Optimisation non-lisse, non-convexe, algorithmes du premier ordre et liens avec le deep learning via les stratégies d'unfolded et PnP

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Focus on the following issues

1. Convergence of the iterates of first order schemes: convex and non-convex.
   - Banach-Picard
   - α-averaged operators
   - KL-based convergence
   - Algorithms dealing with non-linearities
2. From variational approaches to deep learning
   - Unfolded schemes
   - Plug-and-play
3. Illustrations in the context of inverse problem.

Framework

- Non-convexity: f or g non-convex

$$\underset{\mathbf{x}}{\text{minimize}}\ \Psi(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})$$

- Non-convexity: bi-convex problem

$$\underset{\mathbf{x},\mathbf{e}}{\text{minimize}}\ \Psi(\mathbf{x},\mathbf{e}) := f(\mathbf{x}) + g_1(\mathbf{x},\mathbf{e}) + g_2(\mathbf{e})$$

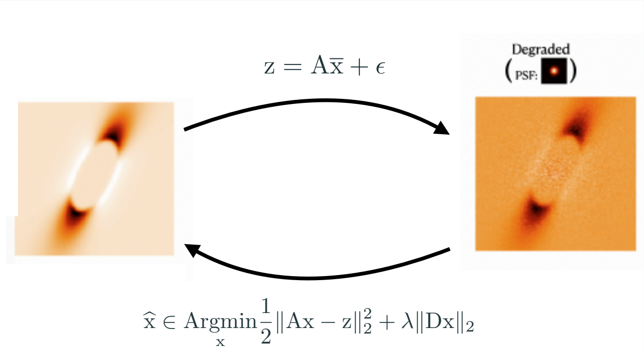
- Non-convexity: non-linear operator

$$\underset{\mathbf{x}}{\text{minimize}}\ \Psi(\mathbf{x}) := f(\mathcal{A}(\mathbf{x})) + g(\mathbf{x})$$

Framework

- Signal/image processing: data $\mathbf{z} = \mathrm{A}\overline{\mathbf{x}} + \varepsilon$
- Standard data-fidelity term + prior formulation

$$\underset{\mathbf{x}}{\text{minimize}}\ \Psi(\mathbf{x}) := f(\mathbf{x};\mathbf{z}) + g(\mathbf{x})$$



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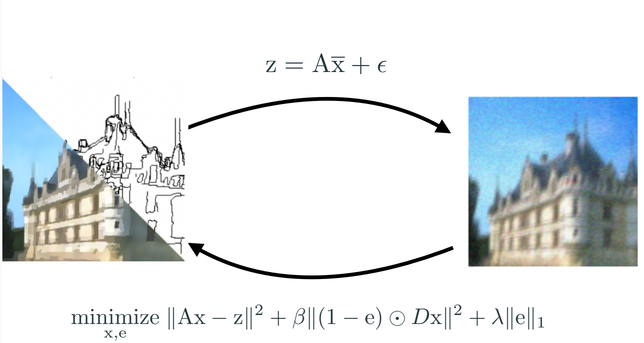

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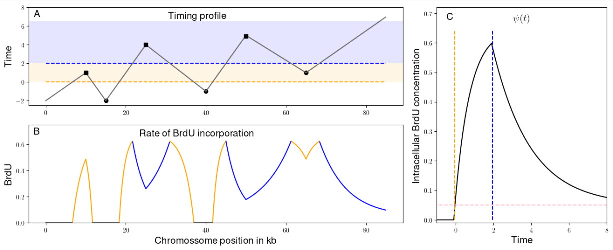

[C. Lage, et al. Identifying a piecewise affine signal from its nonlinear observation - application to DNA replication analysis, 2024]

Nonsmooth convex optimization

Hilbert spaces

A (real) **Hilbert space** $\mathcal{H}$ is a complete real vector space endowed with an inner product $\langle \cdot \mid \cdot \rangle$. The associated norm is

$$(\forall \mathbf{x} \in \mathcal{H}) \qquad \|\mathbf{x}\| = \sqrt{\langle \mathbf{x} \mid \mathbf{x} \rangle}.$$

- Particular case: $\mathcal{H} = \mathbb{R}^N$ (Euclidean space with dimension N).

- Course dedicated to finite dimension.

Norm and adjoint

Let $\mathcal{H}$ and $\mathcal{G}$ be two Hilbert spaces.

A linear operator $\mathrm{D}\colon \mathcal{H} \to \mathcal{G}$ is **bounded** (or continuous) if

$$\|\mathrm{D}\| = \sup_{\|\mathbf{x}\|_{\mathcal{H}} \leqslant 1} \|\mathrm{D}\mathbf{x}\|_{\mathcal{G}} < +\infty$$

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$\mathcal{B}(\mathcal{H}, \mathcal{G})$: Banach space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$.

Norm and adjoint

Let $\mathcal{H}$ and $\mathcal{G}$ be two Hilbert spaces.
Let $\mathrm{D} \in \mathcal{B}(\mathcal{H}, \mathcal{G})$. Its **adjoint** D^* is the operator in $\mathcal{B}(\mathcal{G}, \mathcal{H})$ defined as

$$(\forall (\mathbf{x}, \mathbf{y}) \in \mathcal{H} \times \mathcal{G}) \qquad \langle \mathbf{y} \mid \mathrm{D}\mathbf{x} \rangle_{\mathcal{G}} = \langle \mathrm{D}^* \mathbf{y} \mid \mathbf{x} \rangle_{\mathcal{H}} .$$

Example:

If $\qquad \mathrm{D}\colon \mathcal{H} \to \mathcal{H}^n \colon \mathbf{x} \mapsto (\mathbf{x}, \ldots, \mathbf{x})$

then $\qquad \mathrm{D}^*\colon \mathcal{H}^n \to \mathcal{H} \colon \mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_n) \mapsto \sum_{i=1}^{n} \mathbf{y}_i$

Proof:

$$\langle \mathrm{D}\mathbf{x} \mid \mathbf{y} \rangle = \langle (\mathbf{x}, \ldots, \mathbf{x}) \mid (\mathbf{y}_1, \ldots, \mathbf{y}_n) \rangle = \sum_{i=1}^{n} \langle \mathbf{x} \mid \mathbf{y}_i \rangle = \left\langle \mathbf{x} \mid \sum_{i=1}^{n} \mathbf{y}_i \right\rangle$$

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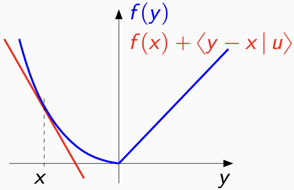
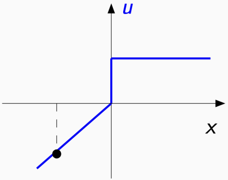
$2^{\mathcal{H}}$ is the power set of $\mathcal{H}$, i.e. the family of all subsets of $\mathcal{H}$.

Let $\Psi : \mathcal{H} \to]-\infty, +\infty]$ be a proper function.
The (Moreau) **subdifferential of Ψ**, is such that

$$\begin{aligned}\partial\Psi : \mathcal{H} &\to 2^{\mathcal{H}}\\ \mathbf{x} &\to \{\mathbf{u} \in \mathcal{H} \mid (\forall \mathbf{y} \in \mathcal{H})\ \langle \mathbf{y} - \mathbf{x} | \mathbf{u} \rangle + \Psi(\mathbf{x}) \leqslant \Psi(\mathbf{y})\}\end{aligned}$$





Subdifferential of function

$\min_{\mathbf{x}} \Psi(\mathbf{x})$

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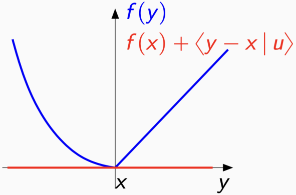

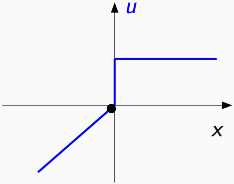

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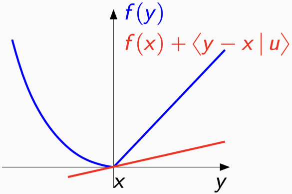

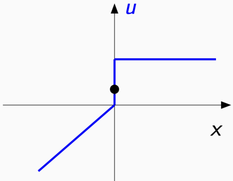

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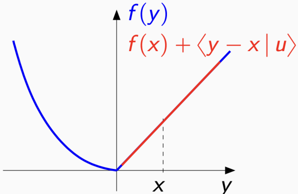

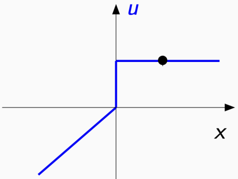

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Fermat's rule: $\quad 0 \in \partial\Psi(\mathbf{x}) \quad \Leftrightarrow \quad \widehat{\mathbf{x}} \in \operatorname{Argmin}_{\mathbf{x}} \Psi(\mathbf{x})$

Let $\mathcal{H}$ be a Hilbert space. Let $\boldsymbol{\Phi}\colon \mathcal{H} \to 2^{\mathcal{H}}$.
The set of **zeros** of $\boldsymbol{\Phi}$ is : $\quad \mathrm{zer}\boldsymbol{\Phi} = \{\mathbf{x} \in \mathcal{H} \,|\, 0 \in \boldsymbol{\Phi}\mathbf{x}\}$.

Fermat's rule:

$$0 \in \partial\Psi(\widehat{\mathbf{x}}) \quad \Leftrightarrow \quad \widehat{\mathbf{x}} \in \mathrm{zer}\partial\Psi \qquad (\text{i.e. } \boldsymbol{\Phi} = \partial\Psi)$$
$$\Leftrightarrow \quad \widehat{\mathbf{x}} \in \operatorname*{Argmin}_{\mathbf{x}} \Psi(\mathbf{x})$$

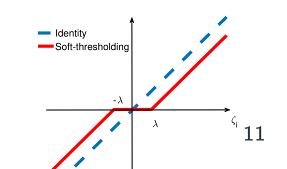
Remark: If Ψ differentiable, $\partial\Psi = \{\nabla\Psi\}$

Definition [Moreau,1965] Let $\Psi\colon \mathcal{H} \to]-\infty, +\infty]$ be a convex, l.s.c., and proper function. The proximity operator of Ψ at point $\mathbf{x} \in \mathcal{H}$ is the **unique point** denoted by $\mathrm{prox}_\Psi \mathbf{x}$ such that

$$(\forall \mathbf{x} \in \mathcal{H}) \qquad \mathrm{prox}_\Psi \mathbf{x} = \arg\min_{\mathbf{v} \in \mathcal{H}} \frac{1}{2} \|\mathbf{x} - \mathbf{v}\|^2 + \Psi(\mathbf{v})$$

➡ **Existing many closed form expressions**

- $\mathrm{prox}_{\lambda\|\cdot\|_1}$: **soft-thresholding** with a fixed threshold $\lambda > 0$.
- exhaustive list: **PROX Repository**



Let $\mathcal{H}$ be a Hilbert space and $\Psi \in \Gamma_0(\mathcal{H})$.

$$(\forall \mathbf{x} \in \mathcal{H}) \quad \mathbf{p} = \mathrm{prox}_{\Psi}(\mathbf{x}) \quad \Leftrightarrow \quad \mathbf{x} - \mathbf{p} \in \partial\Psi(\mathbf{p}).$$

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- Proof:

$$\begin{aligned} & \mathbf{p} = \underset{\mathbf{y}\in\mathcal{H}}{\arg\min}\ \Psi(\mathbf{y}) + \frac{1}{2}\|\mathbf{y} - \mathbf{x}\|^2 \\ \Leftrightarrow \quad & 0 \in \partial\Big(\Psi + \frac{1}{2}\|\cdot - \mathbf{x}\|^2\Big)(\mathbf{p}) \\ \Leftrightarrow \quad & 0 \in \partial\Psi(\mathbf{p}) + \mathbf{p} - \mathbf{x} \end{aligned}$$

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Gradient descent step :

$$\mathbf{x}^{[k+1]} = \mathbf{x}^{[k]} - \gamma \nabla \Psi(\mathbf{x}^{[k]})$$

(Explicit) sub-gradient descent step :

$$\mathbf{x}^{[k+1]} = \mathbf{x}^{[k]} - \mathbf{u}^{[k]} \text{ where } \mathbf{u}^{[k]} \in \gamma \partial \Psi(\mathbf{x}^{[k]})$$

(Implicit) sub-gradient descent step = Proximal step :

$$\begin{aligned} \mathbf{x}^{[k+1]} &= \mathbf{x}^{[k]} - \mathbf{u}^{[k]} \text{ where } \mathbf{u}^{[k]} \in \gamma \partial \Psi(\mathbf{x}^{[k+1]}) \\ &= \mathrm{prox}_{\Psi}(\mathbf{x}^{[k]}) \end{aligned}$$

Iterative scheme

➡ **Minimization problem** :

$$\widehat{\mathbf{x}} \in \underset{\mathbf{x}}{\mathrm{Argmin}} \Psi(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})$$

➡ **Design of a recursive sequence of the form**

$$(\forall k \in \mathbb{N}) \qquad \mathbf{x}^{[k+1]} = \boldsymbol{\Phi}\mathbf{x}^{[k]},$$

Gradient descent	$\boldsymbol{\Phi} = \mathrm{Id} - \gamma(\nabla f + \nabla g)$
Proximal point algorithm	$\boldsymbol{\Phi} = \mathrm{prox}_{\gamma(f+g)}$
Forward-Backward	$\boldsymbol{\Phi} = \mathrm{prox}_{\gamma g}(\mathrm{Id} - \gamma \nabla f)$
Peaceman-Rachford	$\boldsymbol{\Phi} = (2\,\mathrm{prox}_{\gamma g} - \mathrm{Id}) \circ (2\,\mathrm{prox}_{\gamma f} - \mathrm{Id})$
Douglas-Rachford	$\boldsymbol{\Phi} = \mathrm{prox}_{\gamma g}(2\,\mathrm{prox}_{\gamma f} - \mathrm{Id}) + \mathrm{Id} - \mathrm{prox}_{\gamma f}$

Fixed point algorithm: zeros and fixed points

Let $\mathcal{H}$ be a Hilbert space. Let $\mathbf{\Phi} \colon \mathcal{H} \to 2^{\mathcal{H}}$.

The set of **zeros** of $\mathbf{\Phi}$ is : $\quad \mathrm{zer}\mathbf{\Phi} = \{\mathbf{x} \in \mathcal{H} \,|\, 0 \in \Phi\mathbf{x}\}$.

The set of **fixed points** of $\mathbf{\Phi}$ is : $\mathrm{Fix}\mathbf{\Phi} = \{\mathbf{x} \in \mathcal{H} \,|\, \mathbf{x} \in \Phi\mathbf{x}\}$.

Banach-Picard theorem

An operator $\boldsymbol{\Phi}\colon \mathcal{H} \to \mathcal{H}$ is **$\omega-$Lipschitz continuous** for some $\omega \in [0, +\infty[$ if

$$(\forall \mathbf{x} \in \mathcal{H})(\forall \mathbf{y} \in \mathcal{H}) \quad \|\boldsymbol{\Phi}\mathbf{x} - \boldsymbol{\Phi}\mathbf{y}\| \leqslant \omega\|\mathbf{x} - \mathbf{y}\|.$$

$\boldsymbol{\Phi}$ is **nonexpansive** if it is $1-$Lipschitz continuous.

Banach-Picard theorem Let $\omega \in [0, 1[$, let $\boldsymbol{\Phi}\colon \mathcal{H} \to \mathcal{H}$ be a $\omega-$Lipschitz continuous operator, and let $\mathbf{x}_0 \in \mathcal{H}$. Set

$$(\forall k \in \mathbb{N}) \quad \mathbf{x}^{[k+1]} = \boldsymbol{\Phi}\mathbf{x}^{[k]}.$$

Then, $\operatorname{Fix}\boldsymbol{\Phi} = \{\widehat{\mathbf{x}}\}$ for some $\widehat{\mathbf{x}} \in \mathcal{H}$ and we have

$$(\forall k \in \mathbb{N}) \quad \|\mathbf{x}^{[k]} - \widehat{\mathbf{x}}\| \leqslant \omega^k \|\mathbf{x}^{[0]} - \widehat{\mathbf{x}}\|.$$

Moreover, $(\mathbf{x}^{[k]})_{k\in\mathbb{N}}$ converges to $\widehat{\mathbf{x}}$ with linear convergence rate ω.

Averaged nonexpansive operator

An operator $\mathbf{\Phi}\colon \mathcal{H} \to \mathcal{H}$ is **$\alpha-$averaged nonexpansive** for some $\alpha \in]0,1]$ if, for every $\mathbf{x} \in \mathcal{H}$ and $\mathbf{y} \in \mathcal{H}$,

$$\|\mathbf{\Phi x} - \mathbf{\Phi y}\|^2 \leqslant \|\mathbf{x} - \mathbf{y}\|^2 - \left(\frac{1-\alpha}{\alpha}\right) \|(\mathbf{I} - \mathbf{\Phi})\mathbf{x} - (\mathbf{I} - \mathbf{\Phi})\mathbf{y}\|^2,$$

$\mathbf{\Phi}$ is **firmly nonexpansive** if it is $1/2-$averaged.

Theorem Let $\alpha \in]0,1[$, let $\mathbf{\Phi}\colon \mathcal{H} \to \mathcal{H}$ be a $\alpha-$averaged nonexpansive operator such that $\operatorname{Fix}\mathbf{\Phi} \neq \varnothing$, and let $\mathbf{x}^{[0]} \in \mathcal{H}$. Set

$$(\forall k \in \mathbb{N}) \quad \mathbf{x}^{[k+1]} = \mathbf{\Phi x}^{[k]}.$$

Then $(\mathbf{x}^{[k]})_{k\in\mathbb{N}}$ converges weakly to a point in $\operatorname{Fix}\mathbf{\Phi}$.

Averaged operator: example

Let $\mathcal{H}$ be a Hilbert space, $\Gamma_0(\mathcal{H})$ denotes the class of proper, lower semi-continuous, and convex functions from $\mathcal{H}$ to $]-\infty, +\infty]$.

Gradient descent operator

$\Psi \in \Gamma_0(\mathcal{H})$ with ν-Lipschitz gradient with $\nu > 0$.

For some $\gamma \in]0, \frac{2}{\nu}[$, $\mathbf{I} - \gamma \nabla \Psi$ is a $\gamma\nu/2$-averaged operator.

Proximal operator

$\Psi \in \Gamma_0(\mathcal{H})$.

For some $\gamma > 0$, $\mathrm{prox}_{\gamma\Psi}$ is a $1/2$-averaged operator.

