Constructive approaches to the analysis and design of first-order methods for optimization

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Many optimization schemes: usages depend on application requirements (target precision, time budget, memory budget,...).

"Time"
"Error"
Fast, then stall?
Slow, then fast?
Smoothly improving?
Diverging?
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Can we predict their behaviors?
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Can we predict their behaviors?

![Graph showing time vs. error with two possible behaviors: slow then fast? or fast then stall?]
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Can we predict their behaviors?

- Fast, then stall?
- Slow, then fast?
- Smoothly improving?
- Diverging?
How to show that an algorithm works?
How to show that an algorithm works?

Here: principled approach to worst-case analysis.
Important inspiration & reference:

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First part of the presentation:

Important inspiration & reference:


First part of the presentation:


Second part


Informal introduction: https://francisbach.com/computer-aided-analyses/.
Example: minimize differentiable $f : \mathbb{R}^d \rightarrow \mathbb{R}$:

$$x_\star = \arg \min_{x \in \mathbb{R}^d} f(x),$$

where $f$ is $L$-smooth and $\mu$-strongly convex ($0 \leq \mu \leq L < \infty$).
Example: minimize differentiable \( f : \mathbb{R}^d \to \mathbb{R} \):

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where \( f \) is \( L \)-smooth and \( \mu \)-strongly convex \((0 \leq \mu \leq L < \infty)\).

Use gradient descent:

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x_{k+1} = x_k - h \nabla f(x_k).
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Example: minimize differentiable $f : \mathbb{R}^d \to \mathbb{R}$:

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**Question:** what *a priori* guarantees after $N$ iterations?
Example: minimize differentiable $f : \mathbb{R}^d \to \mathbb{R}$:

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Use gradient descent:

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$$

**Question:** what *a priori* guarantees after $N$ iterations?

Examples: how small should $f(x_N) - f(x^\star)$, $\|\nabla f(x_N)\|$, $\|x_N - x^\star\|$ be?
About the assumptions

A differentiable function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is \( \mu \)-strongly convex and \( L \)-smooth iff \( \forall x, y \in \mathbb{R}^d \):

\begin{align*}
(1) \text{(Convexity)} & \quad f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle \\
(1b) \text{ (\( \mu \)-strong convexity)} & \quad f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \| x - y \|^2 \\
(2) \text{ (L-smoothness)} & \quad f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2} \| x - y \|^2 \\
(1&2) & \quad \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 1 + \frac{\mu}{L} + \frac{\mu}{L} \| \nabla f(x) - \nabla f(y) \|^2 + \frac{\mu}{L^2} \| x - y \|^2.
\end{align*}
About the assumptions

A differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ is $\mu$-strongly convex and $L$-smooth iff $\forall x, y \in \mathbb{R}^d$:

(1) (Convexity) $f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle$,

(1b) ($\mu$-strong convexity) $f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|^2$,

(2) (L-smoothness) $f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2} \|x - y\|^2$,

(1&2) $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{1}{L} + \frac{\mu}{2} \|\nabla f(x) - \nabla f(y)\|^2 + \frac{\mu}{2} L \|x - y\|^2$. 
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(1&2) \( \langle \nabla f(x) - \nabla f(y); x - y \rangle \geq \frac{1}{L+\mu} \| \nabla f(x) - \nabla f(y) \|^2 + \frac{\mu L}{L+\mu} \| x - y \|^2 \).
Toy example, take $I$: find $\tau$ such that:

$$\|x_{k+1} - x_*\|^2 \leq \tau \|x_k - x_*\|^2,$$

for all $L$-smooth and $\mu$-strongly convex function $f$ (notation $f \in F_{\mu, L}$),
Toy example, take I: find \( \tau \) such that:

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for all

- \( L \)-smooth and \( \mu \)-strongly convex function \( f \) (notation \( f \in \mathcal{F}_{\mu,L} \)),
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$\|x_{k+1} - x_*\|^2$
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$$\|x_{k+1} - x_*\|^2 = \|x_k - x_*\|^2 - 2h\langle \nabla f(x_k); x_k - x_*\rangle + h^2 \|\nabla f(x_k)\|^2$$
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inequality (1&2)

\[
\leq \left(1 - \frac{2\gamma L \mu}{L + \mu}\right) \|x_k - x_\star\|^2 + h \left(h - \frac{2}{L + \mu}\right) \|\nabla f(x_k)\|^2
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\[\downarrow \text{inequality (1&2)}\]

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\[\downarrow \text{if } 0 \leq h \leq \frac{2}{L + \mu}\]
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inequality (1&2)

$$\leq (1 - \frac{2\gamma L \mu}{L + \mu}) \|x_k - x_*\|^2 + h \left( h - \frac{2}{L + \mu} \right) \|\nabla f(x_k)\|^2$$

if $0 \leq h \leq \frac{2}{L + \mu}$

$$\leq (1 - h \mu)^2 \|x_k - x_*\|^2.$$
Legitimate questions:
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- anything improvable? Realistic analyses?
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- How to choose the right inequalities to combine?
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◊ Why studying this specific quantity?
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- How to study other quantities, e.g., $f(x_k) - f(x_*)$?
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- Unique way to arrive to the desired result?
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◊ anything improvable? Realistic analyses?
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◊ Why studying this specific quantity?
◊ How to study other quantities, e.g., $f(x_k) - f(x_*)$?
◊ Unique way to arrive to the desired result?
◊ How likely are we to find such proofs in more complicated cases?
Take-home messages

Worst-cases are solutions to optimization problems.
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Sometimes, those optimization problems are tractable.
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Often tractable for first-order methods in convex optimization!
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Often tractable for first-order methods in convex optimization!

Acceleration/optimal methods by optimizing worst-cases.
Example

Software

Step-size optimization

Concluding remarks
Example

Software

Step-size optimization

Concluding remarks
Convergence rate of a gradient step

Toy example, take II: What is the smallest $\tau$ such that:

$$\|x_1 - x^\star\|_2 \leq \tau \|x_0 - x^\star\|_2,$$

for all $\nabla f$-smooth and $\mu$-strongly convex function $f$ (notation $f \in F_{\mu, L}$),

$\nabla f(x^\star) = 0$

Optimality of $x^\star$

Variables: $f, x_0, x_1, x^\star$; parameters: $\mu, L, h$. 

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Toy example, take II: What is the smallest $\tau$ such that:

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for all

- $L$-smooth and $\mu$-strongly convex function $f$ (notation $f \in \mathcal{F}_{\mu,L}$),
- $x_0$, and $x_1$ generated by gradient step $x_1 = x_0 - h\nabla f(x_0)$,
- $x_* = \arg\min_{x} f(x)$?

Variables:
- $f, x_0, x_1, x_*$

Parameters:
- $\mu, L, h$
Convergence rate of a gradient step

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First: let’s compute $\tau$!
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First: let’s compute $\tau$!

$$\tau(\mu, L, h) = \max_{f, x_0, x_1, x_*} \frac{\|x_1 - x_*\|^2}{\|x_0 - x_*\|^2}$$

s.t. $f \in \mathcal{F}_{\mu,L}$

Functional class
Convergence rate of a gradient step

**Toy example, take II:** What is the smallest $\tau$ such that:

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Functional class

Algorithm
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\tau(\mu, L, h) = \max_{f,x_0,x_1,x_*} \frac{\|x_1 - x_*\|^2}{\|x_0 - x_*\|^2}
$$

s.t. $f \in \mathcal{F}_{\mu,L}$

- Algorithm
  - $x_1 = x_0 - h\nabla f(x_0)$
- Functional class
  - $\nabla f(x_*) = 0$
- Optimality of $x_*$
Convergence rate of a gradient step

**Toy example, take II:** What is the smallest $\tau$ such that:

$$\|x_1 - x*\|^2 \leq \tau \|x_0 - x*\|^2,$$

for all

- $L$-smooth and $\mu$-strongly convex function $f$ (notation $f \in \mathcal{F}_{\mu,L}$),
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s.t. $f \in \mathcal{F}_{\mu,L}$

**Functional class**

$\nabla f(x) = 0$

**Algorithm**

Optimality of $x*$

**Variables:** $f$, $x_0$, $x_1$, $x*$;
Convergence rate of a gradient step

Toy example, take II: What is the smallest $\tau$ such that:

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- Functional class

- Algorithm

- Optimality of $x_\star$

Variables: $f$, $x_0$, $x_1$, $x_\star$; parameters: $\mu$, $L$, $h$. 

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Performance estimation problem:

$$\max f, x_0, x_1, x^* \|x_1 - x_0\|_2 \|x_0 - x^*\|_2$$

subject to $f$ is $L$-smooth and $\mu$-strongly convex,

$$x_1 = x_0 - h \nabla f(x_0) \nabla f(x^*) = 0.$$ 

Variables: $f, x_0, x_1, x^*$. 

Sampled version: $f$ is only used at $x_0$ and $x^*$ (no need to sample other points)

$$\max x_0, x_1, x^* \|g_0, g^*\|_2 \|f_0, f^*\|_2$$

subject to $\exists f \in F \mu, L$ such that 

$$f_i = f(x_i) \quad i = 0, \ldots, \star$$

$$g_i = \nabla f(x_i) \quad i = 0, \ldots, \star$$

$$x_1 = x_0 - hg_0 \quad g^* = 0.$$ 

Variables: $x_0, x_1, x^*, g_0, g^*, f_0, f^*$. 

Sampled version
Sampled version

- Performance estimation problem:

\[
\begin{align*}
\max_{f, x_0, x_1, x_\star} & \quad \frac{\|x_1 - x_0\|^2}{\|x_0 - x_\star\|^2} \\
\text{subject to} & \quad f \text{ is } L\text{-smooth and } \mu\text{-strongly convex,} \\
& \quad x_1 = x_0 - h\nabla f(x_0) \\
& \quad \nabla f(x_\star) = 0.
\end{align*}
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Sampled version

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- Variables: \(f, x_0, x_1, x_\star\).
Performance estimation problem:

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subject to \( f \) is \( L \)-smooth and \( \mu \)-strongly convex,

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x_1 = x_0 - h\nabla f(x_0)
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\[
\nabla f(x_\star) = 0.
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Variables: \( f, x_0, x_1, x_\star \).

