big)\\ &= \sup_{x\in\mathcal{H}} \big(\langle x \mid u\rangle - f(x)\big)\\ &= f^*(u)\end{aligned}$$

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- If f is even then f^* is even.
- For every $\alpha \in \left]0,+\infty\right[$, $(\alpha f)^* = \alpha f^*(\cdot/\alpha)$.
- For every $(y,v) \in \mathcal{H}^2$ et $\alpha \in \mathbb{R}$, $\big(f(\cdot - y) + \langle \cdot \mid v\rangle + \alpha\big)^* = f^*(\cdot - v) + \langle y \mid \cdot - v\rangle - \alpha$.
- Let $\mathcal{G}$ be a Hilbert space and $L \in \mathcal{B}(\mathcal{G},\mathcal{H})$ be an isomorphism. $(f \circ L)^* = f^* \circ (L^{-1})^*$.
- f^* is l.s.c. and convex.

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Moreau-Fenchel theorem

Let $\mathcal{H}$ be a Hilbert space and $f\colon \mathcal{H} \to \,]-\infty, +\infty]$ be a proper function.

$$f \text{ is l.s.c. and convex} \Leftrightarrow f^{**} = f.$$

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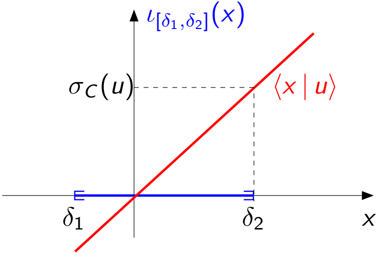
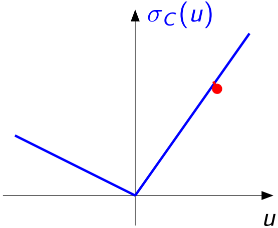
- Consquence: Si $f \in \Gamma_0(\mathbb{R})$ alors f^* est propre et donc $f^* \in \Gamma_0(\mathbb{R})$.

Conjugate: example

Let $\mathcal{H}$ be a Hilbert space and $C \subset \mathcal{H}$.
σ_C is the support function of C if

$$(\forall u \in \mathcal{H}) \qquad \sigma_C(u) = \sup_{x \in C} \langle x \mid u \rangle = \iota_C^*(u).$$





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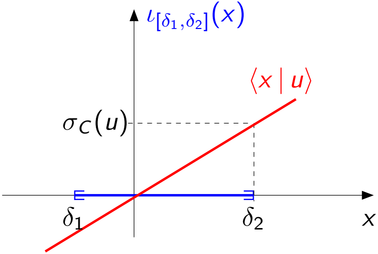

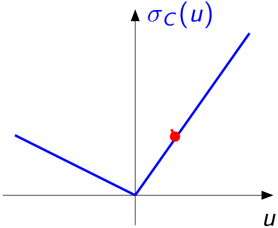

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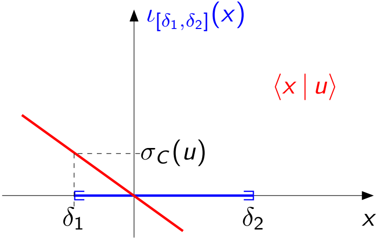

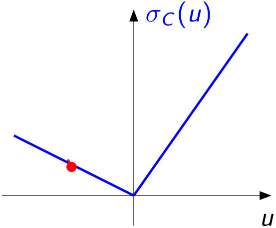

Conjugate: examples

Let $\mathcal{H}$ be a Hilbert space.
$f \colon \mathcal{H} \to \,]-\infty, +\infty]$ is **positively homogeneous** if

$$(\forall x \in \mathcal{H})(\forall \alpha \in \,]0, +\infty[) \qquad f(\alpha x) = \alpha f(x).$$

Conjugate: examples

Let $\mathcal{H}$ be a Hilbert space.
$f : \mathcal{H} \to \,]-\infty, +\infty]$ is **positively homogeneous** if

$$(\forall x \in \mathcal{H})(\forall \alpha \in \,]0, +\infty[) \qquad f(\alpha x) = \alpha f(x).$$

<u>Proof</u>: $(\Leftarrow)$
$f = \iota_C^*$ and $\iota_C \in \Gamma_0(\mathcal{H})$. Consequently $\sigma_C \in \Gamma_0(\mathcal{H})$.
Moreover, $(\forall x \in \mathcal{H})$ $(\forall \alpha \in \,]0, +\infty[)$ $\sigma_C(\alpha x) = \alpha \sigma_C(x)$.

