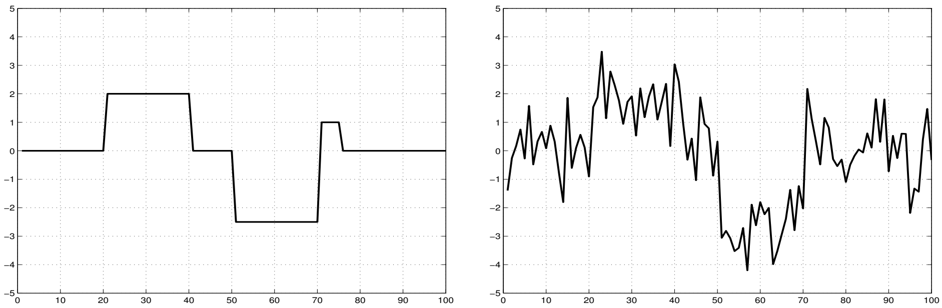
DM optimization: piecewise linear denoising

Let $\overline{x} = (\overline{x}^{(i)})_{1\leq i\leq N} \in \mathbb{R}^N$ be a sampled piecewise constant noisy signal. We denote $y = \overline{x} + \varepsilon$ a noisy version of $\overline{x}$ with $\varepsilon \sim \mathcal{N}(0, \sigma^2)$. An illustration of $\overline{x}$ and y is displayed in Figure 1.



Original signal$\overline{x}$ Noisy signal y

FIGURE 1 – Illustration of a piecewise constant signal with $N = 100$ samples degraded with a white Gaussian noise of variance $\sigma^2 = 1$.

The objective of this exercice is to obtain a piecewise constant estimate $\widehat{x}$ which is the closest to the original $\overline{x}$ from data y. A solution consists in minimizing the following objective function :

$$\widehat{x}_\lambda = \arg\min_{x\in\mathbb{R}^N} \frac{1}{2}\|x - y\|_2^2 + \lambda\|Lx\|_1$$

where $(Lx)^{(i)} = x^{(i+1)} - x^{(i)}$ for every $i \in \{1, \ldots, N-1\}$, $y \in \mathbb{R}^N$ and $\lambda > 0$. $L \in \mathbb{R}^{(N-1)\times N}$ denotes the finite difference operator.

1. Prove that the dual problem can be written as

$$\widehat{u}_\lambda \in \text{Argmin}_{u\in\mathbb{R}^{N-1}} \ \frac{1}{2}\|y - L^*u\|_2^2 \quad \text{s.t.} \quad \|u\|_\infty \leq \lambda,$$

and that the relation with the dual solution is

$$\widehat{x}_\lambda = y - L^*\widehat{u}_\lambda.$$