Sampled version: \( f \) is only used at \( x_0 \) and \( x_\star \) (no need to sample other points).
Sampled version

- Performance estimation problem:

\[
\max_{f, x_0, x_1, x_*} \quad \frac{||x_1 - x_0||^2}{||x_0 - x_*||^2}
\]

subject to \( f \) is \( L \)-smooth and \( \mu \)-strongly convex,

\[ x_1 = x_0 - h \nabla f(x_0) \]
\[ \nabla f(x_*) = 0. \]

- Variables: \( f, x_0, x_1, x_* \).

- Sampled version: \( f \) is only used at \( x_0 \) and \( x_* \) (no need to sample other points)

\[
\max_{x_0, x_1, x_*} \quad \frac{||x_1 - x_0||^2}{||x_0 - x_*||^2}
\]

subject to \( \exists f \in F_{\mu, L} \) such that \( \left\{ \begin{array}{l}
    f_i = f(x_i) \quad i = 0, * \\
    g_i = \nabla f(x_i) \quad i = 0, *
\end{array} \right. \)

\[ x_1 = x_0 - hg_0 \]
\[ g_* = 0. \]
Sampled version

Performance estimation problem:

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\max_{f, x_0, x_1, x_\star} \frac{\|x_1 - x_0\|^2}{\|x_0 - x_\star\|^2} \\
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Variables: \(f, x_0, x_1, x_\star\).

Sampled version: \(f\) is only used at \(x_0\) and \(x_\star\) (no need to sample other points)

\[
\max_{x_0, x_1, x_\star, g_0, g_\star, f_0, f_\star} \frac{\|x_1 - x_0\|^2}{\|x_0 - x_\star\|^2} \\
\text{subject to } \exists f \in \mathcal{F}_{\mu, L} \text{ such that } \left\{ \begin{array}{ll}
  f_i = f(x_i) & i = 0, \star \\
  g_i = \nabla f(x_i) & i = 0, \star 
\end{array} \right.
\]
\[
x_1 = x_0 - hg_0
\]
\[
g_\star = 0.
\]

Variables: \(x_0, x_1, x_\star, g_0, g_\star, f_0, f_\star\).
Smooth strongly convex interpolation (or extension)

Consider an index set $S$, and its associated values $\{(x_i, g_i, f_i)\}_{i \in S}$ with coordinates $x_i$, (sub)gradients $g_i$ and function values $f_i$. 
Smooth strongly convex interpolation (or extension)

Consider an index set $S$, and its associated values $\{(x_i, g_i, f_i)\}_{i \in S}$ with coordinates $x_i$, (sub)gradients $g_i$ and function values $f_i$.

Possible to find $f \in \mathcal{F}_{\mu,L}$ such that

$$f(x_i) = f_i, \quad \text{and} \quad g_i \in \partial f(x_i), \quad \forall i \in S.$$
Smooth strongly convex interpolation (or extension)

Consider an index set $S$, and its associated values $\{(x_i, g_i, f_i)\}_{i \in S}$ with coordinates $x_i$, (sub)gradients $g_i$ and function values $f_i$.

Possible to find $f \in \mathcal{F}_{\mu, L}$ such that

$$f(x_i) = f_i, \quad \text{and} \quad g_i \in \partial f(x_i), \quad \forall i \in S.$$ 

- Necessary and sufficient condition: $\forall i, j \in S$

$$f_i \geq f_j + \langle g_j, x_i - x_j \rangle + \frac{1}{2L} \|g_i - g_j\|^2 + \frac{\mu}{2(1-\mu/L)} \|x_i - x_j - \frac{1}{L} (g_i - g_j)\|^2.$$
Smooth strongly convex interpolation (or extension)

Consider an index set $S$, and its associated values $\{(x_i, g_i, f_i)\}_{i \in S}$ with coordinates $x_i$, (sub)gradients $g_i$ and function values $f_i$.

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- **Necessary and sufficient condition:** $\forall i, j \in S$

  $$f_i \geq f_j + \langle g_j, x_i - x_j \rangle + \frac{1}{2L} \|g_i - g_j\|^2 + \frac{\mu}{2(1-\mu/L)} \|x_i - x_j - \frac{1}{L}(g_i - g_j)\|^2.$$ 

- **Simpler example:** pick $\mu = 0$ and $L = \infty$ (just convexity):

  $$f_i \geq f_j + \langle g_j, x_i - x_j \rangle.$$
Replace constraints
Replace constraints

◊ Interpolation conditions allow removing red constraints

\[
\begin{align*}
\max_{x_0, x_1, x^*_0, x^*_1, f_0, f^*_0, g_0, g^*_0} & \quad \frac{\|x_1 - x^*_1\|^2}{\|x_0 - x^*_0\|^2} \\
\text{subject to} & \quad \exists f \in \mathcal{F}_{\mu, L} \text{ such that} \\
& \quad f_i = f(x_i) \quad i = 0, * \\
& \quad g_i = \nabla f(x_i) \quad i = 0, * \\
& \quad x_1 = x_0 - hg_0 \\
& \quad g^*_0 = 0, \\
\end{align*}
\]
Replace constraints

⋄ Interpolation conditions allow removing red constraints

\[
\begin{align*}
\max_{x_0, x_1, x_*, g_0, g_*, f_0, f_*} & \frac{\|x_1 - x_*\|^2}{\|x_0 - x_*\|^2} \\
\text{subject to} & \exists f \in \mathcal{F}_{\mu, L} \text{ such that } \left\{ \begin{array}{ll} 
  f_i = f(x_i) & i = 0, * \\
  g_i = \nabla f(x_i) & i = 0, * \\
  x_1 = x_0 - h g_0 \\
  g_* = 0,
\end{array} \right.
\end{align*}
\]

⋄ replacing them by

\[
\begin{align*}
f_* & \geq f_0 + \langle g_0, x_* - x_0 \rangle + \frac{1}{2L} \|g_* - g_0\|^2 + \frac{\mu}{2(1-\mu/L)} \|x_* - x_0 - \frac{1}{L} (g_* - g_0)\|^2 \\
f_0 & \geq f_* + \langle g_*, x_0 - x_* \rangle + \frac{1}{2L} \|g_0 - g_*\|^2 + \frac{\mu}{2(1-\mu/L)} \|x_0 - x_* - \frac{1}{L} (g_0 - g_*)\|^2.
\end{align*}
\]
Replace constraints

- Interpolation conditions allow removing red constraints

\[
\max_{x_0, x_1, x^*_{g}, g^*, f_0, f^*_{x}} \frac{\|x_1 - x^*_{x}\|^2}{\|x_0 - x^*_{x}\|^2}
\]

subject to \( \exists f \in \mathcal{F}_{\mu, L} \) such that

\[
\begin{align*}
    f_i &= f(x_i) & i &= 0, * \\
    g_i &= \nabla f(x_i) & i &= 0, *
\end{align*}
\]

\[
x_1 = x_0 - hg_0 \\
g^* = 0,
\]

- replacing them by

\[
f^* \geq f_0 + \langle g_0, x^*_{x} - x_0 \rangle + \frac{1}{2L} \|g^* - g_0\|^2 + \frac{\mu}{2(1 - \mu/L)} \|x^*_{x} - x_0 - \frac{1}{L} (g^* - g_0)\|^2
\]

\[
f_0 \geq f^* + \langle g^*, x_0 - x^*_{x} \rangle + \frac{1}{2L} \|g_0 - g^*\|^2 + \frac{\mu}{2(1 - \mu/L)} \|x_0 - x^*_{x} - \frac{1}{L} (g_0 - g^*)\|^2.
\]

- Same optimal value (no relaxation); but still non-convex quadratic problem.
Semidefinite lifting

Using the new variables $G \succeq 0$ and $F = [\|x_0 - x^\star\|_2 \langle g_0, x_0 - x^\star \rangle \|g_0\|_2, f_0 - f^\star]$, the previous problem can be reformulated as a $2 \times 2$ SDP:

$$\max_G, F \quad G_1, 1 + h_2 G_2, 2 - 2hG_1, 2$$

subject to

$$F + L\mu_2 (L - \mu) G_1, 1 + \frac{1}{2} (L - \mu) G_2, 2 - \mu L L - \mu G_1, 2 \leq 0$$

$$G_1, 1 = 1, G \succeq 0$$

(using an homogeneity argument and substituting $x_1$ and $g^\star$).

Assuming $x_0, x^\star, g_0 \in \mathbb{R}^d$ with $d \geq 2$, same optimal value as original problem!