Conjugate: examples

Let $\mathcal{H}$ be a Hilbert space.
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Proof : $(\Rightarrow)$
Soit $y \in \operatorname{dom} f$.
$f(0) = \lim_{\alpha\to 0, \alpha\geq 0} f\big((1-\alpha)0 + \alpha y\big) = \lim_{\alpha\to 0} \alpha f(y) = 0.$
Let $C = \big\{u \in \mathcal{H} \;\big|\; (\forall x \in \mathcal{H})\, \langle x \mid u\rangle \leq f(x)\big\}$.
We have, for every $u \in C$,
$f^*(u) = \sup_{x\in\mathcal{H}} \langle x \mid u\rangle - f(x) \leq 0 = \langle 0 \mid u\rangle - f(0) \leq f^*(u).$
Consequently, $f^*(u) = 0$.

Conjugate: examples

Let $\mathcal{H}$ be a Hilbert space.
$f\colon \mathcal{H} \to \,]-\infty,+\infty]$ is **positively homogeneous** if

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<u>Proof</u>: ($\Rightarrow$)
Moreover, for every $u \notin C$, there exists $x \in \mathcal{H}$ such that $\langle x \mid u\rangle > f(x)$. We have then, for every $\alpha \in \,]0,+\infty[$, $f^*(u) \geq \langle \alpha x \mid u\rangle - f(\alpha x) = \alpha\big(\langle x \mid u\rangle - f(x)\big)$. By taking α to $+\infty$, we obtain $f^*(u) = +\infty$.
To conclude, $f^* = \iota_C \in \Gamma_0(\mathcal{H}) \Rightarrow f = \sigma_C$ and C is a nonempty closed convex set.

Conjugate: examples

- Let $f\colon \mathbb{R} \to \left]-\infty, +\infty\right] : x \mapsto \begin{cases} \delta_1 x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \delta_2 x & \text{if } x > 0 \end{cases}$

  with $-\infty \leq \delta_1 < \delta_2 \leq +\infty$.
  Then, $f = \sigma_C$ where C is the closed real interval such that $\inf C = \delta_1$ et $\sup C = \delta_2$.

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- Let f be a ℓ^q norm of $\mathbb{R}^N$ with $q \in [1, +\infty]$.
  We have $f = \sigma_C$ where

$$C = \left\{ y \in \mathbb{R}^N \;\middle|\; \|y\|_{q^*} \leq 1 \right\} \qquad \text{with } \frac{1}{q} + \frac{1}{q^*} = 1.$$

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  Particular case: ℓ^1 norm of $\mathbb{R}^N$: $C = [-1,1]^N$.

Conjugate: properties

Fenchel-Young inequality : If f is proper, then

1. $\big(\forall (x,u) \in \mathcal{H}^2\big) \qquad f(x) + f^*(u) \geq \langle x \mid u \rangle$

2. $\big(\forall (x,u) \in \mathcal{H}^2\big) \qquad u \in \partial f(x) \Leftrightarrow \quad f(x) + f^*(u) = \langle x \mid u \rangle.$

If $f \in \Gamma_0(\mathcal{H})$, then

$$\big(\forall (x,u) \in \mathcal{H}^2\big) \qquad u \in \partial f(x) \;\Leftrightarrow\; x \in \partial f^*(u).$$

Exercise

- For every $c \in \mathbb{R}^N$, compute the conjugate of

$$
\begin{aligned}
f : \mathbb{R}^N &\to \left]-\infty, +\infty\right] \\
x &\to c^\top x + \iota_{[0,+\infty[^N}(x)
\end{aligned}
$$

- Same question for

$$
\begin{aligned}
g : \mathbb{R}^N &\to \left]-\infty, +\infty\right] \\
x &\to \iota_{[0,+\infty[^N}(x - c)
\end{aligned}
$$