Solution of Mismatched Monotone+Lipschitz Inclusion Problems

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Workshop DIPOpt

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November 28, 2023

Overview

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Motivation

Optimization Problem

$$\min_{x\in\mathcal{H}} f(x) + \frac{\alpha}{2}\|Lx - c\|^2 \quad (1)$$

- $\mathcal{H}$ and $\mathcal{G}$ are real Hilbert spaces.
- $f \in \Gamma_0(\mathcal{H})$.
- $L : \mathcal{H} \to \mathcal{G}$ is a linear bounded operator.
- $\alpha \in]0, +\infty[$.

Motivation

Optimization Problem

$$\min_{x\in\mathcal{H}} f(x) + \frac{\alpha}{2}\|Lx - c\|^2 \tag{2}$$

Image restoration/reconstruction

- Recover an image x from the observation c
- L: observation process
- f: regularization term
- $\frac{\alpha}{2}\|Lx - c\|^2$: data fidelity term

Motivation

Optimization Problem

$$\min_{x\in\mathcal{H}} f(x) + \frac{\alpha}{2}\|Lx - c\|^2 \tag{3}$$

$$+$$

(standard qualification conditions)

$$\Longleftrightarrow$$

Monotone Inclusion

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in \partial f(x) + \alpha L^*(Lx - c). \tag{4}$$

Motivation

Monotone Inclusion

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in \partial f(x) + \alpha L^*(Lx - c). \tag{5}$$

Forward-Backward

$$(\forall n \in \mathbb{N}) \quad \lfloor z_{n+1} = \operatorname{prox}_{\gamma f}(z_n - \gamma L^*(Lz_n - c)). \tag{6}$$

$$\operatorname{prox}_f : x \mapsto \underset{y \in \mathcal{H}}{\arg\min} \left(f(y) + \frac{1}{2}\|x - y\|^2 \right).$$

- Iterative methods require L^*
- In some practical instances L^* is not available and it must be approximated

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$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in \partial f(x) + \alpha L^*(Lx - c). \tag{5}$$

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$$\text{prox}_f : x \mapsto \arg\min_{y \in \mathcal{H}} \left(f(y) + \frac{1}{2}\|x - y\|^2 \right).$$

- Iterative methods require L^*
- In some practical instances L^* is not available and it must be approximated

$$L^* \longrightarrow {\color{red}K}$$

Motivation

Monotone Inclusion

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in \partial f(x) + \alpha L^*(Lx - c). \tag{7}$$

$$L^* \longrightarrow K$$

Adjoint Mismatch Problem

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in \partial f(x) + \alpha K(Lx - c). \tag{8}$$

- $K : \mathcal{G} \to \mathcal{H}$ is a linear bounded operator approximating L^*.
- The convergence results for classical splitting algorithms in the literature are no longer valid in this context.
- A solution to the adjoint mismatch problem is not necessarily a solution to the original problem.

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Mismatched Forward-Backward (FB) Algorithm

Adjoint Mismatch Monotone Inclusion

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in \partial f(x) + K(Lx - c). \tag{9}$$

- $f \in \Gamma_0(\mathcal{H})$.
- $L : \mathcal{H} \to \mathcal{G}$ and $K : \mathcal{G} \to \mathcal{H}$ are linear bounded operators.

Mismatched Forward-Backward Algorithm (Chouzenoux et al 2021)

$$(\forall n \in \mathbb{N}) \quad \left\lfloor\, x_{n+1} = x_n + \theta_n \big(\operatorname{prox}_{\gamma f}(x_n - \gamma K_n(Lx_n - c)) - x_n\big) \right. \tag{10}$$

In the case when $K_n \equiv L^*$ the classical Forward-Backward Algorithm is recovered.

Remark

Let $T\colon \mathcal{H} \to \mathcal{H}$ and $\beta \in \left]0, +\infty\right[$. The operator T is β-cocoercive if

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x - y \mid Tx - Ty \rangle \geq \beta \| Tx - Ty\|^2.$$

The operator T is β-Lipschitzian if

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \| Tx - Ty\| \leq \beta \|x - y\|.$$

- Every cocoercive operator is a Lipschitzian opeartor, the converse is not true.
- If $T = \nabla f$ where f is a convex differentiable function, the notion of Lipschitz and cocoercive gradients are equivalent (Baillon–Haddad).