For $d = 1$ same as original problem by adding $\text{rank}(G) \leq 1$. 
Semidefinite lifting

◇ Using the new variables $G \succ 0$ and $F$

$$G = \begin{bmatrix} \|x_0 - x_*\|^2 & \langle g_0, x_0 - x_* \rangle \\ \langle g_0, x_0 - x_* \rangle & \|g_0\|^2 \end{bmatrix}, \quad F = f_0 - f_*,$$
Semidefinite lifting

◇ Using the new variables $G \succ 0$ and $F$

$$G = \begin{bmatrix} \|x_0 - x_*\|^2 & \langle g_0, x_0 - x_\star \rangle \\ \langle g_0, x_0 - x_\star \rangle & \|g_0\|^2 \end{bmatrix}, \quad F = f_0 - f_\star,$$

◇ previous problem can be reformulated as a $2 \times 2$ SDP

$$\max_{G, F} \quad \frac{G_{1,1} + h^2 G_{2,2} - 2hG_{1,2}}{G_{1,1}}$$

subject to

$$F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{L}{L-\mu} G_{1,2} \leq 0$$

$$-F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{\mu}{L-\mu} G_{1,2} \leq 0$$

$G \succ 0,$
Semidefinite lifting

- Using the new variables $G \succcurlyeq 0$ and $F$

$$G = \begin{bmatrix} \|x_0 - x_*\|^2 & \langle g_0, x_0 - x_* \rangle \\ \langle g_0, x_0 - x_* \rangle & \|g_0\|^2 \end{bmatrix}, \quad F = f_0 - f_*,$$

- previous problem can be reformulated as a $2 \times 2$ SDP

$$\max_{G, F} \quad G_{1,1} + h^2 G_{2,2} - 2hG_{1,2}$$

subject to

$$F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{L}{L-\mu} G_{1,2} \leq 0$$

$$-F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{\mu}{L-\mu} G_{1,2} \leq 0$$

$$G_{1,1} = 1$$

$$G \succcurlyeq 0,$$

(using an anhomogeneity argument and substituting $x_1$ and $g_*$).
Semidefinite lifting

- Using the new variables $G \succeq 0$ and $F$

  \[ G = \begin{bmatrix} \|x_0 - x^*\|^2 & \langle g_0, x_0 - x^* \rangle \\ \langle g_0, x_0 - x^* \rangle & \|g_0\|^2 \end{bmatrix}, \quad F = f_0 - f^* , \]

- previous problem can be reformulated as a $2 \times 2$ SDP

  \[
  \begin{align*}
  \max_{G, F} \quad & G_{1,1} + h^2 G_{2,2} - 2h G_{1,2} \\
  \text{subject to} \quad & F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{L}{L-\mu} G_{1,2} \leq 0 \\
  \quad & -F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{\mu}{L-\mu} G_{1,2} \leq 0 \\
  \quad & G_{1,1} = 1 \\
  \quad & G \succeq 0,
  \end{align*}
  \]

  (using an an homogeneity argument and substituting $x_1$ and $g^*$).

- Assuming $x_0, x^*, g_0 \in \mathbb{R}^d$ with $d \geq 2$, same optimal value as original problem!
Semidefinite lifting

- Using the new variables $G \succcurlyeq 0$ and $F$

\[
G = \begin{bmatrix} \|x_0 - x^*\|^2 & \langle g_0, x_0 - x^* \rangle \\ \langle g_0, x_0 - x^* \rangle & \|g_0\|^2 \end{bmatrix}, \quad F = f_0 - f^*,
\]

- previous problem can be reformulated as a $2 \times 2$ SDP

\[
\max_{G, F} \quad G_{1,1} + h^2 G_{2,2} - 2h G_{1,2}
\]

subject to

\[
\begin{align*}
F + \frac{L \mu}{2(L - \mu)} G_{1,1} + \frac{1}{2(L - \mu)} G_{2,2} - \frac{L}{L - \mu} G_{1,2} & \leq 0 \\
- F + \frac{L \mu}{2(L - \mu)} G_{1,1} + \frac{1}{2(L - \mu)} G_{2,2} - \frac{\mu}{L - \mu} G_{1,2} & \leq 0 \\
G_{1,1} & = 1 \\
G & \succcurlyeq 0,
\end{align*}
\]

(using an homogeneity argument and substituting $x_1$ and $g^*$).

- Assuming $x_0, x^*, g_0 \in \mathbb{R}^d$ with $d \geq 2$, same optimal value as original problem!

- For $d = 1$ same as original problem by adding $\text{rank}(G) \leq 1$. 

Solving the SDP...

Fix $L = 1, \mu = .1$ and solve the SDP for a few values of $h$. 

\[
\|x_1 - x^\star\|_2^2, \quad \|x_0 - x^\star\|_2^2
\]

\[\text{Observation: numerics match max}\left\{\left(1 - hL\right)^2, \left(1 - h\mu\right)^2\right\}\text{.}
\]

\[\text{We recover the celebrated } 2L + \mu \text{ as the optimal step-size.}\]
Solving the SDP...

Fix $L = 1$, $\mu = .1$ and solve the SDP for a few values of $h$. 

\[
\frac{\|x_1 - x^*\|^2}{\|x_0 - x^*\|^2}
\]

\[\diamond\] Observation: numerics match
\[\max\{ (1 - hL)^2, (1 - h\mu)^2 \}\]

\[\diamond\] We recover the celebrated $2L + \mu$ as the optimal step-size.
Solving the SDP...

Fix $L = 1$, $\mu = .1$ and solve the SDP for a few values of $h$.

\[
\frac{\|x_1 - x_*\|^2}{\|x_0 - x_*\|^2}
\]

Observation: numerics match $\max\{(1 - hL)^2, (1 - h\mu)^2\}$. 
Solving the SDP...

Fix $L = 1, \mu = 0.1$ and solve the SDP for a few values of $h$.

- Observation: numerics match $\max\{(1 - hL)^2, (1 - h\mu)^2\}$.
- We recover the celebrated $\frac{2}{L + \mu}$ as the optimal step-size.
Dual problem

⋄ Dual problem is

\[
\min_{\tau, \lambda_1, \lambda_2 \geq 0} \tau
\]

subject to

\[
S = \begin{bmatrix}
\tau - 1 + \frac{\lambda_1 L \mu}{L - \mu} & h - \frac{\lambda_1 (\mu + L)}{2(L - \mu)} \\
h - \frac{\lambda_1 (\mu + L)}{2(L - \mu)} & \frac{\lambda_1}{L - \mu} - h^2
\end{bmatrix} \succeq 0
\]

\[
0 = \lambda_1 - \lambda_2.
\]
Dual problem

- Dual problem is

\[ \min_{\tau, \lambda_1, \lambda_2 \geq 0} \tau \]

subject to

\[ S = \begin{bmatrix} \tau - 1 + \frac{\lambda_1 L \mu}{L - \mu} & h - \frac{\lambda_1 (\mu + L)}{2(L - \mu)} \\ h - \frac{\lambda_1 (\mu + L)}{2(L - \mu)} & \frac{\lambda_1}{L - \mu} - h^2 \end{bmatrix} \succcurlyeq 0 \]

0 = \lambda_1 - \lambda_2.

- Weak duality: any dual feasible point \(\equiv\) valid worst-case convergence rate

- Strong duality holds (existence of a Slater point): any valid worst-case convergence rate \(\equiv\) valid dual feasible point.
Dual problem

Dual problem is

$$\min_{\tau, \lambda_1, \lambda_2 \geq 0} \tau$$

subject to

$$S = \begin{bmatrix} \tau - 1 + \frac{\lambda_1 L \mu}{L - \mu} & h - \frac{\lambda_1 (\mu + L)}{2(L - \mu)} \\ h - \frac{\lambda_1 (\mu + L)}{2(L - \mu)} & \frac{\lambda_1}{L - \mu} - h^2 \end{bmatrix} \succeq 0$$

$$0 = \lambda_1 - \lambda_2.$$ 

Weak duality: any dual feasible point $\equiv$ valid worst-case convergence rate

Direct consequence: for any $\tau \geq 0$ we have

$$\|x_1 - x_*\|^2 \leq \tau \|x_0 - x_*\|^2$$ for all $f \in F_{\mu, L}$, all $x_0 \in \mathbb{R}^d$, all $d \in \mathbb{N}$, with $x_1 = x_0 - h\nabla f(x_0).$
Dual problem

- Dual problem is

\[
\min_{\tau, \lambda_1, \lambda_2 \geq 0} \tau
\]

subject to

\[
S = \begin{bmatrix}
\tau - 1 + \frac{\lambda_1 L\mu}{L - \mu} & h - \frac{\lambda_1 (\mu + L)}{2(L - \mu)} \\
\h - \frac{\lambda_1 (\mu + L)}{2(L - \mu)} & \lambda_1 \frac{L - \mu}{2(L - \mu)} - h^2
\end{bmatrix} \succeq 0
\]

\[
0 = \lambda_1 - \lambda_2.
\]

- Weak duality: any dual feasible point \(\equiv\) valid worst-case convergence rate (\(\uparrow\)).

- Direct consequence: for any \(\tau \geq 0\) we have

\[
\|x_1 - x_*\|^2 \leq \tau \|x_0 - x_*\|^2 \text{ for all } f \in \mathcal{F}_{\mu, L}, \text{ all } x_0 \in \mathbb{R}^d, \text{ all } d \in \mathbb{N}, \text{ with } x_1 = x_0 - h \nabla f(x_0).
\]

\[
\exists \lambda \geq 0 : \begin{bmatrix}
\tau - 1 + \frac{\lambda L\mu}{L - \mu} & h - \frac{\lambda (\mu + L)}{2(L - \mu)} \\
\h - \frac{\lambda (\mu + L)}{2(L - \mu)} & \lambda \frac{L - \mu}{2(L - \mu)} - h^2
\end{bmatrix} \succeq 0
\]
Dual problem

- Dual problem is

\[
\min_{\tau, \lambda_1, \lambda_2 \geq 0} \tau
\]
subject to

\[
S = \begin{bmatrix}
\tau - 1 + \frac{\lambda_1 L \mu}{L - \mu} & h - \frac{\lambda_1 (\mu + L)}{2(L - \mu)} \\
h - \frac{\lambda_1 (\mu + L)}{2(L - \mu)} & \frac{\lambda_1}{L - \mu} - h^2
\end{bmatrix} \succeq 0
\]

\[0 = \lambda_1 - \lambda_2.\]

- Weak duality: any dual feasible point \(\equiv\) valid worst-case convergence rate \((\uparrow)\).

- Direct consequence: for any \(\tau \geq 0\) we have

\[
\|x_1 - x_*\|^2 \leq \tau \|x_0 - x_*\|^2 \text{ for all } f \in \mathcal{F}_\mu, L, \text{ all } x_0 \in \mathbb{R}^d, \text{ all } d \in \mathbb{N}, \text{ with } x_1 = x_0 - h \nabla f(x_0).
\]

- Strong duality holds (existence of a Slater point): any valid worst-case convergence rate \(\equiv\) valid dual feasible point \((\downarrow)\).
Dual problem

- Dual problem is

\[
\min_{\tau, \lambda_1, \lambda_2 \geq 0} \tau
\]

subject to

\[
S = \begin{bmatrix}
\tau - 1 + \frac{\lambda_1 L \mu}{L - \mu} & h - \frac{\lambda_1 (\mu + L)}{2(L - \mu)} \\
h - \frac{\lambda_1 (\mu + L)}{2(L - \mu)} & \frac{\lambda_1}{L - \mu} - h^2
\end{bmatrix} \succeq 0
\]

\[0 = \lambda_1 - \lambda_2.\]

- Weak duality: any dual feasible point \(\equiv\) valid worst-case convergence rate \(\uparrow\).