Composition of averaged operator

Theorem Let S be a nonempty subset of $\mathcal{H}$. Let $\alpha_1 \in]0,1[$ and $\alpha_2 \in]0,1[$. Let $\mathbf{\Phi}_1 : S \to S$ be α_1-averaged and $\mathbf{\Phi}_2 : S \to S$ be α_2-averaged.
Then $\mathbf{\Phi} = \mathbf{\Phi}_1 \mathbf{\Phi}_2$ is α-averaged with

$$\alpha = \frac{\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2}{1 - \alpha_1\alpha_2} \in]0,1[.$$

Proof: Extracted from Theorem 26.14 [Bauschke-Combettes, 2017]

Forward-backward splitting $\min_{\mathbf{x}} f(\mathbf{x}) + g(\mathbf{x})$

$f \in \Gamma_0(\mathcal{H})$ with ν-Lipschitz gradient and $g \in \Gamma_0(\mathcal{H})$.
For some $\gamma > 0$,

$$\mathbf{\Phi} := \mathrm{prox}_{\gamma g}(\mathbf{I} - \gamma \nabla f)$$

- **Iterations**: $(\forall k \in \mathbb{N}) \quad x^{[k+1]} = \mathrm{prox}_{\gamma g}(x^{[k]} - \gamma \nabla f(x^{[k]}))$.

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- Roots in projected gradient method [Levitin 1966] when $g = \iota_C$ for some closed convex set C.
- Also named **Proximal Gradient** (PG) algorithm or **Iterative Soft Thresholding Algorithm** (ISTA).

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- **Iterations**: $(\forall k \in \mathbb{N}) \quad x^{[k+1]} = \mathrm{prox}_{\gamma g}(x^{[k]} - \gamma \nabla f(x^{[k]}))$.
- $\mathrm{prox}_{\gamma g}(\mathbf{I} - \gamma \nabla f)$ is α-**averaged nonexpansive** where

$$\alpha = \frac{\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2}{1 - \alpha_1\alpha_2}.$$

where $\alpha_2 = \gamma\nu/2$ and $\alpha_1 = 1/2$ leading to

$$\alpha = \frac{1}{2 - \gamma\nu/2} \in]0, 1[.$$

Leading to

$$\gamma < 2/\nu.$$

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- **Iterations**:$(\forall k \in \mathbb{N}) \quad x^{[k+1]} = \mathrm{prox}_{\gamma g}(x^{[k]} - \gamma \nabla f(x^{[k]}))$.
- $\mathrm{prox}_{\gamma g}(\mathbf{I} - \gamma \nabla f)$ is α-**averaged nonexpansive**
- For every $\gamma > 0$, $\mathrm{zer}\,(\nabla f + \partial g) = \mathrm{Fix}\,\mathbf{\Phi}$.
  Proof:

$$\begin{aligned} x \in \mathrm{Fix}\,\mathbf{\Phi} &\Leftrightarrow x = \mathrm{prox}_{\gamma g}(x - \gamma \nabla f(x)) \\ &\Leftrightarrow x - \gamma \nabla f(x) - x \in \gamma \partial g x \\ &\Leftrightarrow 0 \in \nabla f(x) + \partial g(x) \\ &\Leftrightarrow x \in \mathrm{zer}\,(\nabla f + \partial g) \end{aligned}$$

Forward-backward splitting $\min_{\mathbf{x}} f(\mathbf{x}) + g(\mathbf{x})$

$f \in \Gamma_0(\mathcal{H})$ with ν-Lipschitz gradient and $g \in \Gamma_0(\mathcal{H})$.
For some $\gamma > 0$,

$$\mathbf{\Phi} := \mathrm{prox}_{\gamma g}(\mathbf{I} - \gamma \nabla f)$$

- **Iterations**: $(\forall k \in \mathbb{N}) \quad x^{[k+1]} = \mathrm{prox}_{\gamma g}(x^{[k]} - \gamma \nabla f(x^{[k]}))$.
- $\mathrm{prox}_{\gamma g}(\mathbf{I} - \gamma \nabla f)$ is α-**averaged nonexpansive**
- For every $\gamma > 0$, $\mathrm{zer}\,(\nabla f + \partial g) = \mathrm{Fix}\,\mathbf{\Phi}$.
- For every $\gamma \in \left]0, 2\nu^{-1}\right[$, the **FBS method converges** to a point in $\mathrm{zer}\,(\nabla f + \partial g)$.

f is **ρ-strongly convex** with $\rho > 0$ if $f - \frac{\rho}{2}\|\cdot\|_2^2$ is convex.

Properties:

- If f is ρ-strongly convex then

$$(\forall x, y \in \mathcal{H}) \qquad \langle \nabla f(x) - \nabla f(y) | x - y \rangle \geqslant \rho \|x - y\|^2$$

- If f is twice differentiable, then f is ρ-strongly convex if and only if all the eigenvalues of the Hessian matrix of f are at most equal to ρ.

Proposition [Briceno-Arias, Pustelnik, 2021]
$f \in \Gamma_0(\mathcal{H})$ with ν-Lipschitz gradient, $\rho-$**strongly convex** for some $\rho \in \,]0,\nu[$, and $g \in \Gamma_0(\mathcal{H})$ with β-Lipschitz gradient. Let $\gamma > 0$. Then,

1. **FBS** Suppose that $\gamma \in \left]0, 2\nu^{-1}\right[$. Then $\mathbf{\Phi} = \operatorname{prox}_{\gamma g}(\mathbf{I} - \gamma\nabla f)$ is $\omega(\gamma)-$Lipschitz continuous, where

$$\omega(\gamma) := \max\left\{|1-\gamma\rho|, |1-\gamma\nu|\right\} \in \,]0,1[\,.$$

The minimum is achieved at

$$\gamma^* = \frac{2}{\rho+\nu} \quad \text{and} \quad r_{T_1}(\gamma^*) = \frac{\nu-\rho}{\nu+\rho}.$$

Proposition [Briceno-Arias, Pustelnik, 2021]
$f \in \Gamma_0(\mathcal{H})$ with ν-Lipschitz gradient, $\rho-$**strongly convex** for some $\rho \in \,]0, \nu[$, and $g \in \Gamma_0(\mathcal{H})$ with β-Lipschitz gradient. Let $\gamma > 0$. Then,

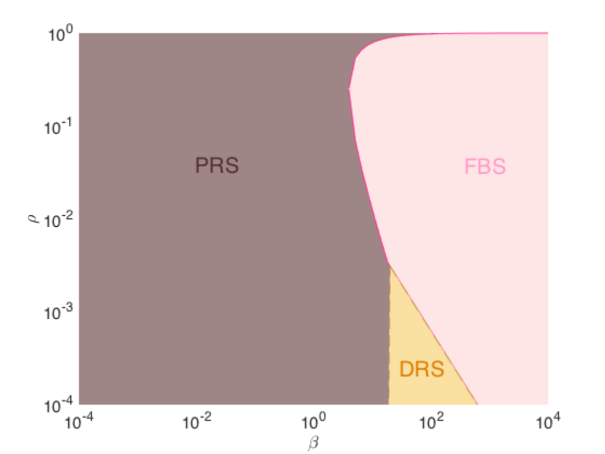
1. **FBS** Suppose that $\gamma \in \left]0, 2\beta^{-1}\right]$. Then $\mathbf{\Phi} = \mathrm{prox}_{\gamma f}(\mathbf{I} - \gamma \nabla g)$ is $\omega(\gamma)-$Lipschitz continuous, where

$$\omega(\gamma) := \frac{1}{1+\gamma\rho} \in \,]0,1[\,.$$

The minimum is achieved at

$$\gamma^* = 2\beta^{-1} \quad \text{and} \quad r_{T_2}(\gamma^*) = \frac{1}{1+2\beta^{-1}\rho}.$$

Regime diagram



Comparison of the convergence rates of Forward-Backward, Peaceman-Rachford, Douglas-Rachford for two choices of $\alpha = \nu_f^{-1}$, $\beta = \nu_g^{-1}$, and ρ.[Briceño-Arias, Pustelnik, 2021]

References nonsmooth convex optimization

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Conclusion nonsmooth convex optimization

- Convergence of the iterates important in signal/image processing.
- Two types of convergence: α-averaged and Lipschitz continuity.
- With linear rate possibility to compare algorithms but one need to be sure that the bound is tight (cf. A. Taylor et al. work).
- Establish systematically diagram regimes when algorithms are compared to capture their regime of efficiency.

Nonsmooth nonconvex optimization: generalities

Let $\Psi : \mathbb{R}^N \to]-\infty, +\infty]$ be a proper function.
The (Fréchet) **subdifferential of Ψ**, denoted by $\partial\Psi$, is such that, for every $\mathbf{x} \in \operatorname{dom}\Psi$,

$$\partial\Psi(\mathbf{x}) = \left\{ \mathbf{u} \in \mathcal{H} \mid \liminf_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\neq\mathbf{x}}} \frac{\Psi(\mathbf{y}) - \Psi(\mathbf{x}) - \langle \mathbf{y} - \mathbf{x} | \mathbf{u} \rangle}{\|\mathbf{x} - \mathbf{y}\|} \geqslant 0 \right\}$$

If $\mathbf{x} \notin \operatorname{dom}\Psi$, then $\partial\Psi(\mathbf{x}) = \varnothing$.

Remarks:

- If Ψ convex, Fréchet subdifferential matches Moreau subdifferential [Rockafellar, Wets, 1997].
- The set of critical points = zeros of ∂f.
- When $f \in \Gamma_0(\mathbb{R}^N)$, the critical points are the global minimizers.
- $\widehat{\mathbf{x}} \in \operatorname{Argmin}_{\mathbf{x}} \Psi(\mathbf{x}) \quad \Rightarrow \quad 0 \in \partial\Psi(\widehat{\mathbf{x}})$

Proximity operator

Proposition [Rockafellar, Wets, 1998] Let $\Psi\colon \mathbb{R}^N \to]-\infty, +\infty]$ be a l.s.c. and proper function. The proximity operator of Ψ at point $\mathbf{x} \in \mathbb{R}^N$ and for $\gamma > 0$ is

$$(\forall \mathbf{x} \in \mathbb{R}^N) \qquad \operatorname{prox}_{\gamma\Psi} \mathbf{x} = \arg\min_{\mathbf{v}\in\mathcal{H}} \frac{1}{2} \|\mathbf{x} - \mathbf{v}\|^2 + \gamma\Psi(\mathbf{v})$$

Proposition (Well-definedness of proximal maps) [Bolte et al., 2014] Let $\Psi\colon \mathbb{R}^N \to]-\infty, +\infty]$ be a l.s.c. and proper function with $\inf \Psi > -\infty$. Then, for every $\gamma > 0$, $\operatorname{prox}_{\gamma\Psi} \mathbf{x}$ is nonempty and compact.

Proximity operator

Proposition [Gribonval, Nikolova, 2020] Let $S \subset \mathcal{H}$ be non-empty. A function $\varrho\colon S \to \mathcal{H}$ is a proximity operator of a function $\Psi\colon \mathcal{H} \to (-\infty, +\infty]$ if, and only if, there exists a convex l.s.c. function $\varphi\colon \mathcal{H} \to (-\infty, +\infty]$ such that for each $y \in S$, $\varrho(y) \in \partial\varphi(y)$.

- If Ψ is continuous, there exists a convex differentiable function φ such that $\mathrm{prox}_\Psi(y) = \nabla\varphi(y)$.

Non-convex case: Lojasiewicz inequality

- Fundamental tool in non-convex optimization to demonstrate the convergence of sequences that are not necessarily convex.
- Inequality first proposed in [Lojasiewicz, 1963].
- Satisfied by any analytic real function.

Let $\Psi\colon \mathcal{H} \to \mathbb{R}$ be an analytic real function, and $\widehat{\mathbf{x}}$ a critical point of this function. There exist $\theta \in [1/2, 1[$ and $\kappa \in]0, +\infty[$ such that Ψ satisfies Lojasiewicz's inequality in a neighborhood $N(\widehat{\mathbf{x}})$ of $\widehat{\mathbf{x}}$, i.e.,

$$(\forall \mathbf{x} \in N(\widehat{\mathbf{x}})) \quad |\Psi(\mathbf{x}) - \Psi(\widehat{\mathbf{x}})|^{\theta} \leqslant \kappa \|\nabla \Psi(\mathbf{x})\|$$

Non-convex case: Kurdyka-Lojasiewicz (KL) inequality

Let $\Psi\colon \mathcal{H} \to]-\infty, +\infty]$ be an analytic real function. Ψ satisfies the KL inequality if, for any $\xi \in \mathbb{R}$, and, for any bounded subset S of $\mathcal{H}$, there exist three constants $\kappa > 0$, $\zeta > 0$, and $\theta \in [0, 1[$ such that

$$(\forall \mathbf{u} \in \partial\Psi(\mathbf{x})) \quad |\Psi(\mathbf{x}) - \xi|^{\theta} \leqslant \kappa \|\mathbf{u}\|$$

for every $\mathbf{x} \in S$ such that $|\Psi(\mathbf{x}) - \xi| \leqslant \zeta$.

- Generalization [Kurdyka, 1998] [Bolte et al., 2006 2007]
- As pointed out in [Attouch et al., 2010], the logarithm and exponential functions are neither real semi-algebraic nor real analytic, and they do not satisfy the inequality. However, there exists a function such that the KL inequality in its most general form is satisfied.

Converge proof receipe

- $\Psi : \mathcal{H} \to (-\infty, +\infty]$ be a proper l.s.c function bounded from below.
- Let $\mathbf{\Phi}$ which generates a sequence $(\mathbf{x}^{[k]})_{k\in\mathbb{N}}$ i.e. $\mathbf{x}^{[k+1]} = \mathbf{\Phi}\mathbf{x}^{[k]}$.

1. **sufficient decrease property**, which requires to find a positive constant ρ_1 such that, for any iteration k
$$\rho_1 \|\mathbf{x}^{[k+1]} - \mathbf{x}^{[k]}\|^2 \leqslant \Psi(\mathbf{x}^{[k]}) - \Psi(\mathbf{x}^{[k+1]})$$
2. **subgradient lower bound for the iterates gap** , which requires to find a positive constant ρ_2 such that
$$\|\mathbf{u}^{[k+1]}\| \leqslant \rho_2 \|\mathbf{x}^{[k+1]} - \mathbf{x}^{[k]}\| \quad \text{with} \quad \mathbf{u}^{[k+1]} \in \partial\Psi(\mathbf{x}^{[k+1]})$$
3. Ψ **constant on a subset of critical points**, From (1)+(2) the set of all limit/accumulation points is a nonempty, compact and connected set. The objective function Ψ is finite and constant on the set of all limit points/subset of the critical points of Ψ.
4. **using the KL property** and show that the generated sequence $(\mathbf{x}^{[k]})_{k\in\mathbb{N}}$ is a Cauchy sequence and hence is a convergent.

Nonsmooth nonconvex optimization: alternated schemes

Gauss-Seidel iterations $\min_{\mathbf{x},\mathbf{e}} \Psi(\mathbf{x}, \mathrm{e}) := f(\mathbf{x}) + g_1(\mathbf{x}, \mathrm{e}) + g_2(\mathrm{e})$

Gauss-Seidel = coordinate descent

Set $\mathbf{e}^{[0]} \in \mathbb{R}^{|\mathbb{E}|}$.

For $k \in \mathbb{N}$

$$\mathbf{x}^{[k+1]} \in \operatorname{Arg\,min}_{\mathbf{x}} \Psi(\mathbf{x}, \mathbf{e}^{[k]})$$

$$\mathbf{e}^{[k+1]} \in \operatorname{Arg\,min}_{\mathbf{e}} \Psi(\mathbf{x}^{[k+1]}, \mathbf{e})$$

- Under technical assumptions, convergence of the sequence $(\mathbf{x}^{[k]}, \mathbf{e}^{[k]})_{k\in\mathbb{N}}$ to a critical point $(\widehat{\mathbf{x}}, \widehat{\mathrm{e}})$ of Ψ.

  Technical assumptions = minimum is attained at each iteration, e.g. by assuming strict convexity w.r.t one argument. [Auslender1976, Bertsekas1999]

PAM = Proximal Alternating Minimization
[Attouch et al 2010]

Set $\mathbf{e}^{[0]} \in \mathbb{R}^{|\mathbb{E}|}$.

For $k \in \mathbb{N}$

$$c_k > 0, d_k > 0$$
$$\mathbf{x}^{[k+1]} \in \operatorname{Arg\,min}_{\mathbf{x}} \Psi(\mathbf{x}, \mathbf{e}^{[k]}) + \frac{c_k}{2}\|\mathbf{x} - \mathbf{x}^{[k]}\|^2$$
$$\mathbf{e}^{[k+1]} \in \operatorname{Arg\,min}_{\mathbf{e}} \Psi(\mathbf{x}^{[k+1]}, \mathbf{e}) + \frac{d_k}{2}\|\mathbf{e} - \mathbf{e}^{[k]}\|^2$$

- Can be rewritten as

For $k \in \mathbb{N}$

$$\mathbf{x}^{[k+1]} \in \operatorname{prox}_{\frac{1}{c_k}\Psi(\cdot,\mathbf{e}^{[k]})}(\mathbf{x}^{[k]})$$
$$\mathbf{e}^{[k+1]} \in \operatorname{prox}_{\frac{1}{d_k}\Psi(\mathbf{x}^{[k+1]},\cdot)}(\mathbf{e}^{[k]})$$

- Under technical assumptions, convergence of the sequence $(\mathbf{x}^{[k]}, \mathbf{e}^{[k]})_{k\in\mathbb{N}}$ to a critical point $(\widehat{\mathbf{x}}, \widehat{\mathrm{e}})$ of Ψ.

PAM $\min_{\mathbf{x},\mathbf{e}} \Psi(\mathbf{x},\mathbf{e}) := f(\mathbf{x}) + g_1(\mathbf{x},\mathbf{e}) + g_2(\mathbf{e})$

PAM = Proximal Alternating Minimization
[Attouch et al 2010]

Set $\mathbf{e}^{[0]} \in \mathbb{R}^{|\mathbb{E}|}$.

For $k \in \mathbb{N}$

$$c_k > 0, d_k > 0$$
$$\mathbf{x}^{[k+1]} \in \operatorname{Arg\,min}_{\mathbf{x}} \Psi(\mathbf{x}, \mathbf{e}^{[k]}) + \frac{c_k}{2}\|\mathbf{x} - \mathbf{x}^{[k]}\|^2$$
$$\mathbf{e}^{[k+1]} \in \operatorname{Arg\,min}_{\mathbf{e}} \Psi(\mathbf{x}^{[k+1]}, \mathbf{e}) + \frac{d_k}{2}\|\mathbf{e} - \mathbf{e}^{[k]}\|^2$$

- Can be rewritten as

For $k \in \mathbb{N}$

$$\mathbf{x}^{[k+1]} \in \operatorname{prox}_{\frac{1}{c_k}\Psi(\cdot,\mathbf{e}^{[k]})}(\mathbf{x}^{[k]}) = \operatorname{prox}_{\frac{1}{c_k}f+g_1(\cdot,\mathbf{e}^{[k]})}(\mathbf{x}^{[k]})$$
$$\mathbf{e}^{[k+1]} \in \operatorname{prox}_{\frac{1}{d_k}\Psi(\mathbf{x}^{[k+1]},\cdot)}(\mathbf{e}^{[k]}) = \operatorname{prox}_{\frac{1}{d_k}g_1(\mathbf{x}^{[k+1]},\cdot)+g_2}(\mathbf{e}^{[k]})$$

- Can be difficult to compute the proximity operator of a sum of two functions.