Mismatched Forward-Backward Algorithm (Chouzenoux et al 2021)

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \; x_{n+1} = x_n + \theta_n\big(\mathrm{prox}_{\gamma f}(x_n - \gamma K_n(Lx_n - c)) - x_n\big) \right. \tag{11}$$

Convergence (Chouzenoux et al 2021)

Suppose that

1. $\{K_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{G},\mathcal{H})$, s.t., $\|K_n - K\| \leq \omega_n$, where $\sum_{n\in\mathbb{N}} \omega_n < +\infty$.
2. $K \circ L$ is η-cocoercive.
3. $\gamma \in]0, 2\eta[$.
4. $\{\theta_n\}_{n\in\mathbb{N}} \subset [0,\delta]$, $\sum \theta_n(\delta - \theta_n) = +\infty$, where $\delta = 2 - \gamma/(2\eta)$.

Then $x_n \rightharpoonup x \in \mathcal{H}$ such that x is a solution to (10).

Cocoercivity of $K \circ L$ (Chouzenoux et al 2021)

Set $T = K \circ L$. Suppose that $\operatorname{ran}(T + T^*)$ is closed. Then, T is η-cocoercive if and only if $\lambda_{\min} \geq 0$, $\ker(T + T^*) = \ker(T)$, and

$$\eta \leq \frac{2}{\|M\|^2}. \tag{12}$$

- $\lambda_{\min}$ is defined by

$$\lambda_{\min} = \inf \Big\{ \langle x \mid KLx \rangle \;\Big|\; x \in \mathcal{H},\ \|x\| = 1 \Big\}. \tag{13}$$

- $M = (\mathrm{Id} + ((T - T^*)(T + T^*)^{\#})(T + T^*)^{1/2})$
- $T^{\#}$ denotes the Moore-Penrose generalized inverse.

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Mismatched Condat–Vũ (CV) Algorithm

Adjoint Mismatch Monotone Inclusion

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in \partial f(x) + N^*\partial g(Nx) + K(Lx - c). \tag{14}$$

- $g \in \Gamma_0(\mathcal{K})$.
- $N : \mathcal{H} \to \mathcal{K}$ is a linear bounded operator.

Mismatched Condat–Vũ Algorithm (Chouzenoux et al 2023)

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_{n+1} = \text{prox}_{\tau f}(x_n - \tau(K_n(Hx_n - c + N^* u_n)) \\ v_{n+1} = \text{prox}_{\sigma \hat{g}^*}(u_n + \sigma N(2y_{n+1} - x_n)) \\ x_{n+1} = x_n + \theta_n(y_{n+1} - x_n) \\ u_{n+1} = u_n \theta_n(v_{n+1} - u_n) \end{array} \right. \tag{15}$$

In the case when $K_n \equiv L^*$ the Condat–Vũ Algorithm is recovered.

Mismatched Condat–Vũ Algorithm (Chouzenoux et al 2023)

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_{n+1} = \mathrm{prox}_{\tau f}(x_n - \tau(K_n(Hx_n - c + N^* u_n)) \\ v_{n+1} = \mathrm{prox}_{\sigma \hat{g}^*}(u_n + \sigma N(2y_{n+1} - x_n)) \\ x_{n+1} = x_n + \theta_n(y_{n+1} - x_n) \\ u_{n+1} = u_n \theta_n(v_{n+1} - u_n) \end{array} \right. \tag{16}$$

Convergence (Chouzenoux et al 2023)

Suppose that

1. $\{K_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{G},\mathcal{H})$, s.t., $\|K_n - K\| \leq \omega_n$, where $\sum_{n\in\mathbb{N}} \omega_n < +\infty$.
2. $K \circ L$ is η-cocoercive.
3. $1 - \tau\delta\|N\|^2 > \tau\|M\|^2/4$
4. $\{\theta_n\}_{n\in\mathbb{N}} \subset [0,\delta]$, $\sum \theta_n(\delta - \theta_n) = +\infty$, where $\delta = 2 - (1/\tau - \sigma\|N\|^2)^{-1}\|M\|^2/4$.