- Direct consequence: for any \(\tau \geq 0\) we have

\[
\|x_1 - x_\star\|^2 \leq \tau \|x_0 - x_\star\|^2 \text{ for all } f \in \mathcal{F}_{\mu, L}, \text{ all } x_0 \in \mathbb{R}^d, \text{ all } d \in \mathbb{N},
\]

with \(x_1 = x_0 - h\nabla f(x_0)\).

\[
\exists \lambda \geq 0 : \begin{bmatrix}
\tau - 1 + \frac{\lambda L \mu}{L - \mu} & h - \frac{\lambda (\mu + L)}{2(L - \mu)} \\
h - \frac{\lambda (\mu + L)}{2(L - \mu)} & \frac{\lambda}{L - \mu} - h^2
\end{bmatrix} \succeq 0
\]

- Strong duality holds (existence of a Slater point): any valid worst-case convergence rate \(\equiv\) valid dual feasible point \(\downarrow\): hence "\(\uparrow\downarrow\)".
Dual solutions

Fix $L = 1$, $\mu = .1$ and solve the dual SDP for a few values of $h$. 
Dual solutions

Fix $L = 1$, $\mu = .1$ and solve the dual SDP for a few values of $h$. 

![Graph showing dual values $\lambda_1$ and $\lambda_2$ vs. step size $h$.]
Fix $L = 1$, $\mu = .1$ and solve the dual SDP for a few values of $h$.

Numerics match $\lambda_1 = \lambda_2 = 2|h|\rho(h)$ with $\rho(h) = \max\{hL - 1, 1 - h\mu\}$. 
Recovering a “standard” proof

Gradient with $h = \frac{1}{L}$. Perform weighted sum of two inequalities
Recovering a “standard” proof

Gradient with $h = \frac{1}{L}$. Perform weighted sum of two inequalities

\[
\begin{align*}
f_0 \geq f_* & \quad + \frac{1}{2L} \| \nabla f(x_0) \|^2 \\
& \quad + \frac{\mu}{2(1-\mu/L)} \| x_0 - x* - \frac{1}{L} \nabla f(x_0) \|^2 \quad : \lambda_1 \\
f_* \geq f_0 & \quad + \langle \nabla f(x_0), x* - x_0 \rangle + \frac{1}{2L} \| \nabla f(x_0) \|^2 \\
& \quad + \frac{\mu}{2(1-\mu/L)} \| x_0 - x* - \frac{1}{L} \nabla f(x_0) \|^2 \quad : \lambda_2
\end{align*}
\]
Recovering a “standard” proof

Gradient with $h = \frac{1}{L}$. Perform weighted sum of two inequalities

\[
\begin{align*}
  f_0 & \geq f_* & + & \frac{1}{2L} \| \nabla f(x_0) \|^2 \\
  & & + & \frac{\mu}{2(1 - \mu / L)} \| x_0 - x_* - \frac{1}{L} \nabla f(x_0) \|^2 : \lambda_1 \\
  f_* & \geq f_0 & + & \langle \nabla f(x_0), x_* - x_0 \rangle + \frac{1}{2L} \| \nabla f(x_0) \|^2 \\
  & & + & \frac{\mu}{2(1 - \mu / L)} \| x_0 - x_* - \frac{1}{L} \nabla f(x_0) \|^2 : \lambda_2
\end{align*}
\]

with $\lambda_1, \lambda_2 \geq 0$. Weighted sum can be reformulated as
Recovering a “standard” proof

Gradient with $h = \frac{1}{L}$. Perform weighted sum of two inequalities

$$f_0 \geq f_\star \quad + \frac{1}{2L} \|\nabla f(x_0)\|^2$$

$$\quad + \frac{\mu}{2(1-\mu/L)} \|x_0 - x_\star - \frac{1}{L} \nabla f(x_0)\|^2$$

$$f_\star \geq f_0 \quad + \langle \nabla f(x_0), x_\star - x_0 \rangle \quad + \frac{1}{2L} \|\nabla f(x_0)\|^2$$

$$\quad + \frac{\mu}{2(1-\mu/L)} \|x_0 - x_\star - \frac{1}{L} \nabla f(x_0)\|^2$$

with $\lambda_1, \lambda_2 \geq 0$. Weighted sum can be reformulated as

$$\lambda_1 = 2h(1 - \mu h)$$

$$\lambda_2 = 2h(1 - \mu h)$$
Recovering a “standard” proof

Gradient with $h = \frac{1}{L}$. Perform weighted sum of two inequalities

$$f_0 \geq f_* + \frac{1}{2L} \| \nabla f(x_0) \|^2 + \frac{\mu}{2(1-\mu/L)} \| x_0 - x_* - \frac{1}{L} \nabla f(x_0) \|^2 : \lambda_1 = 2h(1 - \mu h)$$

$$f_* \geq f_0 + \langle \nabla f(x_0), x_* - x_0 \rangle + \frac{1}{2L} \| \nabla f(x_0) \|^2 + \frac{\mu}{2(1-\mu/L)} \| x_0 - x_* - \frac{1}{L} \nabla f(x_0) \|^2 : \lambda_2 = 2h(1 - \mu h)$$

with $\lambda_1, \lambda_2 \geq 0$. Weighted sum can be reformulated as

$$\| x_1 - x_* \|^2 \leq (1 - \mu h)^2 \| x_0 - x_* \|^2 - h \frac{2 - h(L + \mu)}{L - \mu} \| \mu(x_0 - x_*) - \nabla f(x_0) \|^2 ,$$
Recovering a “standard” proof

Gradient with $h = \frac{1}{L}$. Perform weighted sum of two inequalities

\[
\begin{align*}
f_0 \geq f_* & \quad + \frac{1}{2L} \| \nabla f(x_0) \|^2 \\
& \quad + \frac{\mu}{2(1-\mu/L)} \| x_0 - x_* - \frac{1}{L} \nabla f(x_0) \|^2 & : \lambda_1 = 2h(1 - \mu h) \\
f_* \geq f_0 & \quad + \langle \nabla f(x_0), x_* - x_0 \rangle \\
& \quad + \frac{1}{2L} \| \nabla f(x_0) \|^2 \\
& \quad + \frac{\mu}{2(1-\mu/L)} \| x_0 - x_* - \frac{1}{L} \nabla f(x_0) \|^2 & : \lambda_2 = 2h(1 - \mu h)
\end{align*}
\]

with $\lambda_1, \lambda_2 \geq 0$. Weighted sum can be reformulated as

\[
\begin{align*}
\| x_1 - x_* \|^2 \leq (1 - \mu h)^2 \| x_0 - x_* \|^2 - h \frac{2 - h(L + \mu)}{L - \mu} \| \mu(x_0 - x_*) - \nabla f(x_0) \|^2, \\
\| \mu(x_0 - x_*) - \nabla f(x_0) \|^2 & \geq 0
\end{align*}
\]
Recovering a “standard” proof

Gradient with $h = \frac{1}{L}$. Perform weighted sum of two inequalities

\[
\begin{align*}
    f_0 &\geq f_* + \frac{1}{2L} \|\nabla f(x_0)\|^2 + \frac{\mu}{2(1-\mu/L)} \|x_0 - x_* - \frac{1}{L} \nabla f(x_0)\|^2 : \lambda_1 = 2h(1 - \mu h) \\
    f_* &\geq f_0 + \langle \nabla f(x_0), x_* - x_0 \rangle + \frac{1}{2L} \|\nabla f(x_0)\|^2 + \frac{\mu}{2(1-\mu/L)} \|x_0 - x_* - \frac{1}{L} \nabla f(x_0)\|^2 : \lambda_2 = 2h(1 - \mu h)
\end{align*}
\]

with $\lambda_1, \lambda_2 \geq 0$. Weighted sum can be reformulated as

\[
\|x_1 - x_*\|^2 \leq (1 - \mu h)^2 \|x_0 - x_*\|^2 - h \frac{2 - h(L + \mu)}{L - \mu} \|\mu(x_0 - x_*) - \nabla f(x_0)\|^2, \
\]

\[
\geq 0
\]

\[
\leq (1 - \mu h)^2 \|x_0 - x_*\|^2,
\]
Recovering a “standard” proof

Gradient with $h = \frac{1}{L}$. Perform weighted sum of two inequalities

\[
\begin{align*}
\lambda_1 &= 2h(1 - \mu h) \\
\lambda_2 &= 2h(1 - \mu h)
\end{align*}
\]

with $\lambda_1, \lambda_2 \geq 0$. Weighted sum can be reformulated as

\[
\|x_1 - x_*\|^2 \leq (1 - \mu h)^2 \|x_0 - x_*\|^2 - h \frac{2 - h(L + \mu)}{L - \mu} \|\mu(x_0 - x_*) - \nabla f(x_0)\|^2, \\
\hline
\geq 0
\]

leading to $\|x_1 - x_*\|^2 \leq (1 - \frac{\mu}{L})^2 \|x_0 - x_*\|^2$
Recovering a “standard” proof