Proximity operator of a sum of two functions

$$\mathrm{prox}_{g_1+g_2} = \mathrm{prox}_{g_2} \circ \mathrm{prox}_{g_1}?$$

- [Combettes-Pesquet, 2007] $N = 1$, $g_2 = \iota_C$ of a non-empty closed convex subset of C and g_1 is differentiable at 0 with $h'(0) = 0$.
- [Chaux-Pesquet-Pustelnik,2009] C and g_2 are separable in the same basis.
- [Yu, 2013][Shi et al., 2017] $\partial g_2(x) \subset \partial g_2(\mathrm{prox}\, g_1(x))$.
- Other recent results [Pustelnik, Condat, 2017][Yukawa, Kagami, 2017][del Aguila Pla, Jaldén, 2017]

PALM = Proximal Alternating Linearized Minimization [Bolte et al 2014]

Set $\mathbf{e}^{[0]} \in \mathbb{R}^{|\mathbb{E}|}$.

For $k \in \mathbb{N}$

- Set $\gamma > 1$ and $c_k = \gamma \nu_{g_1}(\mathbf{e}^{[k]})$
- $\mathbf{x}^{[k+1]} \in \mathrm{prox}_{\frac{1}{c_k} f}\left(\mathbf{x}^{[k+1]} - \frac{1}{c_k}\nabla_{\mathbf{x}} g_1(\mathbf{x}^{[k]}, \mathbf{e}^{[k]})\right)$
- Set $\tau > 1$ and $d_k = \tau \nu_{g_1}(\mathbf{x}^{[k+1]})$
- $\mathbf{e}^{[k+1]} \in \mathrm{prox}_{\frac{1}{d_k} g_2}\left(\mathbf{e}^{[k+1]} - \frac{1}{d_k}\nabla_{\mathbf{e}} g_1(\mathbf{x}^{[k+1]}, \mathbf{e}^{[k]})\right)$

- Under technical assumptions, convergence of the sequence $(\mathbf{x}^{[k]}, \mathbf{e}^{[k]})_{k\in\mathbb{N}}$ to a critical point $(\widehat{\mathbf{x}}, \widehat{\mathrm{e}})$ of Ψ.

SL-PAM = Semi-linearized Proximal Alternating Minimization

Set $\mathbf{e}^{[0]} \in \mathbb{R}^{|\mathbb{E}|}$.

For $k \in \mathbb{N}$

- Set $\gamma > 1$ and $c_k = \gamma \nu_{g_1}(\mathbf{e}^{[k]})$
- $\mathbf{x}^{[k+1]} \in \operatorname{prox}_{\frac{1}{c_k} f}\left(\mathbf{x}^{[k+1]} - \frac{1}{c_k}\nabla_{\mathbf{x}} g_1(\mathbf{x}^{[k]}, \mathbf{e}^{[k]})\right)$
- Set $d_k > 0$
- $\mathbf{e}^{[k+1]} \in \operatorname{prox}_{\frac{1}{d_k} g_1(\mathbf{x}^{[k+1]},\cdot)+g_2}(\mathbf{e}^{[k]})$

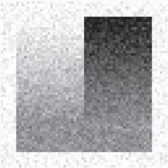
- [Foare et al., 2019]
- Under technical assumptions, convergence of the sequence $(\mathbf{x}^{[k]}, \mathbf{e}^{[k]})_{k\in\mathbb{N}}$ to a critical point $(\widehat{\mathbf{x}}, \widehat{\mathrm{e}})$ of Ψ.

Direct model

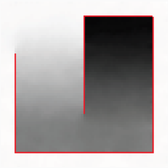
$$\mathbf{z} = \mathcal{D}(\mathrm{A}\bar{\mathbf{x}})$$

where

- $\Omega = \{1, \ldots, N_1\} \times \{1, \ldots, N_2\}$;
- $\bar{\mathbf{x}} \in \mathbb{R}^{|\Omega|}$: original image
- $\mathrm{A} \in \mathbb{R}^{M \times |\Omega|}$: linear degradation (e.g. a blur)
- $\mathcal{D} \colon \mathbb{R}^M \to \mathbb{R}^M$: random degradation (e.g. a Gaussian or Poisson noise)
- $\mathbf{z} \in \mathbb{R}^M$: degraded image



$\mathbf{z}$



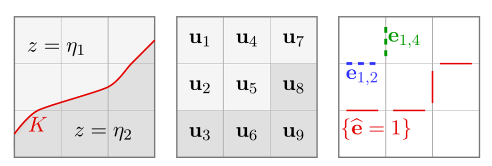
$\widehat{\mathbf{x}}$ and $\widehat{\mathrm{e}}$ (red)

Goal: Recover $\widehat{\mathbf{x}}$ and its associated contours $\widehat{\mathrm{e}}$ from $\mathbf{z}$.

Proposed Discrete Mumford-Shah like (D-MS) model

$$\underset{\mathbf{x},\mathbf{e}}{\text{minimize}}\ \Psi(\mathbf{x},\mathbf{e}) := \mathcal{L}(\mathrm{A}\mathbf{x};\mathbf{z}) + \beta\|(1-\mathbf{e})\odot \mathrm{D}\mathbf{x}\|^2 + \lambda\mathcal{R}(\mathbf{e})$$

- $\mathcal{L}(\mathrm{A}\cdot;\mathbf{z})$: data fidelity term;
- $\mathrm{D} \in \mathbb{R}^{|\mathbb{E}|\times|\Omega|}$: models a finite difference operator;
- $\mathcal{R}$: favors sparse solution (i.e. "short $|K|$");
- $\mathbf{x} \in \mathbb{R}^{|\Omega|}$: piecewise smooth approximation of $\mathbf{z}$;
- $\mathbf{e} \in \mathbb{R}^{|\mathbb{E}|}$: edges between nodes whose value is 1 when a contour change is detected and 0 otherwise.



Proposed Discrete Mumford-Shah like (D-MS) model

$$\underset{\mathbf{x},\mathbf{e}}{\text{minimize}}\ \Psi(\mathbf{x},\mathbf{e}) := \underbrace{\mathcal{L}(\mathrm{A}\mathbf{x};\mathbf{z})}_{f(\mathbf{x})} + \underbrace{\beta\|(1-\mathbf{e})\odot \mathrm{D}\mathbf{x}\|^2}_{g_1(\mathbf{x},\mathbf{e})} + \underbrace{\lambda\mathcal{R}(\mathbf{e})}_{g_2(\mathbf{e})}$$

- $\mathcal{L}(\mathrm{A}\cdot;\mathbf{z})$: data fidelity term;
- $\mathrm{D} \in \mathbb{R}^{|\mathbb{E}|\times|\Omega|}$: models a finite difference operator;
- $\mathcal{R}$: favors sparse solution (i.e. "short $|K|$");
- $\mathbf{x} \in \mathbb{R}^{|\Omega|}$: piecewise smooth approximation of $\mathbf{z}$;
- $\mathbf{e} \in \mathbb{R}^{|\mathbb{E}|}$: edges between nodes whose value is 1 when a contour change is detected and 0 otherwise.

Goal: Identify the **assumptions on $\mathcal{L}$ and $\mathcal{R}$** to design an algorithmic scheme with **convergence guarantees** and **fast** to deal with large scale problems.

Proposed D-MS and algorithmic solution

$$\underset{\mathbf{x},\mathbf{e}}{\text{minimize}}\ \Psi(\mathbf{x},\mathbf{e}) := \mathcal{L}(A\mathbf{x};\mathbf{z}) + \beta\|(1-\mathbf{e})\odot D\mathbf{x}\|^2 + \lambda\mathcal{R}(\mathbf{e})$$

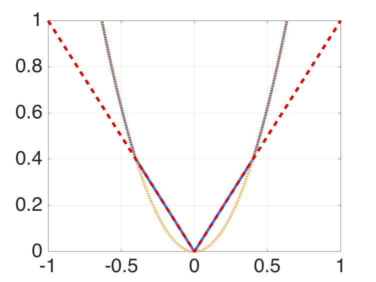
- $\mathcal{R}$: favors sparse solution (i.e. "short $|K|$") and convex.

1. Ambrosio-Tortorelli approximation:

   $\mathcal{R}(\mathbf{e}) = \|\tilde{\mathrm{D}}\mathbf{e}\|_2^2 + \frac{1}{4}\|\mathbf{e}\|_2^2$ with > 0

2. ℓ_1-norm: $\mathcal{R}(\mathbf{e}) = \|\mathbf{e}\|_1$

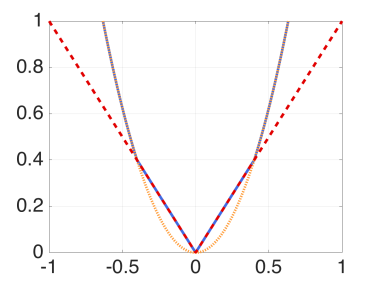
3. Quadratic ℓ_1:
   [Foare, Pustelnik, Condat, 2017]
   $\mathcal{R}(\mathbf{e}) = \sum_{i=1}^{|\mathbb{E}|} \max\left\{|e_i|, \frac{e_i^2}{4}\right\}$.

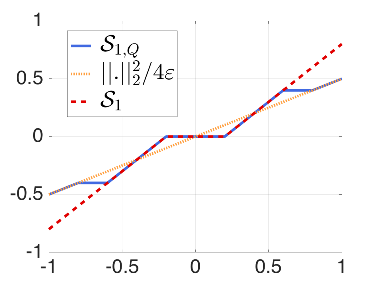


Proposed D-MS and algorithmic solution

Proposition [Foare, Pustelnik, Condat, 2017]
For every $\eta \in \mathbb{R}$ and $\gamma, \epsilon > 0$

$$\mathrm{prox}_{\gamma \max\{|.|, \frac{|.|^2}{4\epsilon}\}}(\eta) = \mathrm{sign}(\eta)\ \max\left\{0, \min\left[|\eta|-\gamma, \max\left(4\epsilon, \frac{|\eta|}{\frac{\gamma}{2\epsilon}+1}\right)\right]\right\}$$





Proposed D-MS and algorithmic solution

Proposition [Foare, Pustelnik, Condat, 2019]

We assume that $\mathcal{R}$ is separable, i.e,

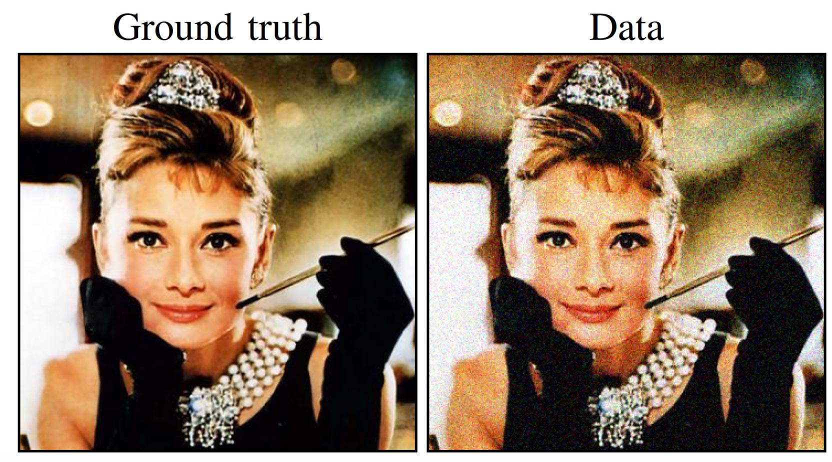
$$(\forall \mathbf{e} = (\mathbf{e}_i)_{1 \leqslant i \leqslant |\mathbb{E}|}) \qquad \mathcal{R}(\mathbf{e}) = \sum_{i=1}^{|\mathbb{E}|} \sigma_i(\mathbf{e}_i),$$

where $\sigma_i : \mathbb{R}^{|\mathbb{E}|} \to]-\infty; +\infty]$ with a closed form proximity operator expression.

Let $\mathrm{d}_k > 0$, then

$$\mathrm{prox}_{\frac{1}{d_k}\lambda\mathcal{R}+\mathcal{S}(\mathrm{D},\cdot)}(\mathbf{e}^{[k]}) = \left(\mathrm{prox}_{\frac{\lambda\sigma_i}{2\beta(\mathrm{D}\mathbf{x}^{[k]})_i^2 + d_k}} \left(\frac{\beta(\mathrm{D}\mathbf{e}^{[k]})_i^2 + \frac{d_k \mathbf{e}_i^{[k]}}{2}}{\beta(\mathrm{D}\mathbf{e}^{[k]})_i^2 + \frac{d_k}{2}} \right) \right)_{i \in \mathbb{E}}$$

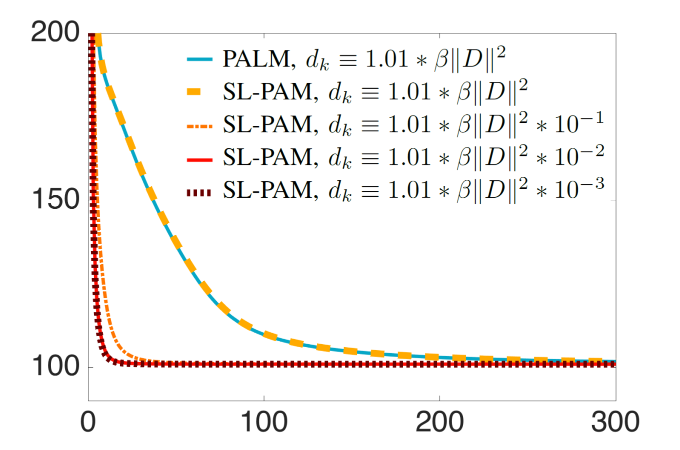
Experiments



Experiments 1

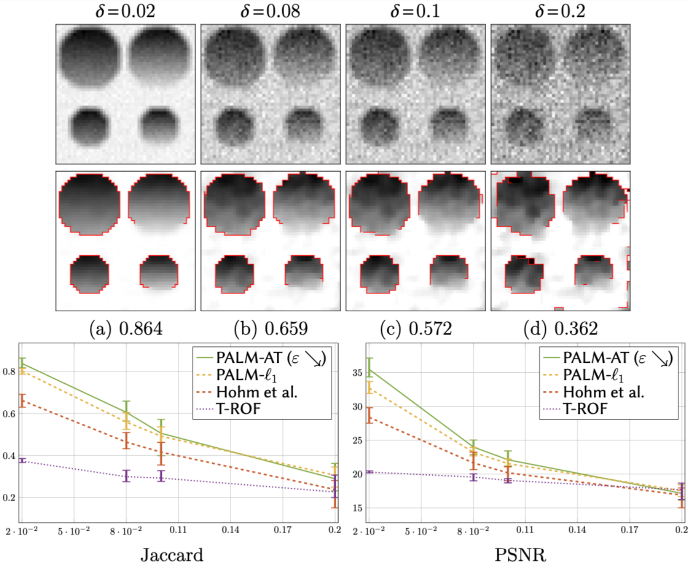
TV	(Strekalovskiy, Cremers, 2014)	Discret AT (Foare et al., 2016)	Quadratic-ℓ_1
SNR = 24.67 dB	SNR = 22.42 dB	SNR = 23.43 dB	**SNR = 23.75 dB**
SSIM = 0.944	SSIM = 0.855	SSIM = 0.867	**SSIM = 0.877**
Time = 29.25 s	**Time = 1.92s**	Time = 1992s	Time = 57.84s

Convergence PALM versus SL-PALM: $\Psi(\mathbf{x}^{[\ell]}, \mathrm{e}^{[\ell]})$ w.r.t. iterations ℓ



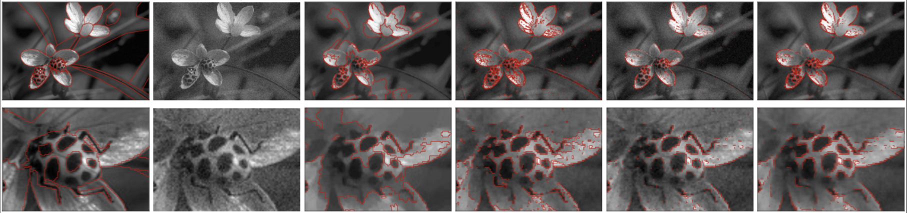
Experiments 3

[Le, Foare, Pustelnik, 2022]



Experiments 4

Ground Truth	Data	T-ROF	BZ	DMS-ℓ_1	Proposed
PSNR / Jaccard	26.0 dB / ND	34.3 dB / 0.068	33.1 dB / 0.084	30.3 dB / 0.081	34.2 dB / **0.101**



Alternative formulation

$$(\widehat{\mathbf{x}}, \widehat{\mathbf{e}}) \in \underset{\substack{\mathbf{x}\in\mathbb{R}^{|\Omega|},\\ \mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}}{\operatorname{Argmin}} \mathcal{L}(\mathrm{A}\mathbf{x}, \mathbf{z}) + \frac{\zeta^2}{2} \sum_{\ell=1}^{|\mathbb{E}|} \phi(\mathrm{D}_\ell \mathbf{x})(1-e_\ell)^2 + \lambda \sum_\ell \psi(e_\ell).$$

$$\widehat{\mathbf{x}} \in \underset{\mathbf{x}\in\mathbb{R}^{|\Omega|}}{\operatorname{Argmin}} \mathcal{L}(\mathrm{A}\mathbf{x}, \mathbf{z}) + \lambda^2 \sum_{\ell=1}^{|E|} \widetilde{\phi}_{\lambda/\zeta^2}\left(\phi(\mathrm{D}_\ell \mathbf{x})\right),$$

and $\widehat{\mathbf{e}} = (\widehat{e}_\ell)_{\ell\in E}$ with $\widehat{e}_\ell = \begin{cases} \operatorname{prox}_{\frac{\lambda}{\zeta^2 \phi(\mathrm{D}_\ell \widehat{\mathbf{x}})}\psi}(1) & \text{if} \quad \phi(\mathrm{D}_\ell \widehat{\mathbf{x}}) > 0, \\ 0 & \text{otherwise,} \end{cases}$

where, for every $\eta > 0$,

$$\widetilde{\phi}_{\lambda/\zeta^2}(\eta) = \frac{\eta}{2}(1 - \operatorname{prox}_{\frac{\lambda}{\zeta^2 \eta}\psi}(1))^2 + \frac{\lambda}{\zeta^2}\psi(\operatorname{prox}_{\frac{\lambda}{\zeta^2 \eta}\psi}(1))$$

and 0 otherwise.