Then $x_n \rightharpoonup x \in \mathcal{H}$ such that x is a solution to (14).

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Mismatched Primal-Dual (PD) Algorithm

Adjoint Mismatch Monotone Inclusion

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in \partial f(x) + K\partial h(Lx). \tag{17}$$

- $h \in \Gamma_0(\mathcal{G})$.

Mismatched Primal-Dual (Lorenz–Schneppe 2022)

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{aligned} x_{n+1} &= \operatorname{prox}_{\tau_n f}(x_n - \tau_n K u_n) \\ y_{n+1} &= x_{n+1} + \lambda_n(x_{n+1} - x_n) \\ u_{n+1} &= \operatorname{prox}_{\sigma_n h^*}(u_n + \sigma_n L y_{n+1}) \end{aligned} \right. \tag{18}$$

In the case when $K = L^*$ the Primal-Dual Algorithm (Chambolle–Pock 2011) is recovered.

Mismatched Primal-Dual (Lorenz–Schneppe 2022)

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} x_{n+1} = \mathrm{prox}_{\tau f}(x_n - \tau K^* u_n) \\ y_{n+1} = x_{n+1} + \lambda(x_{n+1} - x_n) \\ u_{n+1} = \mathrm{prox}_{\sigma h^*}(u_n + \sigma L y_{n+1}) \end{array} \right. \tag{19}$$

Convergence (Lorenz–Schneppe 2022)

If f and h are strongly convex functions, for adequate step-sizes τ, σ and extrapolation parameter λ, the sequence $(x_n)_{n\in\mathbb{N}}$ converges linearly to a solution to (17).

• The authors provide examples where the convergence of the Mismatched Primal-Dual algorithm fails to converge if one of the functions is not strongly convex.

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Main Problem

We will focus on the adjoint mismatch problem associated to the following inclusion

Monotone Inclusion

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in Ax + Cx + \alpha L^*Lx + L^*BLx, \tag{20}$$

where

- $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- C is β-cocoercive
- B is monotone and ζ-Lipschitzian

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where

- $A \colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- C is β-cocoercive
- B is monotone and ζ-Lipschitzian

Example:

- $A = \partial f$
- $C = \nabla g$
- $B = \nabla h$

Main Problem

Denote: $D = \alpha L^* L + L^* B L$.

Monotone Inclusion

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in Ax + Cx + Dx, \tag{21}$$

Note that D is monotone and Lipschitzian

$$\|L^*(\alpha \mathrm{Id} + B)Lx - L^*(\alpha \mathrm{Id} + B)Ly\| \leq \|L\|^2(\alpha + \zeta)\|x - y\|. \tag{22}$$

Main Problem

Denote: $D = \alpha L^* L + L^* BL$.

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$$\|L^*(\alpha \mathrm{Id} + B)Lx - L^*(\alpha \mathrm{Id} + B)Ly\| \leq \|L\|^2(\alpha + \zeta)\|x - y\|. \tag{22}$$

Therefore, the problem can be solved by:

- (FBHF) *Forward-Backward-Half-Forward* (Briceño-Arias–Davis 2018)

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Therefore, the problem can be solved by:

- (FBHF) *Forward-Backward-Half-Forward* (Briceño-Arias–Davis 2018)
- (FDRF) *Forward-Douglas–Rachford-Forward* (Ryu–Vũ 2020)

Forward-Backward-Half-Forward (FBHF)

FBHF (Briceño-Arias–Davis 2018)

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{aligned} u_n &= Dz_n \\ y_n &= z_n - \gamma(C(z_n) + u_n) \\ x_n &= J_{\gamma A}(y_n) \\ z_{n+1} &= x_n + \gamma(u_n - Dx_n). \end{aligned} \right. \tag{23}$$

Convergence (Briceño-Arias–Davis 2018)

If

$$\gamma \in \left]0, \frac{4\beta}{1 + \sqrt{1 + 16\beta^2 \|L\|^4 (\alpha + \zeta)^2}}\right[, \tag{24}$$

then, $z_n \rightharpoonup x \in \mathcal{H}$ where x is a solution to the monotone inclusion.