Gradient with $h = \frac{1}{L}$. Perform weighted sum of two inequalities

\[ f_0 \geq f_\star + \frac{1}{2L} \| \nabla f(x_0) \|^2 \]
\[ + \frac{\mu}{2(1 - \mu/L)} \| x_0 - x_\star - \frac{1}{L} \nabla f(x_0) \|^2 \]
\[ f_\star \geq f_0 + \langle \nabla f(x_0), x_\star - x_0 \rangle + \frac{1}{2L} \| \nabla f(x_0) \|^2 \]
\[ + \frac{\mu}{2(1 - \mu/L)} \| x_0 - x_\star - \frac{1}{L} \nabla f(x_0) \|^2 \]

with $\lambda_1, \lambda_2 \geq 0$. Weighted sum can be reformulated as

\[ \| x_1 - x_\star \|^2 \leq (1 - h)^2 \| x_0 - x_\star \|^2 - h \frac{2 - h(L + \mu)}{L - \mu} \| \mu(x_0 - x_\star) - \nabla f(x_0) \|^2, \]

\[ \geq 0, \text{ or } = 0 \text{ when worst-case is achieved} \]

\[ \leq (1 - h)^2 \| x_0 - x_\star \|^2, \]

leading to \[ \| x_1 - x_\star \|^2 \leq (1 - \frac{\mu}{L})^2 \| x_0 - x_\star \|^2 \]
Recovering a “standard” proof

Gradient with $h = \frac{1}{L}$. Perform weighted sum of two inequalities

\[
\begin{align*}
    f_0 & \geq f_* + \frac{1}{2L} \| \nabla f(x_0) \|^2 \\
    f_* & \geq f_0 + \langle \nabla f(x_0), x_* - x_0 \rangle + \frac{1}{2L} \| \nabla f(x_0) \|^2 \\
\end{align*}
\]

with $\lambda_1, \lambda_2 \geq 0$. Weighted sum can be reformulated as

\[
\| x_1 - x_* \|^2 \leq (1 - \mu h)^2 \| x_0 - x_* \|^2 - \frac{h^2 - h(L + \mu)}{L - \mu} \| \mu (x_0 - x_*) - \nabla f(x_0) \|^2,
\]

\[
\geq 0, \text{ or } = 0 \text{ when worst-case is achieved}
\]

\[
\leq (1 - \mu h)^2 \| x_0 - x_* \|^2,
\]

leading to $\| x_1 - x_* \|^2 \leq (1 - \frac{\mu}{L})^2 \| x_0 - x_* \|^2$ (tight).
What did we do, so far?

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◊ we computed the smallest $\tau(\mu, L, h)$ such that

$$
\|x_1 - x_*\|^2 \leq \tau(\mu, L, h) \|x_0 - x_*\|^2
$$

is satisfied for all $x_0 \in \mathbb{R}^d$, $d \in \mathbb{N}$, $f \in \mathcal{F}_{\mu, L}$, and $x_1 = x_0 - h\nabla f(x_0)$. 

Feasible points to primal SDP correspond to lower bounds on $\tau(\mu, L, h)$.

Feasible points to dual SDP correspond to upper bounds on $\tau(\mu, L, h)$.

Proof via linear combinations of interpolation inequalities (evaluated at the iterates and $x_*$),
Proofs can be rewritten as a “sum-of-squares” certificates.
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  – proof via linear combinations of interpolation inequalities (evaluated at the iterates and $x_\star$),
  
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... what happens beyond gradient descent for smooth strongly convex minimization?
When does it work?

The methodology applies, as is, as soon as:

- performance measure and initial condition are linear in $G$ and $F$,
- interpolation inequalities are linear in $G$ and $F$,
- algorithm can be described linearly in $G$ and $F$. 

When does it work?

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What’s next?

- More iterations?
- Other types of problems?
  - Non-smooth convex functions, non-convex smooth functions, indicator functions, monotone operators, etc.
- Other types of methods?
  - Projections, proximal operators, linear optimization oracles (Frank-Wolfe), mirror descent, approximate versions, momentum, etc.
- Human-readable/simpler proofs?
  - Specialized PEPs looking for Lyapunov functions.
- Step-size optimization?
  - Optimize worst-case performance.
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Example

Software

Step-size optimization

Concluding remarks
Avoiding semidefinite programming modeling steps?
Avoiding semidefinite programming modeling steps?

- Python version: PEPit available at [github.com/PerformanceEstimation/PEPit/](https://github.com/PerformanceEstimation/PEPit/)

Packages contain more than 75 examples!
A few examples

Algorithms for solving: \[
\min_x f(x)
\]

with \( f \) convex and \( L \)-smooth.
A few examples

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We compare: gradient descent vs. heavy-ball vs. Nesterov’s acceleration
A few examples

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△ in terms of worst-cases $\frac{f(x_k) - f(x_*)}{\|x_0 - x_*\|^2}$,
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- in terms of worst-cases
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- in terms of worst-cases
  $$\min_{0 \leq i \leq k} \frac{\|\nabla f(x_i)\|^2}{\|x_0 - x_\star\|^2}.$$
A few examples

Proximal point algorithm for (maximal) monotone inclusion:

find \( x : 0 \in A(x) \)

with \( A : \mathbb{R}^d \rightarrow 2^{\mathbb{R}^d} \) maximal monotone.
A few examples

Proximal point algorithm for (maximal) monotone inclusion:

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with \( A : \mathbb{R}^d \to 2^{\mathbb{R}^d} \) maximal monotone.

What is the worst-case
\[ \frac{\|x_{k+1} - x_k\|^2}{\|x_0 - x_*\|^2} \]
when \( x_{i+1} = J_A(x_i) \)?
Current library of examples within PESTO/PEPit

Includes... but not limited to

- subgradient, gradient, heavy-ball, fast gradient, optimized gradient methods,
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- Krasnoselkii-Mann and Halpern fixed-point iterations,
- mirror descent/Bregman gradient/“NoLips”,
- stochastic methods: Point-SAGA, SAGA, SGD and variants.

... contain most of the recent PEP-related advances (including by other groups).
Among others, see works by Drori, Teboulle, Kim, Fessler, Ryu, Lieder, Lessard, Recht, Packard, Van Scoy, Cyrus, Gu, Yang, etc.
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Back to legitimate questions:

◊ anything improvable? Realistic analyses?
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- anything improvable? Realistic analyses?
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- Unique way to arrive to the desired result?
- How likely are we to find such proofs in more complicated cases?
Recap’
Worst-case guarantees cannot be improved, systematic approach,
Recap'

- Worst-case guarantees cannot be improved, systematic approach,
- allows reaching proofs that could barely be obtained by hand,
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😊 Worst-case guarantees cannot be improved, systematic approach,
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😊 fair amount of scenarios/algorithms (e.g., proximal terms, stochastic, etc.),
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- Worst-case guarantees *cannot be improved*, systematic approach,
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- fair amount of scenarios/algorithms (e.g., proximal terms, stochastic, etc.),
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Worst-case guarantees *cannot be improved*, systematic approach,

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😊 proofs (may be) hard to generalize.
A few instructive examples

Worst-case analysis for fixed-point iterations:


Application to nonconvex optimization:

- Abbaszadehpeivasti, de Klerk, Zamani (’21). “The exact worst-case convergence rate of the gradient method with fixed step lengths for $L$-smooth functions”.

- Rotaru, Glineur, Patrinos (’22). “Tight convergence rates of the gradient method on hypoconvex functions”.

Applications to distributed optimization:

- Sundararajan, Van Scoy, Lessard (’19). “Analysis and design of first-order distributed optimization algorithms over time-varying graphs.”

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Analysis of the proximal-point algorithm for monotone inclusions:

◊ Gu, Yang ('19). “Optimal nonergodic sublinear convergence rate of the proximal point algorithm for maximal monotone inclusion problems”.
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A few instructive examples—shameless advertisement

Applications to mirror descent + lower complexity bound

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Applications to adaptive methods


A few instructive examples—shameless advertisement

Applications to mirror descent + lower complexity bound


Applications to adaptive methods


Lyapunov functions (compact proofs) & counter-examples

◊ Lessard, Recht, Packard (’16). “Analysis and design of optimization algorithms via integral quadratic constraints.”


Example

Software

Step-size optimization

Concluding remarks
Creating new algorithms

Smooth (strongly) convex minimization with more than gradient descent?
Creating new algorithms

Smooth (strongly) convex minimization with more than gradient descent?

\[ x_1 = x_0 - h_{1,0} \nabla f(x_0) \]
Creating new algorithms

Smooth (strongly) convex minimization with more than gradient descent?

\[ x_1 = x_0 - h_{1,0} \nabla f(x_0) \]
\[ x_2 = x_1 - h_{2,0} \nabla f(x_0) - h_{2,1} \nabla f(x_1) \]

How to choose \( \{h_i, j\} \)?

- Pick a performance criterion, for instance \( \|x_N - x^\star\|_2 \|x_0 - x^\star\|_2 \).
- Solve the minimax:

\[
\min_{\{h_i, j\}} \max_{f \in F, \{x_i\}} \|x_N - x^\star\|_2 \|x_0 - x^\star\|_2.
\]

Solution to inner maximization via \( N \times N \) SDP.
Creating new algorithms

Smooth (strongly) convex minimization with more than gradient descent?

\[ x_1 = x_0 - h_{1,0} \nabla f(x_0) \]
\[ x_2 = x_1 - h_{2,0} \nabla f(x_0) - h_{2,1} \nabla f(x_1) \]
\[ \vdots \]

How to choose \{h_i, j\}?

\( \diamond \) pick a performance criterion, for instance
\[ \|x_N - x^\star\|_2 \]
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\( \diamond \) solve the minimax:
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\]
\[
\vdots
\]
\[
x_N = x_{N-1} - h_{N,0} \nabla f(x_0) - \ldots - h_{N,N-1} \nabla f(x_{N-1})
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Creating new algorithms

Smooth (strongly) convex minimization with more than gradient descent?

\[ \begin{align*}
x_1 &= x_0 - h_{1,0} \nabla f(x_0) \\
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\end{align*} \]

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How to choose \( \{h_{i,j}\} \)?