[Pustelnik, 2023]

Alternative formulation

$$(\widehat{\mathbf{x}}, \widehat{\mathbf{e}}) \in \underset{\substack{\mathbf{x}\in\mathbb{R}^{|\Omega|},\\ \mathbf{e}\in\mathbb{R}^{|\mathbb{E}|}}}{\operatorname{Argmin}}\ \mathcal{L}(\mathrm{A}\mathbf{x}, \mathbf{z}) + \frac{\zeta^2}{2}\sum_{\ell=1}^{|\mathbb{E}|}\phi(\mathrm{D}_\ell \mathbf{x})(1-e_\ell)^2 + \lambda\sum_\ell \psi(e_\ell).$$

Reformulation when $\psi = |\cdot|$ and $\phi = |\cdot|^2$,

$$\begin{cases} \widehat{\boldsymbol{u}} \in \underset{\boldsymbol{u}\in\mathbb{R}^{|\Omega|}}{\operatorname{Argmin}}\ \mathcal{L}(\mathrm{A}\mathbf{x}, \mathbf{z}) + \frac{\zeta^2}{2}\sum_{\ell=1}^{|E|}\widetilde{\phi}_{\lambda/\zeta^2}\big((\mathrm{D}_\ell \mathbf{x})^2\big), \\ (\forall \ell) \quad \widehat{e}_\ell = \begin{cases} \operatorname{prox}_{\frac{\lambda}{\zeta^2(\mathrm{D}_\ell\widehat{\boldsymbol{u}})^2}|\cdot|}(1) & \text{if} \quad (\mathrm{D}_\ell\widehat{\mathbf{x}})^2 > 0, \\ 0 & \text{otherwise,} \end{cases} \end{cases}$$

where

$$(\forall \eta \geqslant 0) \quad \widetilde{\phi}_{\lambda/\zeta^2}(\eta) = \begin{cases} \frac{\lambda}{\zeta^2}\left(2 - \frac{\lambda}{\zeta^2\eta}\right) & \text{if} \quad \eta > \frac{\lambda}{\zeta^2}, \\ \eta & \text{if} \quad 0 < \eta \leqslant \frac{\lambda}{\zeta^2}, \\ 0 & \text{if} \quad \eta = 0. \end{cases}$$

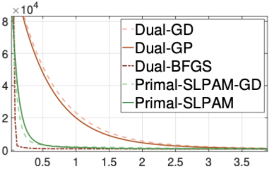
[Pustelnik, 2023]

Experiments

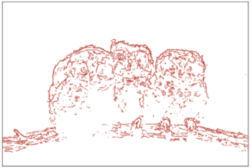
Original image $\overline{\mathbf{x}}$

Degraded image $\mathbf{z} = \overline{\mathbf{x}} + \epsilon$

Restored image $\widehat{\mathbf{x}}$



Primal objective function w.r.t time



Estimated contours $\widehat{\mathbf{e}}$

Nonsmooth nonconvex optimization: nonlinear operator

$$\min_{\mathbf{x}} \Psi(\mathbf{x}) := f(\mathrm{A}\mathbf{x}) + g(\mathbf{x})$$

- **If f is smooth,** forward-backward requires $\mathrm{A}^* \circ \nabla f \circ \mathrm{A}$.
- **If f is nonsmooth,** require the computation of $\mathrm{prox}_{f(\mathrm{A}\cdot)}$.
  - ☛ **Few closed form.**
- **Reformulation in the dual**: $\min_{\mathbf{w} \in \mathcal{G}} f^*(\mathbf{w}) + g^*(-\mathrm{A}^*\mathbf{w})$,

Conjugate: definition

Let $\mathcal{H}$ be a Hilbert space and $f\colon \mathcal{H} \to \,]-\infty, +\infty]$.
The **conjugate** of f is the function $f^*\colon \mathcal{H} \to [-\infty, +\infty]$ such that

$$(\forall \mathbf{u} \in \mathcal{H}) \qquad f^*(\mathbf{u}) = \sup_{\mathbf{x} \in \mathcal{H}} \big(\langle \mathbf{x} \mid \mathbf{u} \rangle - f(\mathbf{x}) \big).$$

Conjugate: definition

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Examples :

- $f = \frac{1}{2}\| \cdot \|^2 \Rightarrow f^* = \frac{1}{2}\| \cdot \|^2$

  Proof : For every $(\mathbf{x}, \mathbf{u}) \in \mathcal{H}^2$,
  $\langle \mathbf{x} \mid \mathbf{u} \rangle - \frac{1}{2}\|\mathbf{u}\|^2 = \frac{1}{2}\|\mathbf{u}\|^2 - \frac{1}{2}\|\mathbf{u} - \mathbf{x}\|^2$ is maximum at $\mathbf{x} = \mathbf{u}$.
  Consequently, $f^*(\mathbf{u}) = \frac{1}{2}\|\mathbf{u}\|^2$.

Conjugate: definition

Let $\mathcal{H}$ be a Hilbert space and $f\colon \mathcal{H} \to]-\infty, +\infty]$.
The **conjugate** of f is the function $f^*\colon \mathcal{H} \to [-\infty, +\infty]$ such that

$$(\forall \mathbf{u} \in \mathcal{H}) \qquad f^*(\mathbf{u}) = \sup_{\mathbf{x} \in \mathcal{H}} \big(\langle \mathbf{x} \mid \mathbf{u} \rangle - f(\mathbf{x}) \big).$$

Examples :

- $f = \frac{1}{2}\|\cdot\|^2 \Rightarrow f^* = \frac{1}{2}\|\cdot\|^2$.
- $(\forall \mathbf{x} \in \mathbb{R}^N)\ f(\mathbf{x}) = \frac{1}{q}\|\mathbf{x}\|_q^q$ with $q \in]1, +\infty[$
  $\Rightarrow (\forall \mathbf{u} \in \mathbb{R}^N)\ f^*(\mathbf{u}) = \frac{1}{q^*}\|\mathbf{u}\|_{q^*}^{q^*}$ **with** $\frac{1}{q} + \frac{1}{q^*} = 1$

Conjugate: definition

Let $\mathcal{H}$ be a Hilbert space and $f\colon \mathcal{H} \to \,]-\infty,+\infty]$.
The **conjugate** of f is the function $f^*\colon \mathcal{H} \to [-\infty,+\infty]$ such that

$$(\forall \mathbf{u} \in \mathcal{H}) \qquad f^*(\mathbf{u}) = \sup_{\mathbf{x}\in\mathcal{H}} \big(\langle \mathbf{x} \mid \mathbf{u} \rangle - f(\mathbf{x}) \big).$$

Moreau-Fenchel theorem

Let $\mathcal{H}$ be a Hilbert space and $f\colon \mathcal{H} \to \,]-\infty,+\infty]$ be a proper function.

$$f \text{ is l.s.c. and convex} \Leftrightarrow f^{**} = f.$$

$$\min_{\mathbf{x}} \Psi(\mathbf{x}) := f(\mathrm{A}\mathbf{x}) + g(\mathbf{x})$$

Assumptions: For $f \in \Gamma_0(\mathcal{G})$ and $g \in \Gamma_0(\mathcal{H})$.

$$\begin{aligned}\min_{\mathbf{x}} \Psi(\mathbf{x}) := f(\mathrm{A}\mathbf{x}) + g(\mathbf{x}) &= \min_{\mathbf{x}} \sup_{\mathbf{u}} \big(\langle \mathrm{A}\mathbf{x} \mid \mathbf{u} \rangle - f^*(\mathbf{u}) \big) + g(\mathbf{x}) \\ &= \max_{\mathbf{u}} \inf_{\mathbf{x}} \big(\langle \mathbf{x} \mid \mathrm{A}^*\mathbf{u} \rangle - f^*(\mathbf{u}) \big) + g(\mathbf{x}) \\ &= \max_{\mathbf{u}} - f^*(\mathbf{u}) - g^*(-\mathrm{A}^*\mathbf{u})\end{aligned}$$

Comment extracted from [Chambolle, Pock, 2016]: *Under very mild conditions on f,g (such as f(0) ¡ â and g continuous at 0 (see e.g. [Ekeland, Témam,1999, (4.21)]; in finite dimensions it is sufficient to have a point x with both Kx in the relative interior of domf and x in the relative interior of dom g [Rockafellar 1997, Corollary 31.2.1)], one can swap the min and sup.*

Primal problem

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces. Let $\mathrm{A} \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.
Let $f \colon \mathcal{G} \to \left]-\infty, +\infty\right]$, $g \colon \mathcal{H} \to \left]-\infty, +\infty\right]$.
We want to

$$\underset{\mathbf{x} \in \mathcal{H}}{\text{minimise}} \;\; f(\mathrm{A}\mathbf{x}) + g(\mathbf{x}).$$

Dual problem

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces. Let $\mathrm{A} \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.
Let $f \colon \mathcal{G} \to \left]-\infty, +\infty\right]$, $g \colon \mathcal{H} \to \left]-\infty, +\infty\right]$.
We want to

$$\underset{\mathbf{v} \in \mathcal{G}}{\text{minimise}} \;\; f^*(\mathbf{v}) + g^*(-\mathrm{A}^*\mathbf{v}).$$

Fenchel-Rockafellar duality

Weak duality

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces. Let $\mathrm{A} \in \mathcal{B}(\mathcal{H}, \mathcal{G})$. Let f be a proper fonction from $\mathcal{G}$ to $]-\infty, +\infty]$, g be a proper function from $\mathcal{H}$ to $]-\infty, +\infty]$.

$$\mu = \inf_{\mathbf{x} \in \mathcal{H}} f(\mathrm{A}\mathbf{x}) + g(\mathbf{x}) \quad \text{and} \quad \mu^* = \inf_{\mathbf{v} \in \mathcal{G}} f^*(\mathbf{v}) + g^*(-\mathrm{A}^*\mathbf{v}).$$

We have $\mu \geqslant -\mu^*$. If $\mu \in \mathbb{R}$, $\mu + \mu^*$ is the **duality gap**.

Fenchel-Rockafellar duality

Weak duality

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces. Let $\mathrm{A} \in \mathcal{B}(\mathcal{H}, \mathcal{G})$. Let f be a proper fonction from $\mathcal{G}$ to $]-\infty, +\infty]$, g be a proper function from $\mathcal{H}$ to $]-\infty, +\infty]$.

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We have $\mu \geqslant -\mu^*$. If $\mu \in \mathbb{R}$, $\mu + \mu^*$ is the **duality gap**.

Proof: According to Fenchel-Young inequality, for every $\mathbf{x} \in \mathcal{H}$ and $\mathbf{v} \in \mathcal{G}$,

$$f(\mathrm{A}\mathbf{x}) + g(\mathbf{x}) + f^*(\mathbf{v}) + g^*(-\mathrm{A}^*\mathbf{v}) \geqslant \langle \mathrm{A}\mathbf{x} \mid \mathbf{v} \rangle + \langle \mathbf{x} \mid -\mathrm{A}^*\mathbf{v} \rangle = 0.$$

Fenchel-Rockafellar duality

Strong duality

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces. Let $\mathrm{A} \in \mathcal{B}(\mathcal{H}, \mathcal{G})$
Let $f \in \Gamma_0(\mathcal{G})$ and $g \in \Gamma_0(\mathcal{H})$
If $\operatorname{int}(\operatorname{dom} f) \cap \mathrm{A}(\operatorname{dom} g) \neq \varnothing$ or $\operatorname{dom} f \cap \operatorname{int}\big(\mathrm{A}(\operatorname{dom} g)\big) \neq \varnothing$, then

$$\mu = \inf_{\mathbf{x} \in \mathcal{H}} f(\mathrm{A}\mathbf{x}) + g(\mathbf{x}) = -\min_{\mathbf{v} \in \mathcal{G}} f^*(\mathbf{v}) + g^*(-\mathrm{A}^*\mathbf{v}) = -\mu^*.$$

Fenchel-Rockafellar duality

Duality theorem

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces. Let $\mathrm{A} \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.
Let $f \in \Gamma_0(\mathcal{G})$, $g \in \Gamma_0(\mathcal{H})$.

- If there exists $\widehat{\mathbf{x}} \in \mathcal{H}$ such that $0 \in \mathrm{A}^* \partial f(\mathrm{A}\widehat{\mathbf{x}}) + \partial g(\widehat{\mathbf{x}})$, then $\widehat{\mathbf{x}}$ is a solution to the primal problem. Moreover, there exists a solution $\widehat{\mathbf{v}}$ to the dual problem such that $-\mathrm{A}^*\widehat{\mathbf{v}} \in \partial g(\widehat{\mathbf{x}})$ and $\mathrm{A}\widehat{\mathbf{x}} \in \partial f^*(\widehat{\mathbf{v}})$.
- If there exists $(\widehat{\mathbf{x}}, \widehat{\mathbf{v}}) \in \mathcal{H} \times \mathcal{G}$ such that $-\mathrm{A}^*\widehat{\mathbf{v}} \in \partial g(\widehat{\mathbf{x}})$ and $\mathrm{A}\widehat{\mathbf{x}} \in \partial f^*(\widehat{\mathbf{v}})$ then $\widehat{\mathbf{x}}$ (resp. $\widehat{\mathbf{v}}$) is a solution to the primal (resp. dual) problem.

If $(\widehat{\mathbf{x}}, \widehat{\mathbf{v}}) \in \mathcal{H} \times \mathcal{G}$ is such that $-\mathrm{A}^*\widehat{\mathbf{v}} \in \partial g(\widehat{\mathbf{x}})$ and $\mathrm{A}\widehat{\mathbf{x}} \in \partial f^*(\widehat{\mathbf{v}})$, then $(\widehat{\mathbf{x}}, \widehat{\mathbf{v}})$ is called a **Kuhn-Tucker point**.

$$\min_{\mathbf{x}} \Psi(\mathbf{x}) := f(\mathrm{A}\mathbf{x}) + g(\mathbf{x})$$

- **If f is smooth,** forward-backward requires $\mathrm{A}^* \circ \nabla f \circ \mathrm{A}$.
- **If f is nonsmooth,** require the computation of $\mathrm{prox}_{f(\mathrm{A}\cdot)}$.

  ☛ **Few closed form.**
- **Reformulation in the dual**: $\min_{\mathbf{v}\in\mathcal{G}} f^*(\mathbf{v}) + g^*(-\mathrm{A}^*\mathbf{v})$,

$$\min_{\mathbf{x}} \Psi(\mathbf{x}) := f(\mathrm{A}\mathbf{x}) + g(\mathbf{x})$$

- **Primal-dual algorithms** [Chambolle-Pock,2011]

Hyperparameters setting: $\tau > 0$, $\gamma > 0$, such that $\tau\gamma\|\mathrm{A}\|^2 < 1$

For $k = 0, 1, \ldots$

$$\left\lfloor \begin{array}{l} \mathbf{x}^{[k+1]} = \mathrm{prox}_{\tau g}\big(\mathbf{x}^{[k]} - \tau \mathrm{A}^* \mathbf{v}^{[k]}\big) \\ \mathbf{v}^{[k+1]} = \mathrm{prox}_{\gamma f^*}\big(\mathbf{v}^{[k]} + \gamma \mathrm{A}(2\mathbf{x}^{[k+1]} - \mathbf{x}^{[k]})\big) \end{array} \right.$$

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces. Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.
Let $f \in \Gamma_0(\mathcal{G})$, $g \in \Gamma_0(\mathcal{H})$.

$$\min_{\mathbf{x}} \max_{\mathbf{u}} \; g(\mathbf{x}) + \langle \mathrm{A}\mathbf{x} \mid \mathbf{u} \rangle - f^*(\mathbf{u})$$

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces. Let $\mathcal{A} \in C^1(\mathcal{H}, \mathcal{G})$.
Let $f \in \Gamma_0(\mathcal{G})$, $g \in \Gamma_0(\mathcal{H})$.

$$\min_{\mathbf{x}} \max_{\mathbf{u}} \; g(\mathbf{x}) + \langle \mathcal{A}(\mathbf{x}) \mid \mathbf{u} \rangle - f^*(\mathbf{u})$$

☛ **1st-order optimality conditions for $(\widehat{\mathbf{x}}, \widehat{\mathbf{v}})$ [Valkonen,2014]**

$$\begin{cases} -\nabla \mathcal{A}(\widehat{\mathbf{x}})^* \widehat{\mathbf{v}} \in \partial g(\widehat{\mathbf{x}}) \\ \mathcal{A}(\widehat{\mathbf{x}}) \in \partial f^*(\widehat{\mathbf{v}}) \end{cases}$$

Let $\mathcal{H}$ and $\mathcal{G}$ be two real Hilbert spaces. Let $\mathcal{A} \in C^1(\mathcal{H}, \mathcal{G})$. Let $f \in \Gamma_0(\mathcal{G})$, $g \in \Gamma_0(\mathcal{H})$.

$$\min_{\mathbf{x}} \max_{\mathbf{u}} \; g(\mathbf{x}) + \langle \mathcal{A}(\mathbf{x}) \mid \mathbf{u} \rangle - f^*(\mathbf{u})$$

☛ **Primal-dual algo. for nonlinear operators** [Valkonen, 2014]

Hyperparameters setting: $\tau > 0$, $\gamma > 0$ + conditions to satisfy.

For $k = 0, 1, \ldots$

$$\mathbf{x}^{[k+1]} = \mathrm{prox}_{\tau g}\big(\mathbf{x}^{[k]} - \tau \nabla \mathcal{A}(\mathbf{x}^{[k]})^* \mathbf{v}^{[k]}\big)$$

$$\mathbf{v}^{[k+1]} = \mathrm{prox}_{\gamma f^*}\big(\mathbf{v}^{[k]} + \gamma(\mathcal{A}(\mathbf{x}^{[k]}) + \nabla \mathcal{A}(\mathbf{x}^{[k]})^*(2\mathbf{x}^{[k+1]} - \mathbf{x}^{[k]})\big)$$

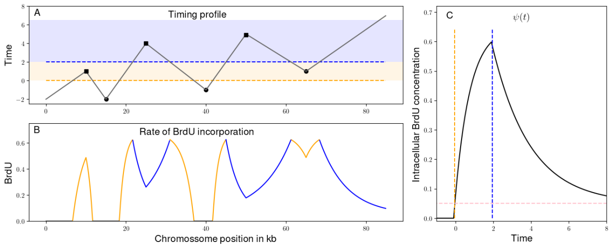
Identifying piecewise affine signal from nonlinear observation

- $\boldsymbol{\tau}$: Timing profile.
- $\mathbf{z}$: Signal provided by FORK-seq.
- $\mathcal{A}$ the coordinatewise composition of nonlinear function φ that measures the concentration of BrdU in time.

$$\mathbf{z} \approx \mathcal{A}(\boldsymbol{\tau})$$

with $(\forall \boldsymbol{\tau} = (\tau_1, ..., \tau_n) \in \mathbb{R}^n) \quad \mathcal{A}(\boldsymbol{\tau}) = (\alpha(\tau_1), ..., \alpha(\tau_n))$.

Assumptions: $\boldsymbol{\tau}$ is in the set of piecewise linear vectors with a maximum of $C > 0$ breakpoints. [Lage, C. et al. 2024]



Identifying piecewise affine signal from nonlinear observation

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$$\mathbf{z} \approx \mathcal{A}(\mathbf{x})$$

with $(\forall \mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n) \quad \mathcal{A}(\mathbf{x}) = (\alpha(x_1), ..., \alpha(x_n))$.

