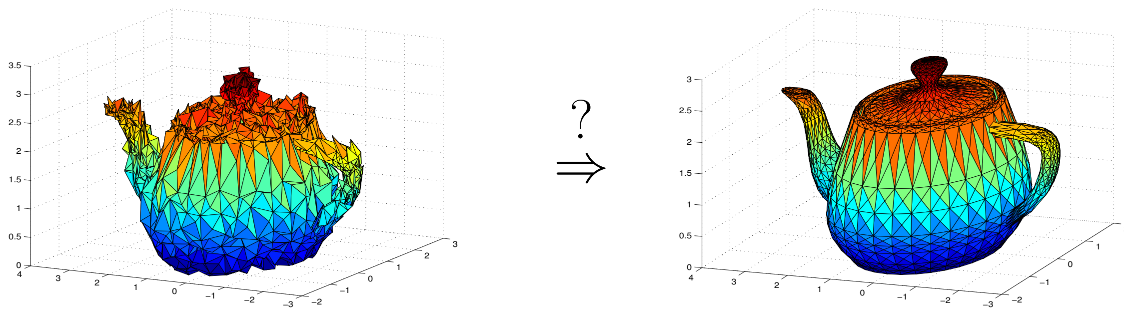
TP : Mesh denoising

1 Degradation model

A mesh can be viewed as a weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v^{(i)} \,|\, i \in \{1, \ldots, M\}\}$ denotes the set of vertices and $\mathcal{E} = \{e^{(i,j)} \,|\, (i,j) \in \mathbb{E}\}$ the set of edges, having cardinality of M and P, respectively. This graph is weighted in the sense that weights are included on both the edges and nodes. At each node of index $i \in \{1, \ldots, M\}$, we measure a 3D coordinates of the i-th vertex that is denoted by $y^{(i)} = (y_1^{(i)}, y_2^{(i)}, y_3^{(i)}) \in \mathbb{R}^3$. This observation results from an original unknown object $\overline{x} = (\overline{x}^{(i)})_{1 \leq i \leq M} \in \mathbb{R}^N$ (with $N = 3M$), the measure being degraded by a noise $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_N)$. An illustration of such a mesh is provided in Figure 1. An edge weight is a value assigned to each $e^{(i,j)}$, and it is denoted by $\omega_{i,j} \in]0, +\infty[$.



Noisy mesh y Original mesh $\overline{x}$

Figure 1: Example of a graph $\mathcal{G}$.

We propose here to find an estimate $\widehat{x} \in \mathbb{R}^N$ of the original mesh $\overline{x}$ by solving the following nonsmooth minimization problem involving only the knowledge of y:

$$\widehat{x} = \arg\min_{x \in \mathbb{R}^N} \frac{1}{2} \sum_{i=1}^{M} \|x^{(i)} - y^{(i)}\|_2^2 + \chi g(x), \tag{1}$$

where $g \in \Gamma_0(\mathbb{R}^N)$ denotes a regularization term and $\chi > 0$.

2 Analysis of the data

1. Load $\overline{x}$ and its associated triangulation mesh:

```
>> xyz = load('teapot_coord.txt');
>> tri = load('teapot_tri.txt');
```

2. Display the data by means of the function `trisurf.m`.

3. What are the values of N and M ?

4. Create a noisy mesh from $\overline{x}$ by means of `randn.m` such that,

$$(\forall k \in \{1,2,3\}) \quad \mathsf{y}_k = \overline{\mathsf{x}}_k + \epsilon_k \quad \text{with} \quad \varepsilon_k \sim \mathcal{N}(0, \sigma^2 I_M),$$

   where $\sigma^2 = 0.36$, and $\overline{\mathsf{x}}_k$ denotes the vector of the k-th coordinates of the object, a similar notation being used for other vectors. Display the resulting noisy mesh $y \in \mathbb{R}^N$.

5. For every vertices v_i with $i \in \{1, \ldots, M\}$, identify its neighbors from the triangulation mesh and compute the cardinality/degree of each. What is the maximum encountered cardinality? Identify the associated vertex on the mesh.

6. Compute the number of edges.

7. Create a table with the edges.

3 A simple constraint approach

A simple approach for mesh denoising consists of minimizing the function $x \mapsto \sum_{i=1}^{M} \|x^{(i)} - y^{(i)}\|_2^2$ subject to the constraint that $x \in C$ where C is a nonempty closed subset of $\mathbb{R}^N$.

1. Show that this problem can be formulated under the form of Problem (1).

2. Can we find a closed form solution to this problem ?

3. Implement it when we impose lower and upper bounds on the 3 coordinates of the object.

4 Tikkonov-like regularization

We add to the previous constraint a quadratic regularization term applied on the difference between neighbors, by setting

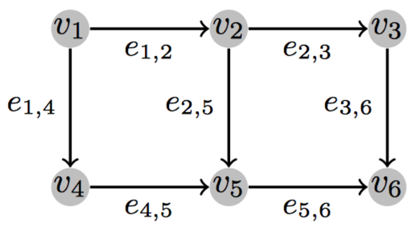
$$(\forall x \in \mathbb{R}^N) \qquad g(x) = \frac{1}{2} \sum_{j \in \mathcal{N}_i} \omega_{i,j}^2 \|x^{(j)} - x^{(i)}\|_2^2 + \iota_C(x),$$

where, for every $i \in \{1, \dots, M\}$, $\mathcal{N}_i$ is the neigborhood of node of index i.