- $J_{\gamma A} = (\mathrm{Id} + \gamma A)^{-1}$, if $A = \partial f$, then $J_{\gamma A} = \mathrm{prox}_{\gamma f}$.

Remark

FBHF combines Forward-Backward (FB) and Forward-Backward-Forward (FBF - Tseng 2000).

$$\left\lfloor \begin{aligned} u_n &= Dz_n \\ y_n &= z_n - \gamma(Cz_n + u_n) \\ x_n &= J_{\gamma A}(y_n) \\ z_{n+1} &= x_n + \gamma(u_n - Dx_n). \end{aligned} \right. \xRightarrow[C=0]{} \underbrace{\left\lfloor \begin{aligned} y_n &= J_{\gamma A}(z_n - \gamma Dz_n) \\ z_{n+1} &= y_n + \gamma(Dz_n - Dy_n). \end{aligned} \right.}_{\text{FBF}}$$

$$\Big\Downarrow D=0$$

$$\underbrace{\lfloor z_{n+1} = J_{\gamma A}(z_n - \gamma C(z_n)).}_{\text{FB}}$$

Forward-Douglas–Rachford-Forward (FDRF)

FDRF (Ryu–Vũ 2020)

$$(\forall n \in \mathbb{N}) \quad \begin{array}{|l} x_n = J_{\gamma C} z_n \\ w_n = D x_n \\ y_n = J_{\gamma A}(2x_n - z_n - \gamma w_n) \\ z_{n+1} = z_n + y_n - x_n - \gamma(D y_n - w_n). \end{array} \tag{25}$$

Convergence

If

$$\gamma \in \left\{ \gamma \in \left]0, +\infty\right[\;\middle|\; \|L\|^4 (\alpha + \zeta)^2 \gamma^2 \left(1 + \frac{\gamma}{2\beta}\right) < 1 \right\}. \tag{26}$$

then, $z_n \rightharpoonup x \in \mathcal{H}$ where $J_{\gamma C} x$ is a solution to the monotone inclusion.

Remark

FDRF combines Forward-Backward-Forward and Douglas–Rachford Splitting (DR - Lions–Mercier 1979).

$$\left\lfloor\begin{aligned} x_n &= J_{\gamma C} z_n \\ w_n &= D x_n \\ y_n &= J_{\gamma A}(2x_n - z_n - \gamma w_n) \\ z_{n+1} &= z_n + y_n - x_n - \gamma(Dy_n - w_n) \end{aligned}\right. \quad \xRightarrow[C=0]{} \quad \underbrace{\left\lfloor\begin{aligned} y_n &= J_{\gamma A}(z_n - \gamma D z_n) \\ z_{n+1} &= y_n + \gamma(Dz_n - Dy_n) \end{aligned}\right.}_{\text{FBF}}$$

$$\Big\Downarrow D=0$$

$$\underbrace{\lfloor z_{n+1} = J_{\gamma A}(2J_{\gamma C} z_n - z_n) + (\mathrm{Id} - J_{\gamma C}) z_n}_{\text{DR}}$$

Mismatch Problem

Monotone Inclusion

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in Ax + Cx + \alpha L^* Lx + L^* BLx, \tag{27}$$

Consider the following relaxed monotone inclusion

Adjoint Mismatch Monotone Inclusion

$$\text{find} \quad x \in \mathcal{H} \quad \text{such that} \quad 0 \in Ax + Cx + \alpha K Lx + K BLx \tag{28}$$

- $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is a maximally ρ-monotone operator for some $\rho \in \mathbb{R}$, i.e., for every $(x, u) \in \operatorname{gra} A$ and $(y, v) \in \operatorname{gra} A$ we have

$$\langle x - y \mid u - v \rangle \geq \rho \|x - y\|^2. \tag{29}$$

- If $\rho > -1$, $J_A = (\mathrm{Id} + A)^{-1}$ is single valued and Lipschitzian.
- We assume that the solution set is not empty.

Mismatch Problem

Adjoint Mismatch Inclusion

$$\text{find} \quad x \in \mathcal{H} \quad \text{such that} \quad 0 \in Ax + Cx + \alpha K Lx + K BLx \tag{30}$$

- $K(\alpha \mathrm{Id} + B)L$ is Lipschitzian but not necessarily monotone.
- The convergence results for FBHF and FDRF are no longer valid in this context.