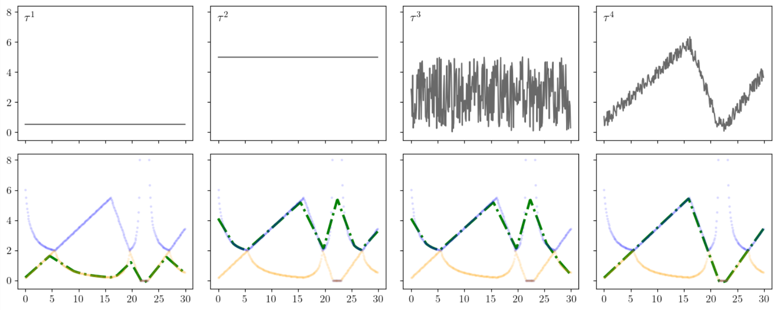
Assumptions: $\mathbf{x}$ is in the set of piecewise linear vectors with a maximum of $C > 0$ breakpoints.

$$\widehat{\mathbf{x}} := \underset{\mathbf{x} \in \mathcal{P}_C}{\operatorname{argmin}} \|\mathbf{z} - \mathcal{A}(\mathbf{x})\|_2^2.$$

with

$$\mathcal{P}_C := \{\mathbf{x} \; : \; \|\mathrm{D}\mathbf{x}\|_0 \leqslant C\}$$

Identifying piecewise affine signal from nonlinear observation



Results of Valkonen's algorithm for different initial points.
(**Top**): different values of $\mathbf{x}^{[0]}$.
(**Bottom**): solutions of the Valkonen's algorithm in green. In orange we observe $\mathbf{z}^0 = \mathcal{A}_0^{-1}(\mathbf{z})$ and in blue $\mathbf{z}^1 = \mathcal{A}_1^{-1}(\mathbf{z})$.

Identifying piecewise affine signal from nonlinear observation

Alternative optimization problem

$$(\mathbf{x}^*, \mathbf{d}^*) := \underset{\{(\mathbf{x},\mathbf{d}) \in \mathcal{P}_C \times \{0,1\}^n\}}{\operatorname{argmin}} \|\mathbf{x} - \mathcal{A}_{\mathbf{d}}^{-1}(\mathbf{z})\|_{w_{\mathbf{d}}}^2,$$

with

$$\mathcal{A}_{\mathbf{d}}^{-1}(\mathbf{z}) := (\varphi_{d_1}^{-1}(\mathrm{z}_1), ..., \varphi_{d_n}^{-1}(\mathrm{z}_n)) \in \mathbb{R}^n$$

and

$$w_{\mathbf{d}} := (1 - \mathbf{d}) \odot w_0 + \mathbf{d} \odot w_1, \text{ for all } i \in \{1, ..., n\},$$

and

$$w_{0,i} := \begin{cases} \varphi^{'}(\varphi_0^{-1}(\mathrm{z}_i)) & \varphi_0^{-1}(\mathrm{z}_i) \neq \varnothing \\ 0 & \varphi_0^{-1}(\mathrm{z}_i) = \varnothing, \end{cases}$$

$$w_{1,i} := \begin{cases} \psi^{'}(\varphi_1^{-1}(\mathrm{z}_i)) & \varphi_1^{-1}(\mathrm{z}_i) \neq \varnothing \\ 0 & \varphi_1^{-1}(\mathrm{z}_i) = \varnothing. \end{cases}$$

Identifying piecewise affine signal from nonlinear observation

Alternative optimization problem

$$(\mathbf{x}^*, \mathbf{d}^*) := \underset{\{(\mathbf{x},\mathbf{d})\in\mathcal{P}_C\times\{0,1\}^n\}}{\operatorname{argmin}} \|\mathbf{x} - \mathcal{A}_{\mathbf{d}}^{-1}(\mathbf{z})\|_{w_{\mathbf{d}}}^2,$$

Algorithm

Initialization: $\mathcal{D}$, w_0, w_1, and $\mathcal{D}_{\text{past}} = \emptyset$

For $\boldsymbol{d} \in \mathcal{D}$

Step 0: $\mathcal{D}_{\text{past}} \leftarrow \mathcal{D}_{\text{past}} \cup \{\boldsymbol{d}\}$.

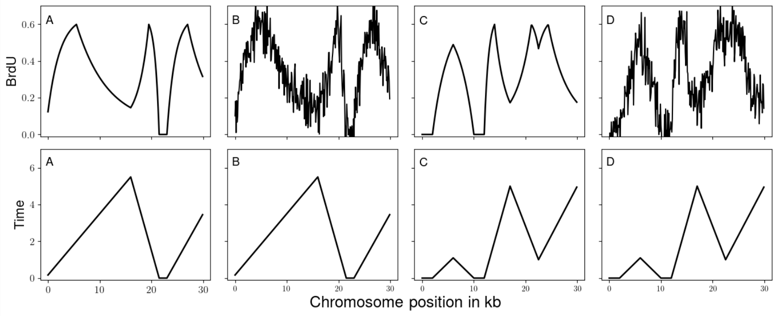
Step 1: Solve the optimization problem

$$\mathbf{x}_{\mathbf{d}}^* = \underset{\mathbf{x}}{\operatorname{argmin}} \;\; \frac{1}{2}\|\mathbf{d} \odot (\mathbf{x} - \mathbf{z}_1)\|_{\mathbf{w}_1}^2 + \frac{1}{2}\|(1 - \mathbf{d}) \odot (\mathbf{x} - \mathbf{z}_0)\|_{\mathbf{w}_0}^2 + \lambda\|\mathrm{D}\mathbf{x}\|_1.$$

Step 2: Extract the optimal $\mathbf{x}_{\mathbf{d}}^*$ for $\boldsymbol{d} \in \mathcal{D}_{\text{past}}$:

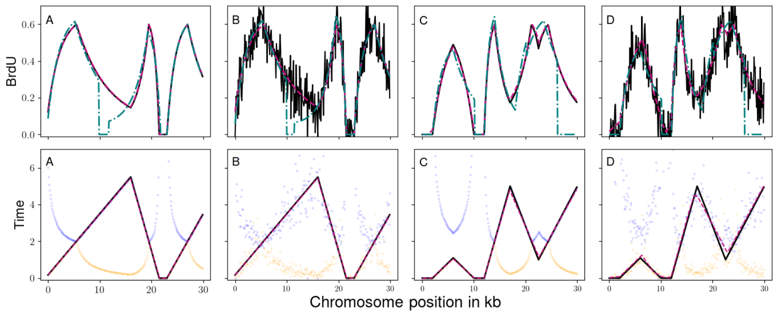
$$(\mathbf{x}^*, \mathbf{d}^*) := \underset{\{\mathbf{x}_{\mathbf{d}}^*,\ \mathbf{d}\in\mathcal{D}_{\text{past}}\}}{\operatorname{argmin}} F(\mathbf{x}_{\mathbf{d}}^*) := \frac{1}{2}\|\mathbf{d}\odot(\mathbf{x}_{\mathbf{d}}^*-\mathbf{z}_1)\|^2+\frac{1}{2}\|(1-\mathbf{d})\odot(\mathbf{x}_{\mathbf{d}}^*-\mathbf{z}_0)\|^2.$$

Identifying piecewise affine signal from nonlinear observation



(**Top**) Simulated signals A,B,C,D (black). Approximations of each signal computed by `DNA-inverse` (magenta) and Matching Pursuit (green).
(**Bottom**) Timing profile obtained via the `DNA-inverse` (magenta) compared to ground truth (black). In orange $\mathcal{A}_0^{-1}(\mathbf{z})$, and in blue $\mathcal{A}_1^{-1}(\mathbf{z})$ for each simulated signal .

Identifying piecewise affine signal from nonlinear observation



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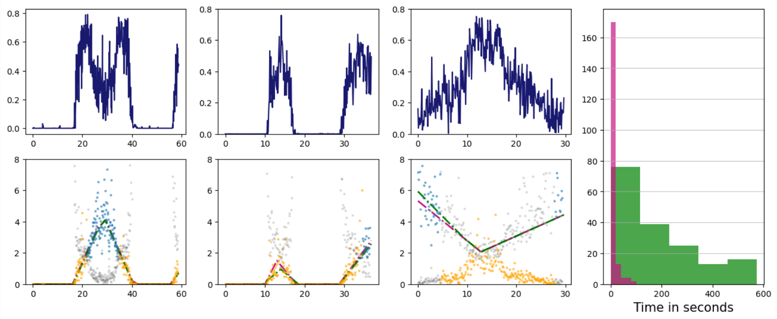
Identifying piecewise affine signal from nonlinear observation



(**Top**) Different reads in blue.

(**Bottom**) Solution $\mathbf{x}^*_{\texttt{DNA-inverse}}$ (magenta) and $\mathbf{x}^*_{\texttt{Adapted Valkonen}}$ (green) for the different reads.

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Conclusions on nonconvex optimization

- ☛ Extension of subdifferential and proximal operator for nonconvex case.
- ☛ Allows for convergence to critical points.
- ☛ Sensitivity to initialization.
- ☛ Main challenge: provide good initialization.
- ☛ Linear convergence rate in the non-convex setting ? Performance diagram ?.

Toward deep learning

Context

Data: $\mathbf{z}$

Direct/acquisition model

Design a cost function

Minimization algorithm

Hyperparameter tuning

Estimated parameters: $\mathbf{x}(\mathbf{z}; \widehat{\lambda})$

Data: $\mathbf{z}$

$$\mathbf{z} = \mathcal{D}(\mathrm{A}\overline{\mathbf{x}})$$

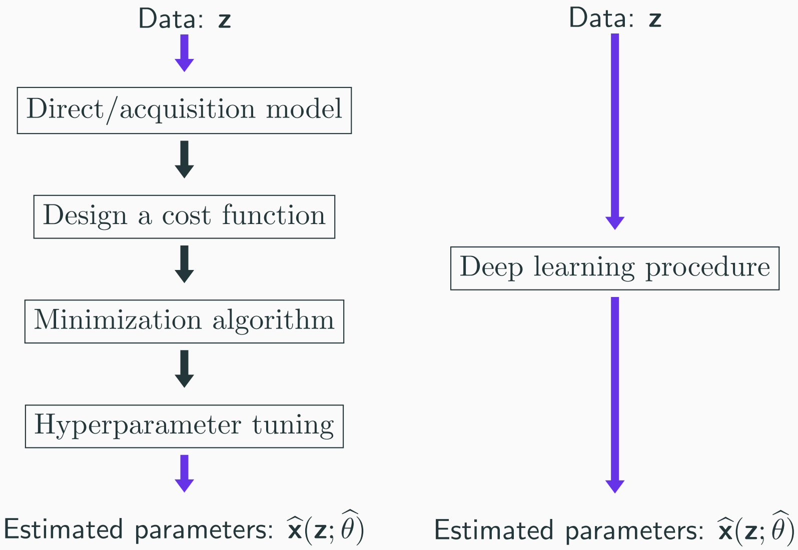
$$\widehat{\mathbf{x}}(\mathbf{z}; \lambda) \in \underset{\mathbf{x}}{\mathrm{Argmin}}\ f(\mathbf{x}; \mathbf{z}) + \lambda g(\mathbf{x})$$

Sequence $\mathbf{x}^{[k+1]} = \boldsymbol{\Phi}\mathbf{x}^{[k]}$

$$\widehat{\lambda} \in \underset{\lambda}{\mathrm{Argmin}} \|\overline{\mathbf{x}} - \widehat{\mathbf{x}}(\mathbf{z}; \lambda)\|_2^2$$

Estimated parameters: $\widehat{\mathbf{x}}(\mathbf{z}; \widehat{\lambda})$

Standard learning and deep learning

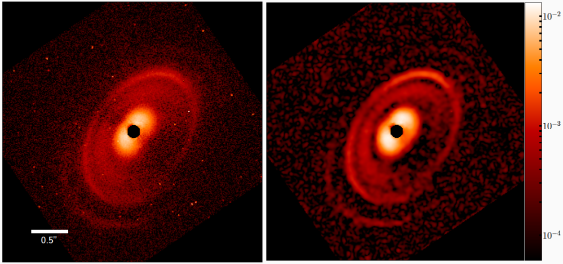


Context: Image restoration

➡ **Data**: $\mathbf{z} \in \mathbb{R}^M$ degraded version of an original image $\overline{\mathbf{x}} \in \mathbb{R}^N$:

$$\mathbf{z} = \mathrm{A}\overline{\mathbf{x}} + \mathbf{w}$$

- $\mathrm{A} \in \mathbb{R}^{M \times N}$: linear degradation (e.g. a blur)
- $\mathbf{w}$: noise (e.g. Gaussian noise)



SPHERE-IRDIS

Training a prediction function for a restoration task

☛ **Database**: $\mathcal{S} = \big\{(\mathbf{z}_i, \overline{\mathbf{x}}_i) \in \mathbb{R}^M \times \mathbb{R}^N \;\big|\; i \in \{1, \ldots, \mathbb{L}\}\big\}$

Training a prediction function for a restoration task

☛ **Database**: $\mathcal{S} = \left\{(\mathbf{z}_i, \bar{\mathbf{x}}_i) \in \mathbb{R}^M \times \mathbb{R}^N \;\middle|\; i \in \{1, \ldots, \mathbb{L}\}\right\}$
We consider two sets of images: the *training set* $(\mathbf{z}_i, \bar{\mathbf{x}}_i)_{i \in \mathbb{I}}$ of size $\sharp\,\mathbb{I}$ and the *testing set* $(\mathbf{z}_j, \bar{\mathbf{x}}_j)_{j \in \mathbb{J}}$ of size $\sharp\,\mathbb{J}$ where

$$(\forall i \in \mathbb{I} \cup \mathbb{J}) \quad \mathbf{z}_i = \mathrm{A}\bar{\mathbf{x}}_i + \mathbf{w}_i$$

Training a prediction function for a restoration task

- **Database**: $\mathcal{S} = \big\{(\mathbf{z}_i, \overline{\mathbf{x}}_i) \in \mathbb{R}^M \times \mathbb{R}^N \;\big|\; i \in \{1, \ldots, \mathbb{L}\}\big\}$
  We consider two sets of images: the *training set* $(\mathbf{z}_i, \overline{\mathbf{x}}_i)_{i\in\mathbb{I}}$ of size $\sharp\,\mathbb{I}$ and the *testing set* $(\mathbf{z}_j, \overline{\mathbf{x}}_j)_{j\in\mathbb{J}}$ of size $\sharp\,\mathbb{J}$ where
  $$(\forall i \in \mathbb{I} \cup \mathbb{J}) \quad \mathbf{z}_i = \mathrm{A}\overline{\mathbf{x}}_i + \mathbf{w}_i$$

- **Training**: A prediction function $f_{\boldsymbol{\Theta}}$ is learned using the *training set*:
  $$\widehat{\boldsymbol{\Theta}} \in \underset{\boldsymbol{\Theta}}{\mathrm{Argmin}}\, \frac{1}{\sharp\,\mathbb{I}} \textstyle\sum_{i\in\mathbb{I}} \|\overline{\mathbf{x}}_i - f_{\boldsymbol{\Theta}}(\mathbf{z}_i)\|^2$$

- **Testing**: The learned $f_{\widehat{\boldsymbol{\Theta}}}$ is then validated on the testing set.
  A properly trained network should satisfy
  $$(\forall j \in \mathbb{J}) \quad \overline{\mathbf{x}}_j \approx f_{\widehat{\boldsymbol{\Theta}}}(\mathbf{z}_j).$$

Variational approach versus Deep learning architecture

Variational approach $f_{\boldsymbol{\Theta}}(\mathbf{z}_i) = \underset{\mathbf{x}\in\mathcal{H}}{\text{Argmin}} \left\{ \mathsf{F}(\mathbf{x}) = f(\mathrm{A}\mathbf{x}, \mathbf{z}_i) + g(\mathrm{D}\mathbf{x}) + \iota_C(\mathbf{x}) \right\}$

- ⊙ Data fidelity term: $f(\mathrm{A}\cdot, \mathbf{z}_i)$
- ⊙ Prior: $g(\mathrm{D}\cdot)$
- ⊙ Constraint set : C

$\Rightarrow$ $\boldsymbol{\Theta} = \{\mathrm{D}, K = \#\text{iter}\}$

Deep learning $f_{\boldsymbol{\Theta}}(\mathbf{z}_i) = \eta^{[K]}\big(W^{[K]} \ldots \eta^{[1]}(W^{[1]}\mathbf{z}_i + b^{[1]}) \ldots + b^{[K]}\big)$

- ⊙ Linear operators: $W^{[1]}, \mathrm{W}^{[2]}, \ldots, W^{[K]}$
- ⊙ Activation functions: $\eta^{[1]}, \eta^{[2]}, \ldots, \eta^{[K]}$
- ⊙ Biais vectors: $b^{[1]}, b^{[2]}, \ldots, b^{[K]}$

$\Rightarrow$ $\boldsymbol{\Theta} = \{W^{[1]}, \ldots, W^{[K]}, b^{[1]}, \ldots, b^{[K]}\}$

Mixing variational approach and Deep learning architecture

➡ **Synthesis formulation:**

$$\boxed{\min_{\mathbf{x}} \frac{1}{2}\|\mathrm{AD}^*\mathbf{x} - \mathbf{z}\|_2^2 + \lambda\|\mathbf{x}\|_1} \text{ where } \mathrm{H} = \mathrm{AD}^* \in \mathbb{R}^{\overline{N}\times N}$$

➡ **Forward-backward iterations:**

$$\mathbf{x}^{[k+1]} = \mathrm{prox}_{\tau\lambda\|\cdot\|_1}(\mathbf{x}^{[k]} - \tau\mathrm{H}^*(\mathrm{H}\mathbf{x}^{[k]} - \mathbf{z}))$$

➡ **Reformulation:**

$$\mathbf{x}^{[k+1]} = \mathrm{prox}_{\tau\lambda\|\cdot\|_1}((\mathbf{I} - \tau\mathrm{H}^*\mathrm{H})\mathbf{x}^{[k]} + \tau\mathrm{H}^*\mathbf{z}))$$

➡ **Layer network:**

$$\mathbf{x}^{[k+1]} = \underbrace{\boxed{\mathrm{prox}_{\tau\lambda\|\cdot\|_1}}}_{\eta^{[k]}}\Big(\underbrace{\boxed{\mathrm{Id} - \tau\mathrm{H}^*\mathrm{H}}}_{\mathrm{W}^{[k]}}\,\mathbf{x}^{[k]} + \underbrace{\boxed{\tau\mathrm{H}^*\mathrm{z}}}_{\mathrm{b}^{[k]}}\Big)$$

[Gregor, LeCun, 2010]

Unfolded schemes: Study case on denoising

D(i)FB algorithm

OBJECTIVE: $\widehat{\mathbf{x}} = \underset{\mathbf{x}\in\mathcal{H}}{\operatorname{argmin}}\left\{ \mathsf{F}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{z}\|_2^2 + g(\mathrm{D}\mathbf{x}) + \iota_C(\mathbf{x}) \right\}$

- $C \subset \mathcal{H}$ is a closed, convex, non-empty.
- $\mathrm{D}\colon \mathcal{H} \to \mathcal{G}$ and g $\in \Gamma_0(\mathcal{G})$

D(i)FB algorithm

OBJECTIVE: $\widehat{\mathbf{x}} = \underset{\mathbf{x}\in\mathcal{H}}{\operatorname{argmin}}\left\{\mathsf{F}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{z}\|_2^2 + g(\mathrm{D}\mathbf{x}) + \iota_C(\mathbf{x})\right\}$

- $C \subset \mathcal{H}$ is a closed, convex, non-empty.
- $\mathrm{D}\colon \mathcal{H} \to \mathcal{G}$ and g $\in \Gamma_0(\mathcal{G})$

ALGORITHM: Let $\mathbf{v}^{[0]} \in \mathcal{G}$,

For $k = 0, 1, \ldots$

$$
\left\lfloor
\begin{aligned}
\mathbf{u}^{[k+1]} &= \mathrm{prox}_{\tau_k(\nu g)^*}\left(\mathbf{v}^{[k]} + \tau_k \mathrm{D}\mathrm{P}_C(\mathbf{z} - \mathrm{D}^\top \mathbf{v}^{[k]})\right)\\
\mathbf{v}^{[k+1]} &= (1+\rho_k)\mathbf{u}^{[k+1]} - \rho_k \mathbf{u}^{[k]}
\end{aligned}
\right.
$$

D(i)FB algorithm

OBJECTIVE: $\widehat{\mathbf{x}} = \underset{\mathbf{x}\in\mathcal{H}}{\operatorname{argmin}} \Big\{ \mathrm{F}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{z}\|_2^2 + g(\mathrm{D}\mathbf{x}) + \iota_C(\mathbf{x}) \Big\}$

- $C \subset \mathcal{H}$ is a closed, convex, non-empty.
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ALGORITHM: Let $\mathbf{v}^{[0]} \in \mathcal{G}$,

For $k = 0, 1, \ldots$

$$\left\lfloor \begin{array}{l} \mathbf{u}^{[k+1]} = \mathrm{prox}_{\tau_k(\nu g)^*}\Big(\mathbf{v}^{[k]} + \tau_k \mathrm{D}\mathrm{P}_C(\mathbf{z} - \mathrm{D}^\top \mathbf{v}^{[k]})\Big) \\ \mathbf{v}^{[k+1]} = (1+\rho_k)\mathbf{u}^{[k+1]} - \rho_k \mathbf{u}^{[k]} \end{array} \right.$$

THEOREM: Assume that one of the following conditions is satisfied.