The incidence matrix of a graph is a fundamental operator for formulating variational problems. Specifically, the edge-node incidence matrix $A \in \mathbb{R}^{P \times M}$ defines the discrete calculus analogue of the gradient, while $A^\top$ is known to define the discrete calculus analogue of the divergence. The incidence matrix can be viewed as an operator mapping functions on nodes (analogue to a scalar field) to functions on edges (analogue to a vector field) and its elements are defined as

$$A_{e_{i,j} v_k} = \begin{cases} -1 & \text{if} \quad i = k, \\ +1 & \text{if} \quad j = k, \\ 0 & \text{otherwise,} \end{cases}$$

for every vertex v_k with $k \in \{1, \dots, M\}$ and edge $e_{i,j}$ with $(i,j) \in \{1, \dots, M\}^2$. An example of a graph represented with its incidence matrix is given in Figure 2. We introduce the weighted incidence matrix $D = \text{diag}(\omega)A$, where $\text{diag}(\omega)$ is a diagonal matrix whose diagonal elements correspond to the square roots of the components of edge weight vector ω.



$A =$

	v_1	v_2	v_3	v_4	v_5	v_6
$e_{1,2}$	-1	1	0	0	0	0
$e_{2,3}$	0	-1	1	0	0	0
$e_{4,5}$	0	0	0	-1	1	0
$e_{5,6}$	0	0	0	0	-1	1
$e_{1,4}$	-1	0	0	1	0	0
$e_{2,5}$	0	-1	0	0	1	0
$e_{3,6}$	0	0	-1	0	0	1

Figure 2: A graph and its incidence matrix $A \in \mathbb{R}^{P \times M}$ with $M = 6$ and $P = 7$.

Using the incidence matrix notation, Problem (1) can be reexpressed as

$$\widehat{x} = \arg \min_{x \in \mathbb{R}^N} \frac{1}{2} \sum_{i=1}^{M} \|x^{(i)} - y^{(i)}\|_2^2 + \frac{\chi}{2} \sum_{k=1}^{3} \|D\mathsf{x}_k\|_2^2 + \iota_C(x).$$

We propose to estimate $\widehat{x}$ using an optimization approach.

1. Let $f \colon x \mapsto \frac{1}{2} \sum_{i=1}^{M} \|x^{(i)} - y^{(i)}\|_2^2 + \frac{\chi}{2} \sum_{k=1}^{3} \|D\mathsf{x}_k\|_2^2$. Is f convex ? proper ? Lipschitz continuous ? If so, how can we compute a Lipschitz constant of its gradient ?

2. Which algorithm would you suggest to solve the optimization problem ? Discuss the choice of the parameters of this algorithm.

3. How the algorithm proposed in the previous part can be adapted ?

4. Implement it and compare its performance by setting the weights to 1.

5. An alternative algorithm for minimizing $f + h$ with $h \in \Gamma_0(\mathbb{R}^N)$ is given below:

   Let β denote the Lipschitz constant of ∇f.
   Take $u_1 = x_0 \in \mathbb{R}^N$, $t_1 = 1$.
   For $n = 1, 2, \ldots$

   $$\left\lfloor \begin{array}{l} x_n = \mathrm{prox}_{\frac{1}{\beta}h}\big(u_n - \frac{1}{\beta}\nabla f(u_n)\big) \\ t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2} \\ u_n = x_n + \Big(\frac{t_n - 1}{t_{n+1}}(x_n - x_{n-1})\Big). \end{array} \right.$$

   Implement this algorithm and compare it to the previous one in terms of convergence speed.

6. What is the best choice for parameter χ in terms of mean square error between the estimated object and the ground truth ?

7. Is there be an alternative way of splitting the objective function ? Test this second solution.

5 Anisotropic TV

In this part, we focus on a sparse regularization term applied on the difference between neighbors, by choosing

$$(\forall x \in \mathbb{R}^N) \qquad g(x) = \chi\big(\sum_{j \in \mathcal{N}_i} \omega_{i,j} \|x^{(j)} - x^{(i)}\|_1\big).$$

We propose to estimate $\widehat{x}$ using an optimization procedure based on PPXA+ algorithm.

1. Recall the closed form expression of $\mathrm{prox}_{\gamma\|\cdot - y^{(i)}\|^2}$ with $\gamma \in \,]0, +\infty[$ and the one associated with the ℓ_1-norm.

2. Implement PPXA+ for estimating $\widehat{x}$ by setting the weights to 1.

3. Evaluate the performance regarding the choice of χ.

4. Evaluate the impact of the choice of parameters γ and $(\lambda_n)_{n\in\mathbb{N}}$ of the algorithm.

5. We now evaluate the performance of the reconstruction according to the choice of the weights. Compute $\omega_{i,j} = \|\widehat{x}^{(j)} - \widehat{x}^{(i)}\|$ where $\widehat{x}$ denotes the anisotropic TV solution with weights set to 1. Implement once again PPXA+ for estimating a solution integrating these estimated weights in the minimization formulation. Comment the results.

6. Redo all the work by replacing the incidence matrix by the graph Laplacian matrix.