◊ pick a performance criterion, for instance

\[
\frac{\|x_N - x_*\|^2}{\|x_0 - x_*\|^2},
\]
Creating new algorithms

Smooth (strongly) convex minimization with more than gradient descent?

\[
x_1 = x_0 - h_{1,0} \nabla f(x_0)
\]

\[
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\]

\[\vdots\]

\[
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\(\diamond\) pick a performance criterion, for instance

\[
\frac{\|x_N - x_*\|^2}{\|x_0 - x_*\|^2},
\]

\(\diamond\) solve the minimax:

\[
\min_{\{h_{i,j}\}_{i,j}} \max_{f \in \mathcal{F}, \{x_i\}} \frac{\|x_N - x_*\|^2}{\|x_0 - x_*\|^2}.
\]

Solution to inner maximization via \(N \times N\) SDP.
Design problem

How to solve the design problem (or proxy of it)?

$$\min_{\{h_{i,j}\}} \max_{f \in \mathcal{F}} \frac{\|x_N - x_*\|^2}{\|x_0 - x_*\|^2}$$
Design problem

How to solve the design problem (or proxy of it)?

\[
\min_{\{h_i, j\}} \max_{f \in F} \frac{\|x_N - x_\star\|^2}{\|x_0 - x_\star\|^2}
\]

◊ brutal approaches
Design problem

How to solve the design problem (or proxy of it)?

\[
\min \max_{\{h_{i,j}\}} \max_{f \in \mathcal{F}} \frac{\|x_N - x_*\|^2}{\|x_0 - x_*\|^2}
\]

◊ brutal approaches

◊ convex relaxations,
How to solve the design problem (or proxy of it)?

\[
\min_{\{h_{i,j}\}} \max_{f \in \mathcal{F}} \frac{\|x_N - x_*\|^2}{\|x_0 - x_*\|^2}
\]

- brutal approaches
- convex relaxations,
- analogies (e.g., with conjugate gradient methods).
Primal problem \((N = 1)\)
Recall primal problem, with step-size optimization

\[
\begin{align*}
\min_{h_{1,0}} \max_{G,F} \quad & G_{1,1} + h_{1,0}^2 G_{2,2} - 2h_{1,0} G_{1,2} \\
\text{subject to} \quad & F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{L}{L-\mu} G_{1,2} \leq 0 \\
& -F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{\mu}{L-\mu} G_{1,2} \leq 0 \\
& G_{1,1} = 1 \\
& G \succcurlyeq 0.
\end{align*}
\]
Recall primal problem, with step-size optimization

\[
\begin{align*}
\min_{h_{1,0}} & \quad \max_{G,F} \quad G_{1,1} + h_{1,0}^2 G_{2,2} - 2h_{1,0} G_{1,2} \\
\text{subject to} & \quad F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{L}{L-\mu} G_{1,2} \leq 0 \\
& \quad -F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{\mu}{L-\mu} G_{1,2} \leq 0 \\
& \quad G_{1,1} = 1 \\
& \quad G \succcurlyeq 0.
\end{align*}
\]

“Simple” minimization problem by dualizing inner maximization.
Primal problem \((N = 1)\)

Recall primal problem, with step-size optimization

\[
\begin{align*}
\min_{h_{1,0}} \ & \ \max_{G, F} \ \ \ G_{1,1} + h_{1,0}^2 G_{2,2} - 2h_{1,0} G_{1,2} \\
\text{subject to} \ & \ \ F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{L}{L-\mu} G_{1,2} \leq 0 \\
& - F + \frac{L\mu}{2(L-\mu)} G_{1,1} + \frac{1}{2(L-\mu)} G_{2,2} - \frac{\mu}{L-\mu} G_{1,2} \leq 0 \\
& G_{1,1} = 1 \\
& G \succcurlyeq 0.
\end{align*}
\]

“Simple” minimization problem by dualizing inner maximization.

Dualize inner maximization \(\rightarrow\) \(\min\min\).
Optimizing the step-sizes ($N = 1$)
Optimizing the step-sizes ($N = 1$)

For $N = 1$, optimizing over step-size $h_{1,0}$ remains convex!
Optimizing the step-sizes \((N = 1)\)

For \(N = 1\), optimizing over step-size \(h_{1,0}\) remains convex!

Indeed:

\[
\begin{align*}
\min_{\tau, \lambda \geq 0} & \quad \tau \\
\text{subject to} & \quad \begin{bmatrix}
\tau - 1 + \frac{\lambda L \mu}{L - \mu} & h_{1,0} - \frac{\lambda (\mu + L)}{2(L - \mu)} \\
{h_{1,0} - \frac{\lambda (\mu + L)}{2(L - \mu)}} & \frac{\lambda}{L - \mu} - h_{1,0}^2
\end{bmatrix} \succeq 0.
\end{align*}
\]
Optimizing the step-sizes ($N = 1$)

For $N = 1$, optimizing over step-size $h_{1,0}$ remains convex!

Indeed:

$$\min_{\tau, \lambda \geq 0, h_{1,0}} \tau$$

subject to

$$\begin{bmatrix}
\tau - 1 + \frac{\lambda L \mu}{L - \mu} & h_{1,0} - \frac{\lambda (\mu + L)}{2(L - \mu)} \\
\lambda (\mu + L) \frac{\mu}{2} & L - \mu - h_{1,0}^2
\end{bmatrix} \succ 0.$$
Optimizing the step-sizes \((N = 1)\)

For \(N = 1\), optimizing over step-size \(h_{1,0}\) remains convex!

Indeed:

\[
\begin{align*}
\min_{\tau, \lambda \geq 0, h_{1,0}} \tau \\
\text{subject to} \begin{bmatrix}
\tau - 1 + \frac{\lambda L \mu}{L - \mu} & h_{1,0} - \frac{\lambda (\mu + L)}{2(L - \mu)} \\
h_{1,0} - \frac{\lambda (\mu + L)}{2(L - \mu)} & \frac{\lambda}{L - \mu} - h_{1,0}^2
\end{bmatrix} \succeq 0.
\end{align*}
\]

Optimize \(h_{1,0}\) “for free” (linear SDP via Schur complement):

\[
\begin{align*}
\min_{\tau, \lambda \geq 0, h_{1,0}} \tau \\
\text{subject to} \begin{bmatrix}
\tau - 1 + \frac{\lambda L \mu}{L - \mu} & -\frac{\lambda (\mu + L)}{2(L - \mu)} & 1 \\
-\frac{\lambda (\mu + L)}{2(L - \mu)} & \frac{\lambda}{L - \mu} - h_{1,0} & 1 \\
1 & -h_{1,0} & 1
\end{bmatrix} \succeq 0.
\end{align*}
\]
Optimizing the step-sizes \((N = 2)\)

When \(N = 2\), the problem becomes

\[
\min_{\tau, \lambda_1, \ldots, \lambda_6 \geq 0} \tau
\]

subject to

\[
\begin{bmatrix}
S_{1,1}, S_{1,2}, S_{1,3}, S_{2,1}, S_{2,2}, S_{2,3}, S_{3,1}, S_{3,2}, S_{3,3}
\end{bmatrix} \succeq 0,
\]

for some \(S_{1,1}, S_{1,2}, S_{1,3}, S_{2,1}, S_{2,2}, S_{2,3}, S_{3,1}, S_{3,2}, S_{3,3}\) (functions of \(\tau, \lambda_1, \ldots, \lambda_6\) and \(\{h_{i,j}\}\)).

In particular

\[
S_{1,2} = -L \lambda_3 - 2(L - \mu) h_{2,0} + \mu \lambda_1 + L \mu (\lambda_2 + \lambda_5),
\]

\[
S_{2,2} = -2(\mu \lambda_6 + L \lambda_4) h_{1,0} - 2(L - \mu) h_{2,1} + L \mu (\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6) + \lambda_1 + \lambda_3 + \lambda_4 + \lambda_6.
\]
Optimizing the step-sizes ($N = 2$)

When $N = 2$, the problem becomes

$$
\min_{\tau, \lambda_1, \ldots, \lambda_6 \geq 0} \tau
$$

subject to

$$
\begin{bmatrix}
\lambda_1 + \lambda_2 - \lambda_3 - \lambda_5 \\
-\lambda_1 + \lambda_3 + \lambda_4 - \lambda_6
\end{bmatrix} = 0,
$$

for some $S_{1,1}, S_{1,2}, \ldots, S_{3,3}$ (functions of $\tau, \lambda_1, \ldots, \lambda_6$ and $\{h_{i,j}\}$).
Optimizing the step-sizes \((N = 2)\)

When \(N = 2\), the problem becomes

\[
\min_{\tau, \lambda_1, \ldots, \lambda_6 \geq 0, \{h_{i,j}\}} \tau
\]

subject to

\[
\begin{bmatrix}
S_{1,1} & S_{1,2} & S_{1,3} \\
S_{1,2} & S_{2,2} & S_{2,3} \\
S_{1,3} & S_{2,3} & S_{3,3}
\end{bmatrix} \succeq 0
\]

\[
\begin{bmatrix}
\lambda_1 + \lambda_2 - \lambda_3 - \lambda_5 \\
-\lambda_1 + \lambda_3 + \lambda_4 - \lambda_6
\end{bmatrix} = 0,
\]

for some \(S_{1,1}, S_{1,2}, S_{1,3}, \ldots, S_{3,3}\) (functions of \(\tau, \lambda_1, \ldots, \lambda_6\) and \(\{h_{i,j}\}\)).

In particular

\[
S_{1,2} = -L\lambda_3 - 2(L - \mu)h_{2,0} + \mu\lambda_1 + L\mu(\lambda_2 + \lambda_5)h_{1,0} - 2(L - \mu)h_{2,0} + L\mu(\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6)h_{2,0} + \lambda_1 + \lambda_3 + \lambda_4 - \lambda_6
\]

\[
S_{2,2} = -2(\mu\lambda_6 + L\lambda_4)h_{1,0} - 2(L - \mu)h_{2,0} + L\mu(\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6)h_{2,0} + \lambda_1 + \lambda_3 + \lambda_4 - \lambda_6
\]

LMI convex in some step-sizes \((h_{2,0}, h_{2,1})\) but not in the others.
When $N = 2$, the problem becomes

$$\min_{\tau, \lambda_1, \ldots, \lambda_6 \geq 0} \tau$$

subject to

$$\begin{bmatrix} S_{1,1} & S_{1,2} & S_{1,3} \\ S_{1,2} & S_{2,2} & S_{2,3} \\ S_{1,3} & S_{2,3} & S_{3,3} \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} \lambda_1 + \lambda_2 - \lambda_3 - \lambda_5 \\ -\lambda_1 + \lambda_3 + \lambda_4 - \lambda_6 \end{bmatrix} = 0,$$

for some $S_{1,1}, S_{1,2}, \ldots, S_{3,3}$ (functions of $\tau, \lambda_1, \ldots, \lambda_6$ and $\{h_{i,j}\}$).