- (DFB): $\forall k \in \mathbb{N}$, $\tau_k \in (0, 2/\|\mathrm{D}\|_S^2)$, and $\rho_k = 0$.
- (DIFB): $\forall k \in \mathbb{N}$, $\tau_k \in (0, 1/\|\mathrm{D}\|_S^2)$, $\rho_k = \frac{t_k - 1}{t_{k+1}}$ with $t_k = \frac{k+a-1}{a}$ and $a > 2$.

Then we have $\widehat{\mathbf{x}} = \lim_{k\to\infty} \mathrm{P}_C(\mathbf{z} - \mathrm{D}^\top \mathbf{u}^{[k]})$.

(Sc)CP algorithm

Objective: $\widehat{\mathbf{x}} = \underset{\mathbf{x}\in\mathcal{H}}{\operatorname{argmin}} \left\{ \mathsf{F}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{z}\|_2^2 + g(\mathrm{D}\mathbf{x}) + \iota_C(\mathbf{x}) \right\}$

- $C \subset \mathcal{H}$ is a closed, convex, non-empty.
- $\mathrm{D}\colon \mathcal{H} \to \mathcal{G}$ and g $\in \Gamma_0(\mathcal{G})$

(Sc)CP algorithm

OBJECTIVE: $\widehat{\mathbf{x}} = \underset{\mathbf{x}\in\mathcal{H}}{\operatorname{argmin}} \Big\{ \mathsf{F}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{z}\|_2^2 + g(\mathrm{D}\mathbf{x}) + \iota_C(\mathbf{x}) \Big\}$

- $C \subset \mathcal{H}$ is a closed, convex, non-empty.
- $\mathrm{D}\colon \mathcal{H} \to \mathcal{G}$ and g $\in \Gamma_0(\mathcal{G})$

ALGORITHM: Let $\mathbf{x}^{[0]} \in \mathcal{H}$ and $\mathbf{u}^{[0]} \in \mathcal{G}$.

For $k = 0, 1, \ldots$

$$\left\lfloor \begin{array}{l} \mathbf{x}^{[k+1]} = \mathrm{P}_C\left(\frac{\mu_k}{1+\mu_k}(\mathbf{z} - \mathrm{D}^\top \mathbf{u}^{[k]}) + \frac{1}{1+\mu_k}\mathbf{x}^{[k]}\right) \\ \mathbf{u}^{[k+1]} = \mathrm{prox}_{\tau_k(\nu g)^*}\left(\mathbf{u}^{[k]} + \tau_k \mathrm{D}\Big((1+\alpha_k)\mathbf{x}^{[k+1]} - \alpha_k \mathbf{x}^{[k]}\Big)\right) \end{array} \right.$$

(Sc)CP algorithm

OBJECTIVE: $\widehat{\mathbf{x}} = \underset{\mathbf{x}\in\mathcal{H}}{\operatorname{argmin}}\Big\{\mathsf{F}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{z}\|_2^2 + g(\mathrm{D}\mathbf{x}) + \iota_C(\mathbf{x})\Big\}$

- $C \subset \mathcal{H}$ is a closed, convex, non-empty.
- $\mathrm{D}\colon \mathcal{H} \to \mathcal{G}$ and g $\in \Gamma_0(\mathcal{G})$

ALGORITHM: Let $\mathbf{x}^{[0]} \in \mathcal{H}$ and $\mathbf{u}^{[0]} \in \mathcal{G}$.

For $k = 0, 1, \ldots$

$$\left\lfloor\begin{array}{l} \mathbf{x}^{[k+1]} = \mathrm{P}_C\left(\frac{\mu_k}{1+\mu_k}(\mathbf{z} - \mathrm{D}^\top \mathbf{u}^{[k]}) + \frac{1}{1+\mu_k}\mathbf{x}^{[k]}\right) \\ \mathbf{u}^{[k+1]} = \mathrm{prox}_{\tau_k(\nu g)^*}\left(\mathbf{u}^{[k]} + \tau_k \mathrm{D}\Big((1+\alpha_k)\mathbf{x}^{[k+1]} - \alpha_k \mathbf{x}^{[k]}\Big)\right) \end{array}\right.$$

THEOREM: Assume that one of the following conditions is satisfied.

- (CP): $\tau_k \mu_k \|\mathrm{D}\|_S^2 < 1$, and $\alpha_k = 1$.
- (ScCP): $\alpha_k = \sqrt{1+2\mu_k}^{-1}$, $\mu_{k+1} = \alpha_k \mu_k$, $\tau_{k+1} = \tau_k \alpha_k^{-1}$ with $\mu_0 \tau_0 \|\mathrm{D}\|_S^2 \leqslant 1$.

Then we have $\widehat{\mathbf{x}} = \lim_{k\to\infty} \mathbf{x}^{[k]}$.

S(c)CP to D(i)FB

OBJECTIVE: $\widehat{\mathbf{x}} = \underset{\mathbf{x}\in\mathcal{H}}{\operatorname{argmin}} \Big\{ \mathsf{F}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{z}\|_2^2 + g(\mathrm{D}\mathbf{x}) + \iota_C(\mathbf{x}) \Big\}$

ALGORITHM: For $k = 0, 1, \ldots$

$$\mathbf{x}^{[k+1]} = \mathrm{P}_C \left(\frac{\mu_k}{1+\mu_k}(\mathbf{z} - \mathrm{D}^\top \mathbf{u}^{[k]}) + \frac{1}{1+\mu_k}\mathbf{x}^{[k]} \right)$$

$$\mathbf{u}^{[k+1]} = \mathrm{prox}_{\tau_k(\nu g)^*} \left(\mathbf{u}^{[k]} + \tau_k \mathrm{D}\Big((1+\alpha_k)\mathbf{x}^{[k+1]} - \alpha_k \mathbf{x}_k \Big) \right)$$

☛ **S(c)CP:** Starting point.

S(c)CP to D(i)FB

OBJECTIVE: $\widehat{\mathbf{x}} = \underset{\mathbf{x}\in\mathcal{H}}{\operatorname{argmin}} \Big\{ \mathsf{F}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x} - \mathbf{z}\|_2^2 + g(\mathrm{D}\mathbf{x}) + \iota_C(\mathbf{x}) \Big\}$

ALGORITHM: For $k = 0, 1, \ldots$

$$\mathbf{x}^{[k+1]} = \mathrm{P}_C\left(\frac{\mu_k}{1+\mu_k}(\mathbf{z} - \mathrm{D}^\top \mathbf{u}^{[k]}) + \frac{1}{1+\mu_k}\mathbf{x}^{[k]}\right)$$

$$\mathbf{u}^{[k+1]} = \mathrm{prox}_{\tau_k(\nu g)^*}\left(\mathbf{u}^{[k]} + \tau_k \mathrm{D}\Big((1+\alpha_k)\mathbf{x}^{[k+1]} - \alpha_k \mathbf{x}_k\Big)\right)$$

- **S(c)CP:** Starting point.
- **Arrow-Hurwicz iterations:** $\alpha_k \equiv 0$.

S(c)CP to D(i)FB

OBJECTIVE: $\widehat{\mathbf{x}} = \underset{\mathbf{x}\in\mathcal{H}}{\text{argmin}} \left\{ \mathsf{F}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{z}\|_2^2 + g(\mathrm{D}\mathbf{x}) + \iota_C(\mathbf{x}) \right\}$

ALGORITHM: For $k = 0, 1, \ldots$

$$\mathbf{x}^{[k+1]} = \mathrm{P}_C\left(\tfrac{\mu_k}{1+\mu_k}(\mathbf{z} - \mathrm{D}^\top \mathbf{u}^{[k]}) + \tfrac{1}{1+\mu_k}\mathbf{x}^{[k]} \right)$$

$$\mathbf{u}^{[k+1]} = \mathrm{prox}_{\tau_k(\nu g)^*}\left(\mathbf{u}^{[k]} + \tau_k \mathrm{D}\mathbf{x}^{[k+1]} \right)$$

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S(c)CP to D(i)FB

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- **S(c)CP:** Starting point.
- **Arrow-Hurwicz iterations:** $\alpha_k \equiv 0$.
- **DFB:** $\mu_k \to +\infty$.

S(c)CP to D(i)FB

OBJECTIVE: $\widehat{\mathbf{x}} = \underset{\mathbf{x}\in\mathcal{H}}{\operatorname{argmin}} \left\{ \mathsf{F}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{z}\|_2^2 + g(\mathrm{D}\mathbf{x}) + \iota_C(\mathbf{x}) \right\}$

ALGORITHM: For $k = 0, 1, \ldots$

$$\mathbf{x}^{[k+1]} = \mathrm{P}_C\left(\mathbf{z} - \mathrm{D}^\top \mathbf{u}^{[k]}\right)$$
$$\mathbf{u}^{[k+1]} = \operatorname{prox}_{\tau_k(\nu g)^*}\left(\mathbf{u}^{[k]} + \tau_k \mathrm{D}\mathbf{x}^{[k+1]}\right)$$

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S(c)CP to D(i)FB

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ALGORITHM: For $k = 0, 1, \ldots$

$$\mathbf{x}^{[k+1]} = \mathrm{P}_C\left(\mathbf{z} - \mathrm{D}^\top \mathbf{u}^{[k]}\right)$$

$$\mathbf{u}^{[k+1]} = \mathrm{prox}_{\tau_k(\nu g)^*}\left(\mathbf{u}^{[k]} + \tau_k \mathrm{D}\mathbf{x}^{[k+1]}\right)$$

- **S(c)CP:** Starting point.
- **Arrow-Hurwicz iterations:** $\alpha_k \equiv 0$.
- **DFB:** $\mu_k \to +\infty$.
- **DiFB:** Inertia step on the dual variable.

S(c)CP to D(i)FB

OBJECTIVE: $\widehat{\mathbf{x}} = \underset{\mathbf{x}\in\mathcal{H}}{\text{argmin}} \left\{ \mathsf{F}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{z}\|_2^2 + g(\mathrm{D}\mathbf{x}) + \iota_C(\mathbf{x}) \right\}$

ALGORITHM: For $k = 0, 1, \ldots$

$$
\begin{aligned}
\mathbf{x}^{[k+1]} &= \mathrm{P}_C\left(\mathbf{z} - \mathrm{D}^\top \mathbf{v}^{[k]}\right)\\
\mathbf{u}^{[k+1]} &= \mathrm{prox}_{\tau_k(\nu g)^*}\left(\mathbf{u}^{[k]} + \tau_k \mathrm{D}\mathbf{x}^{[k+1]}\right)\\
\mathbf{v}^{[k+1]} &= (1+\rho_k)\mathbf{u}^{[k+1]} - \rho_k \mathbf{u}^{[k]}
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$$

- **S(c)CP:** Starting point.
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- **DFB:** $\mu_k \to +\infty$.
- **DiFB:** Inertia step on the dual variable.

Arrow-Hurwicz building block

ITERATION: Arrow-Hurwicz iteration can be written as:

$$\mathrm{L}_{\mathbf{z},\nu,\Theta_k} : \quad \mathcal{H} \times \mathcal{G} \to \mathcal{H}$$
$$(\mathbf{x}^{[k]}, \mathbf{u}^{[k]}) \mapsto \mathrm{L}_{\mathbf{z},\Theta_{k,\mathcal{P}},\mathcal{P}}(\mathbf{x}, \mathrm{L}_{\Theta_{k,\mathcal{D}},\mathcal{D}}(\mathbf{x}^{[k]}, \mathbf{u}^{[k]}))$$

with

$$\mathrm{L}_{\nu,\Theta_{k,\mathcal{D}},\mathcal{D}}(\mathbf{x}, \mathbf{u}) = \mathrm{prox}_{\tau_k(\nu g)^*}\left(\tau_k \mathrm{D}\mathbf{x} + \mathbf{u}\right),$$
$$\mathrm{L}_{\mathbf{z},\Theta_{k,\mathcal{P}},\mathcal{P}}(\mathbf{x}, \mathbf{u}) = \mathrm{P}_C\left(\frac{1}{1+\mu_k}\mathbf{x} - \frac{\mu_k}{1+\mu_k}\mathrm{D}^\top \mathbf{u} + \frac{\mu_k}{1+\mu_k}\mathbf{z}\right)$$

DEEP LEARNING NOTATION:

$$f_{\boldsymbol{\Theta}} = \eta^{[K]}\left(W^{[K]} \ldots \eta^{[1]}(W^{[1]} \cdot + b^{[1]}) \ldots + b^{[K]}\right)$$

Deep Arrow-Hurwicz building block

LAYER: Arrow-Hurwicz layer can be written as:

$$
\begin{aligned}
\mathrm{L}_{\mathbf{z},\nu,\Theta_k} : \ & \mathcal{H} \times \mathcal{G} \to \mathcal{H} \\
& (\mathbf{x}^{[k]}, ^{[k]}) \mapsto \mathrm{L}_{\mathbf{z},\Theta_{k,\mathcal{P}},\mathcal{P}}(\mathbf{x}^{[k]}, \mathrm{L}_{\Theta_{k,\mathcal{D}},\mathcal{D}}(\mathbf{x}^{[k]}, \mathbf{u}^{[k]})),
\end{aligned}
$$

with

$$
\mathrm{L}_{\nu,\Theta_{k,\mathcal{D}},\mathcal{D}}(\mathbf{x}, \mathbf{u}) = \mathrm{prox}_{\tau_k(\nu g)^*} \left(\tau_k \mathrm{D}_{k,\mathcal{D}} \mathbf{x} + \mathbf{u} \right),
$$

$$
\mathrm{L}_{\mathbf{z},\Theta_{k,\mathcal{P}},\mathcal{P}}(\mathbf{x}, \mathbf{u}) = \mathrm{P}_C \left(\frac{1}{1+\mu_k} \mathbf{x} - \frac{\mu_k}{1+\mu_k} \mathrm{D}_{k,\mathcal{P}}^\top \mathbf{u} + \frac{\mu_k}{1+\mu_k} \mathbf{z} \right),
$$

HardTanh versus proximity operator of ℓ_1

☛ **Most of activation functions are proximity operator** :
ReLU, Unimodal sigmoid, Softmax . . .
[Combettes, Pesquet 2020]

Proposition [Le, Pustelnik, Foare 2022]: The proximity operator of the conjugate of the ℓ_1-norm scaled by parameter $\lambda > 0$ fits the HardTanh activation function, i.e., for every $\mathbf{x} = (\mathbf{x}_i)_{1 \leqslant i \leqslant N}$:

$$\mathrm{prox}_{(\lambda\|\cdot\|_1)^*}(\mathbf{x}) = \mathrm{P}_{\|\cdot\|_\infty \leqslant \lambda}(\mathbf{x}) = \mathrm{HardTanh}_\lambda(\mathbf{x}) = (\mathrm{p}_i)_{1 \leqslant i \leqslant N}$$

where

$$\mathrm{p}_i = \begin{cases} -\lambda & \text{if} \quad \mathrm{p}_i < -\lambda, \\ \lambda & \text{if} \quad \mathrm{p}_i > \lambda, \\ \mathrm{p}_i & \text{otherwise.} \end{cases}$$

Deep Arrow-Hurwicz building block + skip connections

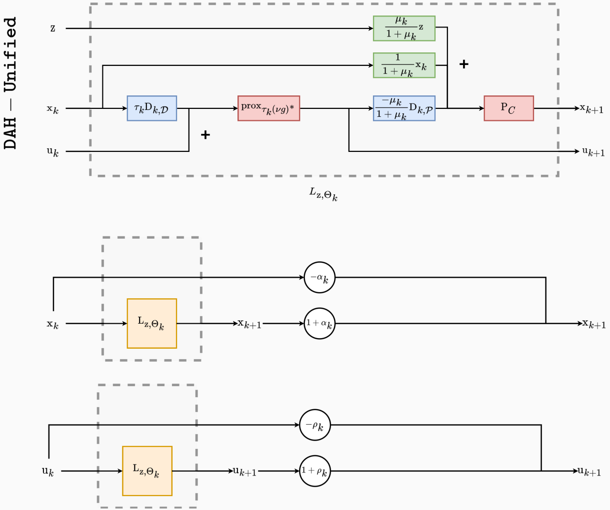

Figure 1: Architecture of the proposed DAH-Unified block for the k-th layer. Inertial step for ScCP (top) and DiFB (bottom).