Mismatch Problem

Adjoint Mismatch Inclusion

$$\text{find} \quad x \in \mathcal{H} \quad \text{such that} \quad 0 \in Ax + Cx + \alpha KLx + KBLx \tag{30}$$

- $K(\alpha \mathrm{Id} + B)L$ is Lipschitzian but not necessarily monotone.
- The convergence results for FBHF and FDRF are no longer valid in this context.

Goals:

- Characterize the Monotone+Lipschitzian mismatch
- Extend the convergence of FBHF and FDRF for solving the mismatch problem allowing approximation on the iterations
- Relax hypotheses in previous results

Notation

Notation

- $D_K : x \mapsto \alpha KLx + KBLx$
- $D_{K_n} : x \mapsto \alpha K_n Lx + K_n BLx$
- κ_K *be the Lipschitz constant of* D_K
- ζ_K *be the Lipschitz constant of* $(K - L^*)BL$
- $$\lambda_{\min} = \inf \left\{ \langle x \mid KLx \rangle \;\middle|\; x \in \mathcal{H},\ \|x\| = 1 \right\}. \tag{31}$$

Adjoint Mismatch Monotone Inclusion

$$\text{find} \quad x \in \mathcal{H} \quad \text{such that} \quad 0 \in Ax + Cx + D_K x. \tag{32}$$

Comparison with the original solution

Proposition

Assume that $\rho + \alpha\lambda_{\min} > 0$. Then, there exists a unique solution z^* to the monotone inclusion (with L^*). Furthermore, every solution z to the adjoint mismatch monotone inclusion is such that

$$\|z - z^*\| \leq \frac{1}{\rho + \alpha\lambda_{\min}} \|L^* - K\| \, \|(\alpha \mathrm{Id} + BL)z\|. \tag{33}$$

- $\rho + \alpha\lambda_{\min} > 0$ is associated to the strongly monotonicity of $A + D$.

Assumptions

We introduce the following assumptions:

Assumption

1. $\hat{\rho} = \rho + \alpha\lambda_{\min} - \zeta_K \geq 0$
2. $(K_n)_{n\in\mathbb{N}}$ *is a sequence of* $\mathcal{B}(\mathcal{G},\mathcal{H})$ *such that, for every* $n \in \mathbb{N}$, $\|K_n - K\| \leq \omega_n$, *where* $\{\omega_n\}_{n\in\mathbb{N}} \subset [0,+\infty[$ *and* $\sum_{n\in\mathbb{N}} \omega_n < +\infty$.

Remark

- *Note that* $\zeta_K \leq \|L^* - K\|\|L\|\zeta$.
- *1 guarantees the monotonicity of* $A + D_K$.
- *If* $B = 0$. *1:* $\rho + \alpha\lambda_{\min} \geq 0$.
- *If* $\alpha = 0$. *1:* $\rho - \zeta_K \geq 0 \Rightarrow \rho > 0$.

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Mismatched Forward-Backward-Half-Forward

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{aligned} u_n &= D_{K_n} z_n \\ y_n &= z_n - \gamma_n (C z_n + u_n) \\ x_n &= J_{\gamma_n A}(y_n) \\ z_{n+1} &= x_n + \gamma_n (u_n - D_{K_n} x_n). \end{aligned} \right. \tag{34}$$

Theorem

Let $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, \chi - \varepsilon]$*, for some* $\varepsilon \in]0, \chi/2[$*, such that* $\gamma_n \rho > -1$*, where* $\chi = 4\beta/(1 + \sqrt{1 + 16\beta^2\kappa_K^2})$ *if* $\rho \geq 0$ *and* $\chi = \min\left\{4\beta/(1 + \sqrt{1 + 16\beta^2\kappa_K^2}), -1/\rho\right\}$ *otherwise. Therefore*

1. $(z_n)_{n\in\mathbb{N}}$ *converges weakly to a zero of* $(A + C + D_K)$.
2. *If* $\hat{\rho} > 0$ *and there exists* $\overline{\eta} \in [0, 1[$ *such that, for every* $n \in \mathbb{N}$*,* $\omega_n = \omega_0\, \overline{\eta}^n$*, then* $(z_n)_{n\in\mathbb{N}}$ *converges linearly to the unique zero of* $(A + C + D_K)$.