Optimizing the step-sizes ($N = 2$)
Optimizing the step-sizes \((N = 2)\)

When \(N = 2\), the problem becomes

\[
\min_{\tau, \lambda_1, \ldots, \lambda_6 \geq 0, \{h_{i,j}\}} \tau
\]

subject to

\[
\begin{bmatrix}
S_{1,1} & S_{1,2} & S_{1,3} \\
S_{1,2} & S_{2,2} & S_{2,3} \\
S_{1,3} & S_{2,3} & S_{3,3}
\end{bmatrix} \succeq 0
\]

\[
\begin{bmatrix}
\lambda_1 + \lambda_2 - \lambda_3 - \lambda_5 \\
-\lambda_1 + \lambda_3 + \lambda_4 - \lambda_6
\end{bmatrix} = 0,
\]

for some \(S_{1,1}, S_{1,2}, \ldots, S_{3,3}\) (functions of \(\tau, \lambda_1, \ldots, \lambda_6\) and \(\{h_{i,j}\}\)).

In particular

\[
S_{1,2} = -\frac{L\lambda_3 - 2(L - \mu)h_{2,0} + \mu \lambda_1 + L\mu(\lambda_2 + \lambda_5)h_{1,0}}{L - \mu}
\]

\[
S_{2,2} = \frac{-2(\mu \lambda_6 + L\lambda_4)h_{1,0} - 2(L - \mu)h_{2,0}^2 + L\mu(\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6)h_{1,0}^2 + \lambda_1 + \lambda_3 + \lambda_4 + \lambda_6}{L - \mu}
\]
Optimizing the step-sizes ($N = 2$)

When $N = 2$, the problem becomes

$$\min_{\tau, \lambda_1, \ldots, \lambda_6 \geq 0} \{ h_{i,j} \} \tau$$

subject to

$$\begin{bmatrix} S_{1,1} & S_{1,2} & S_{1,3} \\ S_{1,2} & S_{2,2} & S_{2,3} \\ S_{1,3} & S_{2,3} & S_{3,3} \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} \lambda_1 + \lambda_2 - \lambda_3 - \lambda_5 \\ -\lambda_1 + \lambda_3 + \lambda_4 - \lambda_6 \end{bmatrix} = 0,$$

for some $S_{1,1}, S_{1,2}, \ldots, S_{3,3}$ (functions of $\tau, \lambda_1, \ldots, \lambda_6$ and $\{ h_{i,j} \}$).

In particular

$$S_{1,2} = -\frac{L\lambda_3 - 2(L - \mu)h_{2,0} + \mu \lambda_1 + L\mu(\lambda_2 + \lambda_5)h_{1,0}}{L - \mu}$$

$$S_{2,2} = -\frac{2(\mu \lambda_6 + L\lambda_4)h_{1,0} - 2(L - \mu)h_{2,1,0} + L\mu(\lambda_2 + \lambda_4 + \lambda_5 + \lambda_6)h_{1,0}^2 + \lambda_1 + \lambda_3 + \lambda_4 + \lambda_6}{L - \mu}$$

LMI convex in some step-sizes ($h_{2,0}$ and $h_{2,1}$) but not in the others.
Numerical examples I

Example for $L = 1$ and $\mu = .1$
Example for $L = 1$ and $\mu = .1$

- For $N = 1$, we reach $\frac{\|x_1 - x_\star\|^2}{\|x_0 - x_\star\|^2} \leq 0.6694$ with step-sizes $[h_{i,j}^\star] = [1.8182]$. 
Example for $L = 1$ and $\mu = .1$

- For $N = 1$, we reach $\frac{\|x_1 - x^*\|^2}{\|x_0 - x^*\|^2} \leq 0.6694$ with step-sizes $[h^*_{i,j}] = [1.8182]$.

- For $N = 2$, we reach $\frac{\|x_2 - x^*\|^2}{\|x_0 - x^*\|^2} \leq 0.3769$ with $[h^*_{i,j}] = \begin{bmatrix} 1.5466 & 0.2038 \\ 0.2038 & 2.4961 \end{bmatrix}$. 
Numerical examples I

Example for $L = 1$ and $\mu = .1$

\begin{itemize}
  \item For $N = 1$, we reach $\frac{\|x_1 - x^*\|^2}{\|x_0 - x^*\|^2} \leq 0.6694$ with step-sizes
    $$[h_{i,j}^*] = [1.8182].$$
  \item For $N = 2$, we reach $\frac{\|x_2 - x^*\|^2}{\|x_0 - x^*\|^2} \leq 0.3769$ with
    $$[h_{i,j}^*] = \begin{bmatrix} 1.5466 \\ 0.2038 & 2.4961 \end{bmatrix}.$$\end{itemize}

\begin{itemize}
  \item For $N = 3$, we reach $\frac{\|x_3 - x^*\|^2}{\|x_0 - x^*\|^2} \leq 0.1932$ with
    $$[h_{i,j}^*] = \begin{bmatrix} 1.5466 \\ 0.1142 & 1.8380 \\ 0.0642 & 0.4712 & 2.8404 \end{bmatrix}.$$\end{itemize}
Numerical examples I

Example for $L = 1$ and $\mu = .1$

- For $N = 1$, we reach $\frac{\|x_1 - x^*\|^2}{\|x_0 - x^*\|^2} \leq 0.6694$ with step-sizes
  
  $[h^*_{i,j}] = [1.8182]$.

- For $N = 2$, we reach $\frac{\|x_2 - x^*\|^2}{\|x_0 - x^*\|^2} \leq 0.3769$ with
  
  $[h^*_{i,j}] = \begin{bmatrix} 1.5466 \\ 0.2038 \\ 2.4961 \end{bmatrix}$.

- For $N = 3$, we reach $\frac{\|x_3 - x^*\|^2}{\|x_0 - x^*\|^2} \leq 0.1932$ with
  
  $[h^*_{i,j}] = \begin{bmatrix} 1.5466 & 1.8380 \\ 0.1142 & 0.4712 \\ 0.0642 & 2.8404 \end{bmatrix}$.

- For $N = 4$, we reach $\frac{\|x_4 - x^*\|^2}{\|x_0 - x^*\|^2} \leq 0.0944$ with
  
  $[h^*_{i,j}] = \begin{bmatrix} 1.5466 & 1.8380 & 1.9501 \\ 0.1142 & 0.2432 & 1.9501 \\ 0.0331 & 0.2432 & 3.0093 \\ 0.0217 & 0.1593 & 0.6224 \end{bmatrix}$.
Numerical examples II

What about different performance measure? Example \( \frac{f(x_N) - f_\star}{f(x_0) - f_\star} \) and \( L = 1, \mu = .1 \).
Numerical examples II

What about different performance measure? Example $\frac{f(x_N) - f^*}{f(x_0) - f^*}$ and $L = 1$, $\mu = .1$.

⋄ For $N = 1$, we obtain $\frac{f(x_1) - f^*}{f(x_0) - f^*} \leq 0.6694$ with step-size $[h_{i,j}] = [1.8182]$.
Numerical examples II

What about different performance measure? Example $\frac{f(x_N) - f_\star}{f(x_0) - f_\star}$ and $L = 1$, $\mu = .1$.

- For $N = 1$, we obtain $\frac{f(x_1) - f_\star}{f(x_0) - f_\star} \leq 0.6694$ with step-size

$$[h_{i,j}] = [1.8182].$$

- For $N = 2$, we obtain $\frac{f(x_2) - f_\star}{f(x_0) - f_\star} \leq 0.3554$ with

$$[h_{i,j}] = \begin{bmatrix} 2.0095 & 0.4229 \\ 0.4229 & 2.0095 \end{bmatrix}.$$
Numerical examples II

What about different performance measure? Example $\frac{f(x_N) - f^*}{f(x_0) - f^*}$ and $L = 1$, $\mu = .1$.

- For $N = 1$, we obtain $\frac{f(x_1) - f^*}{f(x_0) - f^*} \leq 0.6694$ with step-size $[h_{i,j}] = [1.8182]$.

- For $N = 2$, we obtain $\frac{f(x_2) - f^*}{f(x_0) - f^*} \leq 0.3554$ with $[h_{i,j}] = \begin{bmatrix} 2.0095 \\ 0.4229 \\ 2.0095 \end{bmatrix}$.

- For $N = 3$, we obtain $\frac{f(x_3) - f^*}{f(x_0) - f^*} \leq 0.1698$ with $[h_{i,j}] = \begin{bmatrix} 1.9470 \\ 0.4599 \\ 2.2406 \\ 0.1705 \\ 0.4599 \\ 1.9470 \end{bmatrix}$. 
Numerical examples II

What about different performance measure? Example $\frac{f(x_N) - f_\star}{f(x_0) - f_\star}$ and $L = 1$, $\mu = .1$.