Variations on the proposed architecture

	Θ_k	Comments
DDFB-LFO	$\mathrm{D}_{k,\mathcal{P}}$, $\mathrm{D}_{k,\mathcal{D}}$	absorb τ_k in $\mathrm{D}_{k,\mathcal{D}}$
DDFB-LNO	$\mathrm{D}_{k,\mathcal{P}} = \mathrm{D}_{k,\mathcal{D}}^\top$	define $\tau_k = 1.99\|\mathrm{D}_k\|^{-2}$

Variations on the proposed architecture

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DDFB-LFO	$\mathrm{D}_{k,\mathcal{P}}$, $\mathrm{D}_{k,\mathcal{D}}$	absorb τ_k in $\mathrm{D}_{k,\mathcal{D}}$
DDFB-LNO	$\mathrm{D}_{k,\mathcal{P}} = \mathrm{D}_{k,\mathcal{D}}^\top$	define $\tau_k = 1.99\|\mathrm{D}_k\|^{-2}$
DCP-LFO	$\mathrm{D}_{k,\mathcal{P}}$, $\mathrm{D}_{k,\mathcal{D}}$, μ	learn $\mu = \mu_0 = \cdots = \mu_K$, and absorb τ_k in $\mathrm{D}_{k,\mathcal{D}}$
DCP-LNO	$\mathrm{D}_{k,\mathcal{P}} = \mathrm{D}_{k,\mathcal{D}}^\top$, μ	learn $\mu = \mu_0 = \cdots = \mu_K$, and fix $\tau_k = 0.99\mu^{-1}\|\mathrm{D}_k\|^{-2}$

Variations on the proposed architecture

	Θ_k	Comments
DDFB-LFO	$\mathrm{D}_{k,\mathcal{P}}$, $\mathrm{D}_{k,\mathcal{D}}$	absorb τ_k in $\mathrm{D}_{k,\mathcal{D}}$
DDiFB-LFO	$\mathrm{D}_{k,\mathcal{P}}$, $\mathrm{D}_{k,\mathcal{D}}$, α_k	fix α_k, and absorb τ_k in $\mathrm{D}_{k,\mathcal{D}}$
DDFB-LNO	$\mathrm{D}_{k,\mathcal{P}} = \mathrm{D}_{k,\mathcal{D}}^\top$	define $\tau_k = 1.99\|\mathrm{D}_k\|^{-2}$
DDiFB-LNO	$\mathrm{D}_{k,\mathcal{P}} = \mathrm{D}_{k,\mathcal{D}}^\top$	fix $\alpha_k = \frac{t_k-1}{t_{k+1}}$, $t_{k+1} = \frac{k+a-1}{a}$, $a > 2$, and $\tau_k = 0.99\|\mathrm{D}_k\|^{-2}$
DCP-LFO	$\mathrm{D}_{k,\mathcal{P}}$, $\mathrm{D}_{k,\mathcal{D}}$, μ	learn $\mu = \mu_0 = \cdots = \mu_K$, and absorb τ_k in $\mathrm{D}_{k,\mathcal{D}}$
DCP-LNO	$\mathrm{D}_{k,\mathcal{P}} = \mathrm{D}_{k,\mathcal{D}}^\top$, μ	learn $\mu = \mu_0 = \cdots = \mu_K$, and fix $\tau_k = 0.99\mu^{-1}\|\mathrm{D}_k\|^{-2}$

Variations on the proposed architecture

	Θ_k	Comments
DDFB-LFO	$\mathrm{D}_{k,\mathcal{P}}$, $\mathrm{D}_{k,\mathcal{D}}$	absorb τ_k in $\mathrm{D}_{k,\mathcal{D}}$
	$\mathrm{D}_{k,\mathcal{P}}$, $\mathrm{D}_{k,\mathcal{D}}$, α_k	fix α_k, and absorb τ_k in $\mathrm{D}_{k,\mathcal{D}}$
DDFB-LNO	$\mathrm{D}_{k,\mathcal{P}} = \mathrm{D}_{k,\mathcal{D}}^\top$	define $\tau_k = 1.99\|\mathrm{D}_k\|^{-2}$
DDiFB-LNO	$\mathrm{D}_{k,\mathcal{P}} = \mathrm{D}_{k,\mathcal{D}}^\top$	fix $\alpha_k = \frac{t_k - 1}{t_{k+1}}$, $t_{k+1} = \frac{k+a-1}{a}$, $a > 2$, and $\tau_k = 0.99\|\mathrm{D}_k\|^{-2}$
DCP-LFO	$\mathrm{D}_{k,\mathcal{P}}$, $\mathrm{D}_{k,\mathcal{D}}$, μ	learn $\mu = \mu_0 = \cdots = \mu_K$, and absorb τ_k in $\mathrm{D}_{k,\mathcal{D}}$
DScCP-LFO	$\mathrm{D}_{k,\mathcal{P}}$, $\mathrm{D}_{k,\mathcal{D}}$, μ_0	learn μ_0, absorb τ_k in $\mathrm{D}_{k,\mathcal{D}}$, and fix $\alpha_k = (1 + 2\mu_k)^{-1/2}$, and $\mu_{k+1} = \alpha_k \mu_k$
DCP-LNO	$\mathrm{D}_{k,\mathcal{P}} = \mathrm{D}_{k,\mathcal{D}}^\top$, μ	learn $\mu = \mu_0 = \cdots = \mu_K$, and fix $\tau_k = 0.99\mu^{-1}\|\mathrm{D}_k\|^{-2}$
DScCP-LNO	$\mathrm{D}_{k,\mathcal{P}} = \mathrm{D}_{k,\mathcal{D}}^\top$, μ_k	learn μ_k, and fix $\alpha_k = (1 + 2\mu_k)^{-1/2}$, and $\tau_k = 0.99\mu_k^{-1}\|\mathrm{D}_k\|^{-2}$

Limit case for deep unfolded NNs

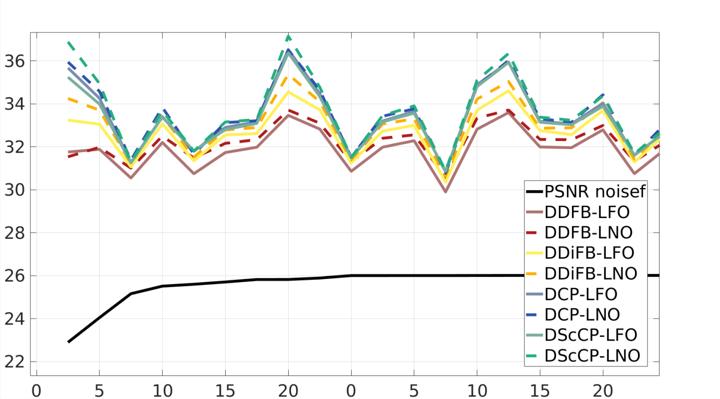
[Le, Repetti, Pustelnik 2023]
We consider the unfolded NNs DD(i)FB and D(Sc)CP. Assume that, for every $k \in \{1, \ldots, K\}$, $\mathrm{D}_{k,\mathcal{D}} = \mathrm{D}$ and $\mathrm{D}_{k,\mathcal{P}} = \mathrm{D}^\top$, for $\mathrm{D}\colon \mathbb{R}^N \to \mathbb{R}^{|\mathbb{F}|}$. In addition, for each architecture, we further assume that, for every $k \in \{1, \ldots, K\}$,

- DDFB: $\tau_k \in (0, 2/\|\mathrm{D}\|_S^2)$.
- DDiFB: $\tau_k \in (0, 1/\|\mathrm{D}\|_S^2)$ and $\rho_k = \frac{t_k - 1}{t_{k+1}}$ with $t_k = \frac{k+a-1}{a}$ and $a > 2$.
- DCP: $(\tau_k, \mu_k) \in (0, +\infty)^2$ such that $\tau_k \mu_k \|\mathrm{D}\|_S^2 < 1$.
- DScCP: $\alpha_k = (1 + 2\mu_k)^{-1/2}$, $\mu_{k+1} = \alpha_k \mu_k$, and $\tau_{k+1} = \tau_k \alpha_k^{-1}$ with $\tau_0 \mu_0 \|\mathrm{D}\|_S^2 \leqslant 1$.

Then, we have $\mathbf{x}_K \to \widehat{\mathbf{x}}$ when $K \to +\infty$, where $\mathbf{x}_K$ is the output of either of the unfolded NNs DD(i)FB or D(Sc)CP, and $\widehat{\mathbf{x}}$ is a solution to

$$\min_{\mathbf{x} \in \mathcal{H}} \frac{1}{2} \|\mathbf{x} - \mathbf{z}\|_2^2 + g(\mathrm{D}\mathbf{x}) + \iota_C(\mathbf{x}).$$

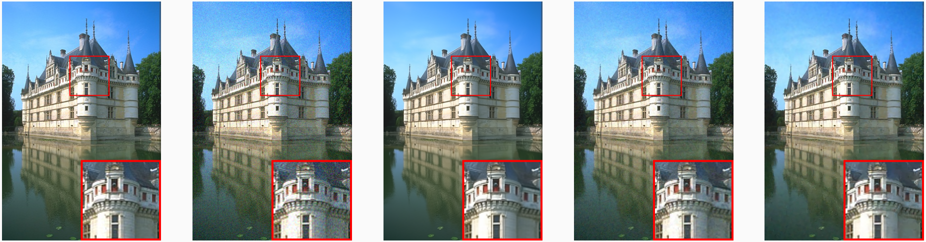
Denoising performance



Denoising performance.

PSNR (with $(K, J) = (20, 64)$), for 20 images of BSDS500 validation set, degraded with noise level $\delta = 0.05$.

Denoising performance



	Noisy	**DRUnet**	DDFB-LNO	DScCP-LNO
	26.03dB	**35.81dB**	32.81dB	34.74dB

Denoising performance on Gaussian noise. Example of denoised images (and PSNR values) for Gaussian noise $\delta = 0.05$ obtained with DRUnet and the proposed DDFB-LNO and DScCP-LNO, with $(K, J) = (20, 64)$.

Complexity of the models

		Time (msec)	$\|\Theta\|$	FLOPs ($\times 10^3$ G)
BM3D		$13{\times}10^3 \pm 317$	–	–
DRUnet		96 ± 21	$32,640,960$	137.24
LNO	DDFB	3 ± 1.5	$34,560$	2.26
	DDiFB	3 ± 0.5	$34,560$	
	DDCP	6 ± 1	$34,561$	
	DDScCP	7 ± 1	$34,580$	
LFO	DDFB	4 ± 17	$69,120$	2.26
	DDiFB	5 ± 15	$69,121$	
	DDCP	7 ± 14	$69,121$	
	DDScCP	9 ± 15	$69,160$	

NN robustness

☛ Given an input $\mathbf{z}$ and a perturbation ϵ, the error on the output can be upper bounded :

$$\|f_\Theta(\mathbf{z}+\epsilon) - f_\Theta(\mathbf{z})\| \leqslant \chi\|\epsilon\|.$$

where χ certificated of the robustness.

☛ [Combettes, Pesquet, 2020]: χ can be upper bounded by:

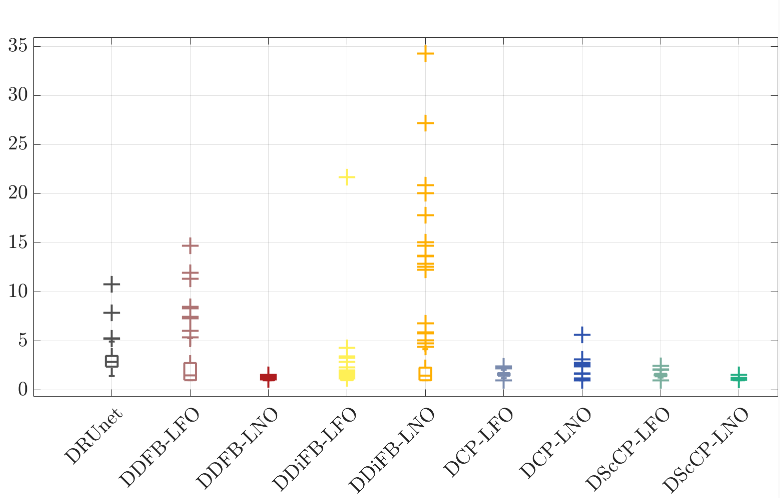
$$\chi \leqslant \prod_{k=1}^{K} \Big(\|W_{k,\mathcal{P}}\|_S \times \|W_{k,\mathcal{D}}\|_S\Big).$$

☛ [Pesquet, Repetti, Terris, Wiaux, 2020]: tighter bound by Lipschitz continuity:

$$\chi \approx \max_{(\mathbf{z}_s)_{s\in\mathbb{I}}} \|\,\mathrm{J}\, f_\Theta(\mathbf{z}_s)\|_S.$$

where J denotes the Jacobian operator.

NN robustness



Distribution of $(\| \mathrm{J} f_\Theta(\mathbf{z}_s)\|_S)_{s\in\mathbb{J}}$ for 100 images extracted from BSDS500 validation dataset $\mathbb{J}$, for the proposed PNNs and DRUnet.

Plug and play algorithms

- **Variational approach** ($k \to \infty$):

$$\mathbf{x}^{[k+1]} = \mathrm{prox}_{\gamma g}(\mathbf{x}^{[k]} - \gamma \nabla f(\mathbf{x}^{[k]})))$$

- **PnP** ($k \to \infty$):

$$\mathbf{x}^{[k+1]} = \mathrm{D}(\mathbf{x}^{[k]} - \gamma \nabla f(\mathbf{x}^{[k]})))$$

- Convergence of $(\mathbf{x}^{[k]})_{k\in\mathbb{N}}$ ensured if denoiser firmly nonexpansive (i.e. $\alpha = 1/2$).
  - Most of the existing denoisers used in PnP do not satisfy this condition
  - Some recent works propose denoisers that can be built to satisfy this condition ([Hasannasab et al, 2020], [Terris et al, 2020], [Terris et al, 2021])

Bayesian interpretation

- $\mathbf{x}$: realization of a random vector $\mathbf{X}$.
- $\mathbf{z}$: realization of a random vector $\mathbf{Z}$.

MAP estimator (Maximum A Posteriori)

$$
\begin{aligned}
& \mathrm{D}_{\mathrm{MAP}}(\mathbf{z}) \in \operatorname*{Argmax}_{\mathbf{x}} \; p(\mathbf{x}|\mathbf{z}) \\
\Leftrightarrow \quad & \mathrm{D}_{\mathrm{MAP}}(\mathbf{z}) \in \operatorname*{Argmax}_{\mathbf{x}} \; \frac{p(\mathbf{z}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{z})} \\
\Leftrightarrow \quad & \mathrm{D}_{\mathrm{MAP}}(\mathbf{z}) \in \operatorname*{Argmin}_{\mathbf{x}} \; \underbrace{-\log p(\mathbf{z}|\mathbf{x})}_{\text{Likelihood}} \; \underbrace{-\log p(\mathbf{x})}_{\text{Prior}}
\end{aligned}
$$

Assuming $\mathbf{z} = \mathbf{x} + \mathbf{b}$ where $\mathbf{b} \sim \mathcal{N}(0, \sigma^2 \, \mathbf{I})$,

$$
\mathrm{D}_{\mathrm{MAP}}(\mathbf{z}) = \mathrm{prox}_{-\log p_{\mathbf{X}}}(\mathbf{x})
$$

MMSE and Tweedie

Assuming that $\mathbf{x}$ has been sampled from a prior density $p_{\mathbf{X}}$ and that $\mathbf{z} = \mathbf{x} + \mathbf{b}$ where $\mathbf{b} \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$, then

$$\mathrm{D}_{\mathrm{MMSE}}(\mathbf{z}) = \mathbb{E}[\mathbf{X}|\mathbf{Z} = \mathbf{z}] = \mathbf{z} + \sigma^2 \nabla_{\mathbf{z}} \log p_{\mathbf{Z}}(\mathbf{z}).$$

- $\mathrm{p}_{\mathbf{Z}}$ is given by a standard convolution between pdfs $p_{\mathbf{Z}} = p_{\mathbf{B}} \star p_{\mathbf{X}}$.
- Identity was first reported by Robbins in 1956.
- $\mathbb{E}[\mathbf{X}|\mathbf{Z} = \mathbf{z}]$ posterori mean (i.e. MMSE estimator, conditional expectation) of $\mathbf{x}$ given $\mathbf{z}$.

Denoiser versus gradient

Proposition [Gribonval-Nikolova, 2020, Hurault et al. 2023]
Let $g_\sigma : \mathbb{R}^N \to \mathbb{R}$ is a C^2 function with ∇g_σ be L_{g_σ}-Lipschitz with $L_{g_\sigma} < 1$.

Then, for $\mathrm{D}_\sigma := \mathbf{I} - \nabla g_\sigma$, there exists a potential $\phi_\sigma : \mathbb{R}^N \to (-\infty, +\infty]$, such that $\mathrm{prox}_{\phi_\sigma}$ is one-to-one and $D_\sigma = \mathrm{prox}_{\phi_\sigma}$.

Moreover ϕ_σ is $\frac{L_{g_\sigma}}{L_{g_\sigma}+1}$-weakly convex and it can be written $\phi_\sigma = \widehat{\phi}_\sigma + K$ on $\mathrm{Im}(D_\sigma)$ (which is open) for some constant $K \in \mathbb{R}$ and

$$\widehat{\phi}_\sigma(\mathbf{x}) = \begin{cases} g_\sigma(\mathrm{D}_\sigma{}^{-1}(\mathbf{x})) - \frac{1}{2}\|D_\sigma^{-1}(\mathbf{x}) - \mathbf{x}\|^2 & \text{if} \quad \mathbf{x} \in \mathrm{Im}(D_\sigma) \\ +\infty & \text{otherwise.} \end{cases}$$

Additionally $\widehat{\phi}_\sigma(\mathbf{x}) \geqslant g_\sigma(\mathbf{x})$.

Weakly convex

For $\Psi : \mathbb{R}^N \to]-\infty, +\infty]$ be a proper, l.s.c., M-weakly convex function with $M > 0$.
We have, forall $(\mathbf{x}, \mathbf{y})$ in $\mathbb{R}^N$,

$$(\forall \mathbf{u} \in \partial\Psi(\mathbf{x}))\ \langle \mathbf{y} - \mathbf{x} | \mathbf{u} \rangle + \Psi(\mathbf{x}) - \frac{M}{2}\|\mathbf{x} - \mathbf{y}\|^2 \leqslant \Psi(\mathbf{y})\}$$

and, for $t \in [0, 1]$,

$$\Psi(t\mathbf{x} + (1-t)\mathbf{y}) \leqslant t\Psi(\mathbf{x}) + (1-t)\Psi(\mathbf{y}) + \frac{M}{2}t(1-t)\|\mathbf{x} - \mathbf{y}\|^2$$

PnP: Characterization of the limit point

- Characterization of the limit point as solution to **a monotone inclusion problem** i.e.