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Mismatched Forward-Douglas–Rachford-Forward

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{aligned} x_n &= J_{\gamma C} z_n \\ w_n &= D_{K_n} x_n \\ y_n &= J_{\gamma A}(2x_n - z_n - \gamma w_n) \\ z_{n+1} &= z_n + y_n - x_n - \gamma(D_{K_n} y_n - w_n). \end{aligned} \right. \tag{35}$$

Theorem

Suppose that $\gamma \in \left\{ \gamma \in \left]0, +\infty\right[\;\middle|\; \kappa_K^2 \gamma^2 \left(1 + \frac{\gamma}{2\beta}\right) < 1 \text{ and } \rho\gamma > -1 \right\}$.
Then, the following hold.

1. $(z_n)_{n\in\mathbb{N}}$ *converges weakly to some* $\overline{z} \in \mathcal{H}$ *and* $(x_n)_{n\in\mathbb{N}}$ *converges weakly to* $J_{\gamma C}\overline{z} \in \operatorname{zer}(A + C + D_K)$.
2. *If* $\hat{\rho} > 0$ *and there exists* $\overline{\eta} \in [0, 1[$ *such that, for every* $n \in \mathbb{N}$, $\omega_n = \omega_0 \overline{\eta}^n$, *then* $(z_n)_{n\in\mathbb{N}}$ *converges linearly to* $\overline{z} \in \mathcal{H}$ *and* $(x_n)_{n\in\mathbb{N}}$ *converges linearly to* $J_{\gamma C}\overline{z}$ *the unique zero of* $(A + C + D_K)$.

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Sketch of proof

Let $Q^\gamma : \mathcal{H} \to \mathcal{H}$ be the operator associated to FBHF or FDRF ($Q^\gamma = Q^\gamma(K)$). We can deduce the following Fejér type inequality.

$$(\forall z \in \mathcal{H})(\forall z^* \in \boldsymbol{Z}) \quad \|Q^\gamma z - z^*\|^2 \leq \|z - z^*\|^2 - \phi^\gamma(z), \qquad (36)$$

where $\phi^\gamma : \mathcal{H} \to [0, +\infty[$.

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where $\phi^\gamma : \mathcal{H} \to [0, +\infty[$.

In order to consider approximations on K we consider an operator $Q_n^\gamma : \mathcal{H} \to \mathcal{H}$ ($Q_n^\gamma = Q_n^\gamma(K_n)$) approximating Q and we prove that

$$(\forall z \in \mathcal{H}) \quad \|Q_n^\gamma z - Q^\gamma z\| \leq \xi_n(z^*)\|z - z^*\| + \eta_n(z^*), \tag{37}$$

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where $\phi^\gamma : \mathcal{H} \to [0, +\infty[$.

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$$(\forall z \in \mathcal{H}) \quad \|Q_n^\gamma z - Q^\gamma z\| \leq \xi_n(z^*)\|z - z^*\| + \eta_n(z^*), \tag{37}$$

where, for every $z^* \in \boldsymbol{Z}$,

- $\{\xi_n(z^*)\}_{n\in\mathbb{N}} \subset [0, +\infty[$ such that $\sum_{n\in\mathbb{N}} \xi_n(z^*) < +\infty$
- $\{\eta_n(z^*)\}_{n\in\mathbb{N}} \subset [0, +\infty[$ such that $\sum_{n\in\mathbb{N}} \eta_n(z^*) < +\infty$.