- For $N = 1$, we obtain $\frac{f(x_1) - f_\star}{f(x_0) - f_\star} \leq 0.6694$ with step-size

$$[h_{i,j}] = [1.8182].$$

- For $N = 2$, we obtain $\frac{f(x_2) - f_\star}{f(x_0) - f_\star} \leq 0.3554$ with

$$[h_{i,j}] = \begin{bmatrix} 2.0095 \\ 0.4229 \\ 2.0095 \end{bmatrix}.$$

- For $N = 3$, we obtain $\frac{f(x_3) - f_\star}{f(x_0) - f_\star} \leq 0.1698$ with

$$[h_{i,j}] = \begin{bmatrix} 1.9470 \\ 0.4599 \\ 2.2406 \\ 0.1705 \\ 0.4599 \\ 1.9470 \end{bmatrix}.$$

- For $N = 4$, we obtain $\frac{f(x_4) - f_\star}{f(x_0) - f_\star} \leq 0.0789$ with

$$[h_{i,j}] = \begin{bmatrix} 1.9187 \\ 0.4098 \\ 2.1746 \\ 0.1796 \\ 0.5147 \\ 2.1746 \\ 0.0627 \\ 0.1796 \\ 0.4098 \\ 1.9187 \end{bmatrix}.$$
Numerical examples III

Worst-case performance $\frac{f(x_N) - f_*}{\|x_0 - x_*\|_2^2}$ with $L = 1$ and $\mu = .01$. We compare
Numerical examples III

Worst-case performance \( \frac{f(x_N) - f_*}{\|x_0 - x_*\|^2} \) with \( L = 1 \) and \( \mu = .01 \). We compare

- worst-case performance of known methods, namely Triple Momentum Method (TMM) and Accelerated/Fast Gradient Method (FGM) computed using PEPs,
Worst-case performance $\frac{f(x_N) - f_*}{\|x_0 - x_*\|^2}$ with $L = 1$ and $\mu = .01$. We compare

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- worst-case performance of optimized method (numerically generated),
Numerical examples III

Worst-case performance \( \frac{f(x_N) - f_*}{\|x_0 - x_*\|_2} \) with \( L = 1 \) and \( \mu = .01 \). We compare

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- worst-case performance of optimized method \textit{(numerically generated)},
- \textbf{Lower complexity bound} \textit{(numerically generated)}. 
Numerical examples III

Worst-case performance $\frac{f(x_N) - f^*}{\|x_0 - x^*\|^2}$ with $L = 1$ and $\mu = .01$. We compare

- worst-case performance of known methods, namely Triple Momentum Method (TMM) and Accelerated/Fast Gradient Method (FGM) computed using PEPs,
- worst-case performance of optimized method (numerically generated),
- Lower complexity bound (numerically generated).

![Graph showing the convergence of the TMM method over iterations](image-url)
Numerical examples III

Worst-case performance $\frac{f(x_N) - f_*}{\|x_0 - x_*\|^2}$ with $L = 1$ and $\mu = .01$. We compare

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- worst-case performance of optimized method (numerically generated),
- Lower complexity bound (numerically generated).

![Graph showing comparison between TMM and FGM](image-url)
Numerical examples III

Worst-case performance \( \frac{f(x_N) - f^*}{\|x_0 - x^*\|^2} \) with \( L = 1 \) and \( \mu = .01 \). We compare

- worst-case performance of known methods, namely Triple Momentum Method (TMM) and Accelerated/Fast Gradient Method (FGM) computed using PEPs,
- worst-case performance of optimized method (numerically generated),
- Lower complexity bound (numerically generated).

![Graph showing comparison of TMM, FGM, and Optimized method over iterations]
Numerical examples III

Worst-case performance $\frac{f(x_N) - f_*}{\|x_0 - x_*\|^2}$ with $L = 1$ and $\mu = .01$. We compare

- worst-case performance of known methods, namely Triple Momentum Method (TMM) and Accelerated/Fast Gradient Method (FGM) computed using PEPs,
- worst-case performance of optimized method (numerically generated),
- Lower complexity bound (numerically generated).
Analytical solutions

⋄ It turns out that for $\frac{\|x_N - x^*\|^2}{\|x_0 - x^*\|^2}$, we can also solve the minimax in closed-form.
It turns out that for $\|x_N - x^\star\|^2$, we can also solve the minimax in closed-form. The method referred to as “Information-Theoretic Exact Method” (ITEM)

$$y_k = (1 - \beta_k)z_k + \beta_k \left( y_{k-1} - \frac{1}{L} \nabla f(y_{k-1}) \right)$$

$$z_{k+1} = (1 - \frac{\mu}{L} \delta_k)z_k + \frac{\mu}{L} \delta_k \left( y_k - \frac{1}{\mu} \nabla f(y_k) \right),$$

for some sequences $\{\beta_k\}, \{\delta_k\}$ (depending on $\mu, L, \text{and } k$).
Analytical solutions

- It turns out that for $\|x_N - x^*\|^2$, we can also solve the minimax in closed-form.
- The method referred to as “Information-Theoretic Exact Method” (ITEM)

\[
y_k = (1 - \beta_k)z_k + \beta_k \left( y_{k-1} - \frac{1}{L} \nabla f(y_{k-1}) \right)
\]

\[
z_{k+1} = (1 - \frac{\mu}{L}\delta_k)z_k + \frac{\mu}{L}\delta_k \left( y_k - \frac{1}{\mu} \nabla f(y_k) \right),
\]

for some sequences $\{\beta_k\}, \{\delta_k\}$ (depending on $\mu, L$, and $k$).
- The worst-case guarantee matches exactly a lower complexity bound.
Analytical solutions

- It turns out that for \( \| x_N - x_\star \|_2^2 / \| x_0 - x_\star \|_2^2 \), we can also solve the minimax in closed-form.
- The method referred to as “Information-Theoretic Exact Method” (ITEM)

\[
y_k = (1 - \beta_k) z_k + \beta_k \left( y_{k-1} - \frac{1}{L} \nabla f(y_{k-1}) \right)
\]

\[
z_{k+1} = (1 - \frac{\mu}{L} \delta_k) z_k + \frac{\mu}{L} \delta_k \left( y_k - \frac{1}{\mu} \nabla f(y_k) \right),
\]

for some sequences \( \{\beta_k\} \), \( \{\delta_k\} \) (depending on \( \mu \), \( L \), and \( k \)).
- The worst-case guarantee matches exactly a lower complexity bound.
- Worst-case guarantee of order

\[
\frac{\| z_N - z_\star \|_2^2}{\| z_0 - z_\star \|_2^2} = O \left( \left( 1 - \sqrt{\frac{\mu}{L}} \right)^{2N} \right).
\]
Analytical solutions

⋄ It turns out that for \( \frac{\|x_N - x^*\|^2}{\|x_0 - x^*\|^2} \), we can also solve the minimax in closed-form.

⋄ The method referred to as “Information-Theoretic Exact Method” (ITEM)

\[
y_k = (1 - \beta_k)z_k + \beta_k \left( y_{k-1} - \frac{1}{L} \nabla f(y_{k-1}) \right)
\]

\[
z_{k+1} = (1 - \frac{\mu}{L} \delta_k)z_k + \frac{\mu}{L} \delta_k \left( y_k - \frac{1}{\mu} \nabla f(y_k) \right),
\]

for some sequences \( \{\beta_k\}, \{\delta_k\} \) (depending on \( \mu, L, \) and \( k \)).

⋄ The worst-case guarantee matches exactly a lower complexity bound.

⋄ Worst-case guarantee of order

\[
\frac{\|z_N - z^*\|^2}{\|z_0 - z^*\|^2} = O \left( \left(1 - \sqrt{\frac{\mu}{L}}\right)^{2N} \right).
\]

⋄ The proof is “simple”!
A few observations/limitations

Were we lucky? Some pieces are missing!
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◊ Why/when are optimal step-sizes \( \{h^*_{i,j}\} \) independent of horizon \( N \)?
A few observations/limitations

Were we lucky? Some pieces are missing!

◊ Why/when are optimal step-sizes \( \{h_{i,j}^*\} \) independent of horizon \( N \)?
◊ Why/when can the optimal method be expressed efficiently? (eg. using second order recursions)
A few observations/limitations

Were we lucky? Some pieces are missing!

◊ Why/when are optimal step-sizes $\{h_{i,j}^*\}$ independent of horizon $N$?

◊ Why/when can the optimal method be expressed efficiently? (eg. using second order recursions)

The situation seems quite involved in general, apart from a few cases
A few observations/limitations

Were we lucky? Some pieces are missing!

- Why/when are optimal step-sizes \( \{h^*_i,j\} \) independent of horizon \( N \)?
- Why/when can the optimal method be expressed efficiently? (eg. using second order recursions)

The situation seems quite involved in general, apart from a few cases

- \( \frac{f(x_N) - f^*}{\|x_0 - x^*\|_2^2} \) with \( \mu = 0 \): optimized gradient method (OGM, Kim & Fessler '16),

- Information-theoretic exact method (ITEM, T & Drori '21),

- \( \|\nabla f(x_N)\|_2^2 \) with \( \mu = 0 \) (via Chebyshev polynomials),

- Asymptotically Polyak's Heavy-Ball method

- See e.g.: A. Nemirovsky's "Information-based complexity of convex programming." (lecture notes, 1995)
A few observations/limitations

Were we lucky? Some pieces are missing!
- Why/when are optimal step-sizes \( \{ h_{i,j}^* \} \) independent of horizon \( N \)?
- Why/when can the optimal method be expressed efficiently? (eg. using second order recursions)

The situation seems quite involved in general, apart from a few cases
- \( \frac{f(x_N) - f_*}{\|x_0 - x_*\|^2} \) with \( \mu = 0 \): optimized gradient method (OGM, Kim & Fessler ’16),
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- see e.g.: A. Nemirovsky’s “Information-based complexity of convex programming.” (lecture notes, 1995)
A few instructive examples

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Software

Step-size optimization

Concluding remarks
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  – proofs are linear combinations of certain specific inequalities.
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  - proofs are linear combinations of certain specific inequalities.

Byproducts:

- computer-assisted design of proofs,
- computer-assisted design of numerical methods,
- step towards reproducible theory
  - validation & benchmark tool for proofs (also for reviews 😊).
Concluding remarks

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♦ adaptative algorithms, high-order, beyond worst-cases,
♦ many open setups: bi-level optimization, multi-objective optimization, etc.
Take-home messages

Optimization can be seen as the science of proving inequalities

...including complexity bounds for numerical methods.

Powerful framework for designing methods and guarantees.
Thanks! Questions?

PerformanceEstimation/Performance-Estimation-Toolbox on Github
PerformanceEstimation/PEPit on Github