Sketch of proof

Mismatched FBHF and Mismatched FDRF are defined by the recurrence:

$$(\forall n \in \mathbb{N}) \quad z_{n+1} = Q_n^\gamma z_n. \tag{38}$$

We can deduce the following quasi-Fejér inequality

$$\|z_{n+1} - z^*\| \leq (1 + \xi_n(z^*))\|z_n - z^*\| + \eta_n(z^*). \tag{39}$$

Mismatched FBHF and Mismatched FDRF are defined by the recurrence:

$$(\forall n \in \mathbb{N}) \quad z_{n+1} = Q_n^{\gamma} z_n. \tag{38}$$

We can deduce the following quasi-Fejér inequality

$$\|z_{n+1} - z^*\| \leq (1 + \xi_n(z^*))\|z_n - z^*\| + \eta_n(z^*). \tag{39}$$

Then, we deduce that

$$(\|z_n - z^*\|)_{n\in\mathbb{N}} \text{ is convergent.} \tag{40}$$

Sketch of proof

Mismatched FBHF and Mismatched FDRF are defined by the recurrence:

$$(\forall n \in \mathbb{N}) \quad z_{n+1} = Q_n^{\gamma} z_n. \tag{38}$$

We can deduce the following quasi-Fejér inequality

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- For linear convergence, a similar procedure is followed, using the strong monotonicity of $A + D_K$.

Remark

Adjoint Mismatch Inclusion

$$\text{find} \quad x \in \mathcal{H} \quad \text{such that} \quad 0 \in Ax + Cx + \alpha K Lx + K BLx \tag{41}$$

Mismatched Alg.	A	C	B	$\lambda_{\min}$
FB	$\mathcal{M}$	0	0	$\lambda_{\min} \geq 0$ (+ others)
CV	$\mathcal{M}$	$N^* \mathcal{M} N$	0	$\lambda_{\min} \geq 0$ (+ others)
CP	$\mathcal{S}$	0	$\mathcal{S}$	$\alpha = 0$
FBHF & FDRF	ρ-$\mathcal{M}$	$\mathcal{C}$	ζ-$\mathcal{L}$	$\rho + \alpha\lambda_{\min} \geq \|L^* - K\|\|L\|\zeta$

- $\mathcal{M}$: Maximally monotone
- $\mathcal{S}$: Strongly monotone
- $\mathcal{C}$: Cocoercive
- $\mathcal{L}$: Lipschitzian

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Numerical Example

Optimization Problem

$$\min_{x \in \mathbb{R}^N} f(x) + g(x) + \alpha \frac{\|Lx - c\|^2}{2} + h(Lx). \tag{42}$$

Image reconstruction

- Recover an image x from the observation c.
- $f(x) = \iota_{[0, x_{\max}]^N}(x) + \frac{\rho}{2}\|x\|^2$
- $g(x) = (\Phi_\delta \circ W)(x)$ where W is an orthonormal wavelet transform and Φ_δ is the Huber function.
- $L \in \mathbb{R}^{N \times M}$ is a discretized Radon projector.
- h is the generalized Anscombe function (smoothed approximation of the neg-log-likelihood associated to a Gauss-Poisson noise model).

Numerical Example

Optimization Problem

$$\min_{x\in\mathbb{R}^N} f(x) + g(x) + \alpha\frac{\|Lx - c\|^2}{2} + h(Lx). \tag{43}$$

$\downarrow$

Monotone Inclusion

$$\text{find } x \in \mathcal{H} \text{ such that } \partial f(x) + \partial g(x) + \alpha L^*(Lx - c) + L^*\nabla h(Lx). \tag{44}$$

$\downarrow$

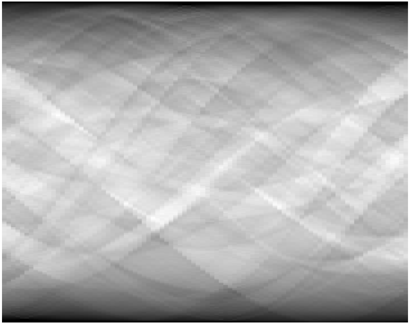
Mismatch Monotone Inclusion

$$\text{find } x \in \mathcal{H} \text{ such that } \partial f(x) + \partial g(x) + \alpha K(Lx - c) + K\nabla h(Lx). \tag{45}$$

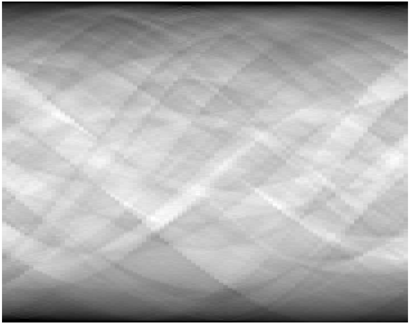
- L: line length ray-drive projector (ASTRA[1] toolbox in MATLAB). 180°, 90 regularly spaced angular steps. Distance source to object 800 mm, source-to-image distance is 1200 mm.
- K: adjoint of the strip fan-beam projector.
- $\lambda_{\min} \approx -6.0082$. We set $\hat{\rho} = -\alpha\lambda_{\min} + \hat{\zeta}_{L^*-K} + 10^{-3}$.
- $\bar{x}$ represents a part of 128×128 pixels of a high resolution scan of a phase-separated barium borosilicate glass imaged at the ESRF synchrotron[2].
- $c = z + e$, where z is a Poisson distribution with mean $L\bar{x}$, and e is a Gaussian distribution with zero-mean and variance σ^2 ($\sigma = 200$).
- $\lambda = 150$, $\delta = 5$, and $\alpha = 0.1$.

[1] W. van Aarle, W.J. Palenstijn, J. De Beenhouwer, T. Altantzis, S. Bals, K.J. Batenburg, J. Sijbers: The astra toolbox: A platform for advanced algorithm development in electron tomography. Ultramicroscopy **157**, 35–47 (2015). https://doi.org/10.1016/j.ultramic.2015.05.002

[2] https://www.esrf.fr/ - The dataset is a courtesy of David Bouttes.

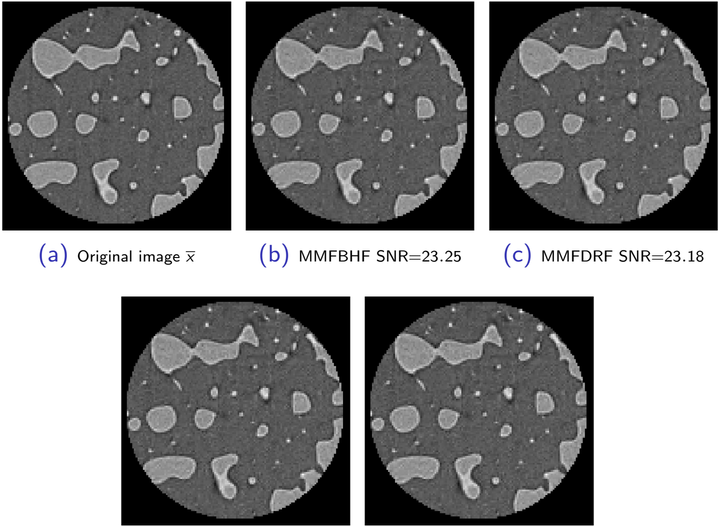


(a) Tomographic projection $L\bar{x}$



(b) Noisy projection c.

Figure: Clean projection and its noisy version, with $\mathrm{SNR}_{\mathrm{input}} = 42.18$ dB



(a) Original image $\overline{x}$ (b) MMFBHF SNR=23.25 (c) MMFDRF SNR=23.18

(d) FBHF SNR=23.31 (e) FDRF SNR=23.23

Figure: Original and reconstructed images after 1500 iterations.

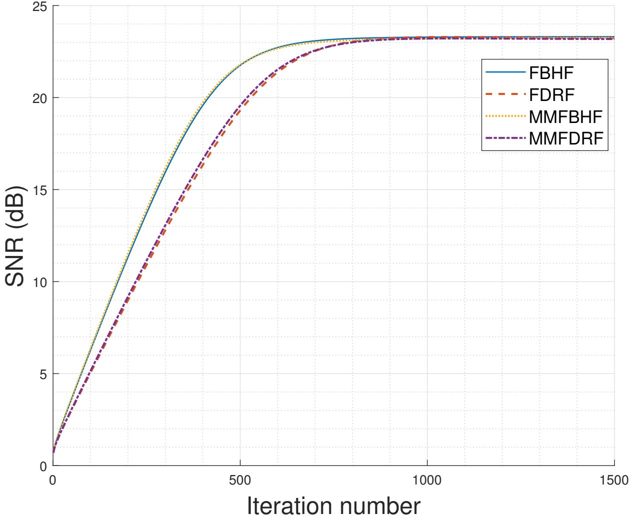


Figure: SNR (dB) evolution along iterations.

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Thanks!