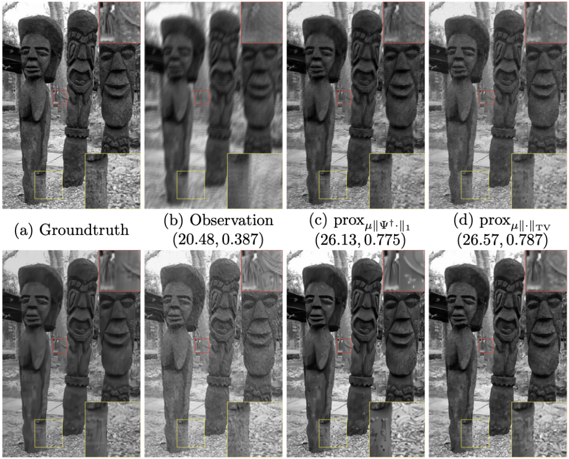
$$0 \in \partial f(\widehat{\mathbf{x}}) + M(\widehat{\mathbf{x}})$$

with $\mathrm{D} = \frac{\mathbf{I}+Q}{2}$ with Q non-expansive and $M = 2(\mathbf{I}+Q)^{-1} - \mathbf{I}$ a maximally monotone operators [Terris et al, 2021].

- Characterization of the limit point as solution to **a variational (non-convex) minimisation** problem
  - Denoiser used in the gradient step ([Laumont et al, 2021], [Hurault et al, 2021])
  - Denoiser used in the proximity step ([Hurault et al, 2022])

PnP imposing firmly non-expansivity of the denoiser

[Pesquet et al., 2021]



(a) Groundtruth

(b) Observation (20.48, 0.387)

(c) $\text{prox}_{\mu\|\Psi^\dagger\cdot\|_1}$ (26.13, 0.775)

(d) $\text{prox}_{\mu\|\cdot\|_{\text{TV}}}$ (26.57, 0.787)

(e) BM3D (26.09, 0.732)

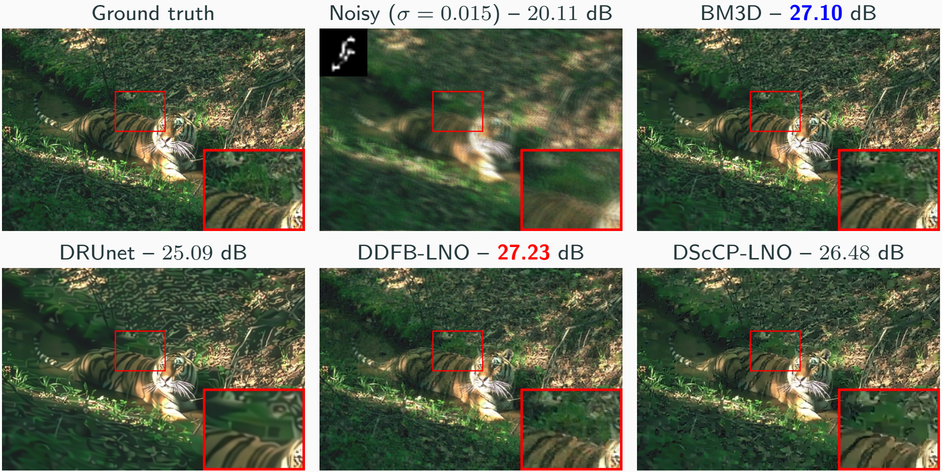
(f) RealSN (24.68, 0.726)

(g) DnCNN (26.12, 0.643)

(h) Proposed (**27.09**, **0.789**)

PnP based on PNN

[Le et al., 2023]



Restoration performance.

Restoration example for $\sigma = 0.015$, with parameters $\gamma = 1.99$ and β chosen optimally

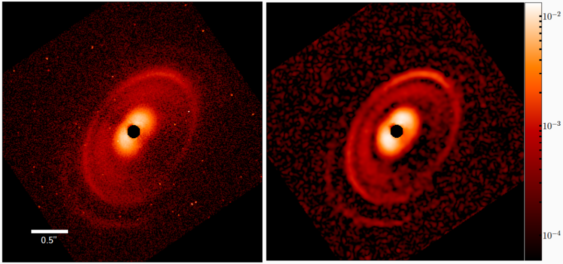
Unfolded proximal schemes with non-linear operators

Context: Image restoration

➡ **Data**: $\mathbf{z} \in \mathbb{R}^M$ degraded version of an original image $\overline{\mathbf{x}} \in \mathbb{R}^N$:

$$\mathbf{z} = \mathrm{A}\overline{\mathbf{x}} + \mathbf{w}$$

- $\mathrm{A} \in \mathbb{R}^{M\times N}$: linear degradation (e.g. a blur)
- $\mathbf{w}$: noise (e.g. Gaussian noise)



SPHERE-IRDIS

Context: Image restoration (astronomy context)

- **Studying circumstellar environments**: crucial for understanding exoplanets and stellar systems.
- **High contrast imagery**: high contrast between environment and host star.
- **Instrument**: Spectro-Polarimetric High-contrast Exoplanet REsearch (SPHERE) and its instrument InfraRed Dual Imaging and Spectrograph (IRDIS) installed on the Very Large Telescope (VLT).
- **Direct model**:

$$\mathbf{z}_{j,\ell} = T_{j,\ell} A \left(\frac{1}{2} I^u + I^p \cos^2(\theta - 2\alpha_\ell - \psi_j) \right) + \varepsilon_{j,\ell},$$

or

$$\mathbf{z}_{j,\ell} = \sum_{m=1}^{3} \nu_{j,\ell,m} \, T_{j,\ell} A \, S_m + \varepsilon_{j,\ell},$$

DeepPDNet for high-contrast recovery

☛ **Analysis formulation:** $$\min_{\mathbf{x}} \frac{1}{2}\|\mathrm{A}\mathbf{x} - \mathbf{z}\|_2^2 + \|\mathrm{D}\mathbf{x}\|_1$$

☛ **Condat-Vũ iterations:**

$$\left|\begin{array}{ll} \mathbf{x}^{[k+1]} & = \mathbf{x}_k - \tau \mathrm{A}^*(\mathrm{A}\mathbf{x}^{[k]} - \mathbf{z}) - \tau \mathrm{D}^* \mathbf{u}^{[k]} \\ \mathbf{u}^{[k+1]} & = \mathrm{prox}_{\gamma\|\cdot\|_1^*}\left(\mathbf{u}^{[k]} + \gamma \mathrm{D}(2\mathbf{x}^{[k+1]} - \mathbf{x}^{[k]})\right) \end{array}\right.$$

☛ **Reformulation:**

$$\left|\begin{array}{l} \mathbf{x}^{[k+1]} = (\mathrm{Id} - \tau \mathrm{A}^*\mathrm{A})\mathbf{x}^{[k]} - \tau \mathrm{D}^*\mathbf{u}^{[k]} + \tau \mathrm{A}^*\mathbf{z} \\ \mathbf{u}^{[k+1]} = \mathrm{prox}_{\gamma\|\cdot\|_1^*}\left(\gamma \mathrm{D}(\mathrm{Id} - 2\tau \mathrm{A}^*\mathrm{A})\mathbf{x}^{[k]} + (\mathrm{Id} - 2\tau\gamma \mathrm{D}\mathrm{D}^*)\mathbf{u}^{[k]} + 2\tau\gamma \mathrm{D}\mathrm{A}^*\mathrm{z}\right). \end{array}\right.$$

☛ **Layer network:** [Jiu, Pustelnik, 2021]

$$\begin{bmatrix} \mathbf{x}^{[k+1]} \\ \mathbf{u}^{[k+1]} \end{bmatrix} = \underbrace{\begin{matrix} \mathbf{I} \\ \mathrm{prox}_{\gamma\|\cdot\|_1^*} \end{matrix}}_{\eta^{[k]}} \left(\underbrace{\begin{bmatrix} \mathrm{Id} - \tau \mathrm{A}^*\mathrm{A} & -\tau \mathrm{D}^* \\ \gamma \mathrm{D}(\mathrm{Id} - 2\tau \mathrm{A}^*\mathrm{A}) & \mathrm{Id} - 2\tau\gamma \mathrm{D}\mathrm{D}^* \end{bmatrix}}_{W^{[k]}} \begin{bmatrix} \mathbf{x}^{[k]} \\ \mathbf{u}^{[k]} \end{bmatrix} + \underbrace{\begin{bmatrix} \tau \mathrm{A}^*\mathbf{z} \\ 2\tau\gamma \mathrm{D}\mathrm{A}^*\mathbf{z} \end{bmatrix}}_{b^{[k]}} \right)$$

DeepPDNet for high-contrast recovery

$$f_{\Theta}(\mathbf{z}) = \eta^{[K]}\left(W^{[K]}\ldots\eta^{[1]}(W^{[1]}\mathbf{z}+b^{[1]})\ldots+b^{[K]}\right)$$

☛ **Network with fixed layer:** $\Theta = \{D, \tau, \gamma\}$

$$\begin{bmatrix}\mathbf{x}^{[k+1]}\\ \mathbf{u}^{[k+1]}\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{I}\\ \operatorname{prox}_{\gamma\|\cdot\|_1^*}\end{bmatrix}}_{\eta^{[k]}}\left(\underbrace{\begin{bmatrix}\mathrm{Id}-\tau A^*A & -\tau D^*\\ \gamma D(\mathrm{Id}-2\tau A^*A) & \mathrm{Id}-2\tau\gamma DD^*\end{bmatrix}}_{W^{[k]}}\begin{bmatrix}\mathbf{x}^{[k]}\\ \mathbf{u}^{[k]}\end{bmatrix} + \underbrace{\begin{bmatrix}\tau A^*z\\ 2\tau\gamma DA^*z\end{bmatrix}}_{b^{[k]}}\right)$$

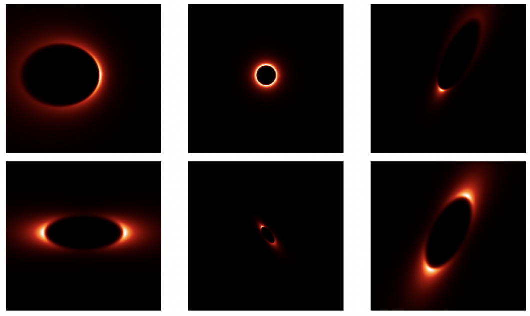
☛ **Network with variable layers:** $\Theta = \{D_k, \tau_k, \gamma_k, \}_{1\leqslant k\leqslant K}$

$$\begin{bmatrix}\mathbf{x}^{[k+1]}\\ \mathbf{u}^{[k+1]}\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{I}\\ \operatorname{prox}_{\gamma_k\|\cdot\|_1^*}\end{bmatrix}}_{\eta^{[k]}}\left(\underbrace{\begin{bmatrix}\mathrm{Id}-\tau_k A^*A & -\tau_k D_k^*\\ \gamma_k D_k(\mathrm{Id}-2\tau_k A^*A) & \mathrm{Id}-2\tau\gamma_k D_k D_k^*\end{bmatrix}}_{W^{[k]}}\begin{bmatrix}\mathbf{x}^{[k]}\\ \mathbf{u}^{[k]}\end{bmatrix} + \underbrace{\begin{bmatrix}\tau_k A_k^*z\\ 2\tau_k\gamma_k D_k A^*z\end{bmatrix}}_{b^{[k]}}\right)$$

$+$ specificities for the first and last layers.

Dataset

- **DDIT**: Debris DIsks Tools library produces synthetic images of $(I^u_{\text{disk}}, I^p, \theta)$.
- I^u_{star} has been obtained from real observational high-contrast coronagraphic data from the SPHERE.
- Different semi-major axis of the disk, inclination, eccentricity, and ratio between the star and disk intensity.



Dataset

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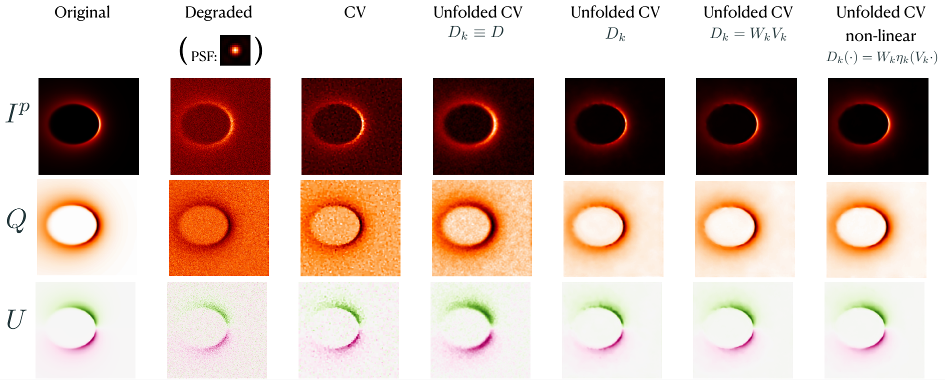
- I^u_{star} has been obtained from real observational high-contrast coronagraphic data from the SPHERE.
- Different semi-major axis of the disk, inclination, eccentricity, and ratio between the star and disk intensity.

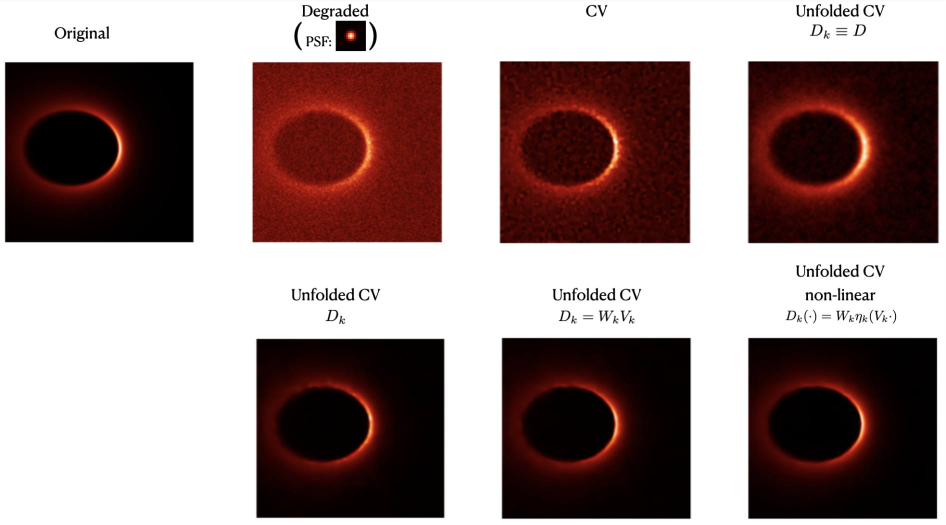
- **Synthetic dataset** – Prescribe blur and Gaussian noise with a standard deviation of 0.1.

- **More realistic dataset** – z obtained from RHAPSODIE forward model.

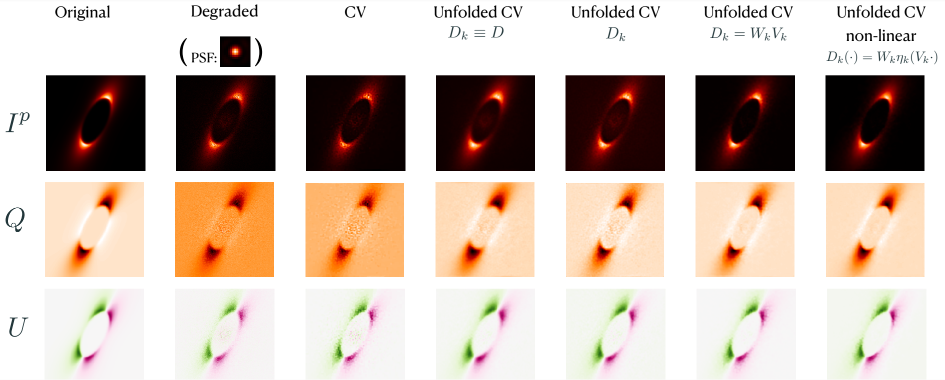
DeepPDNet for high-contrast recovery



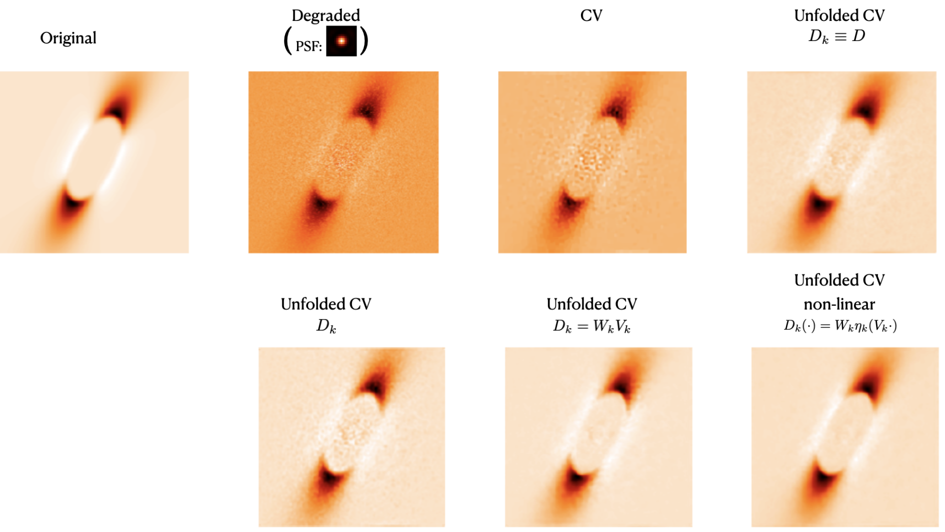
DeepPDNet for high-contrast recovery

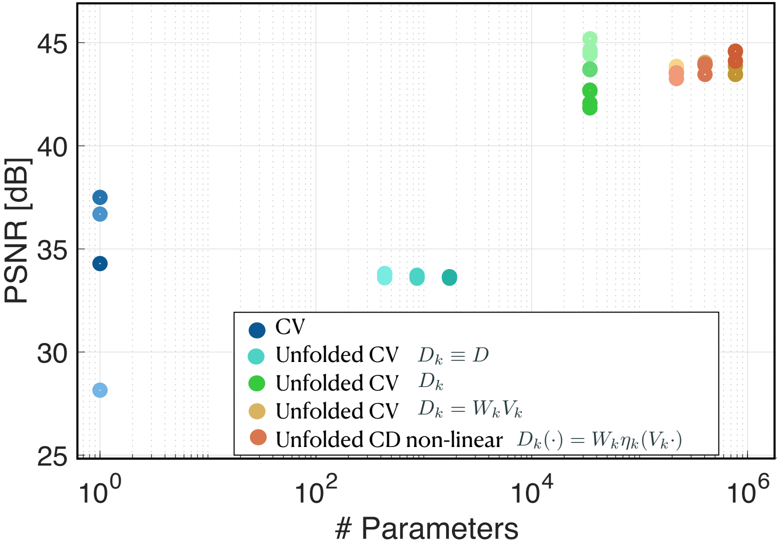


DeepPDNet for high-contrast recovery

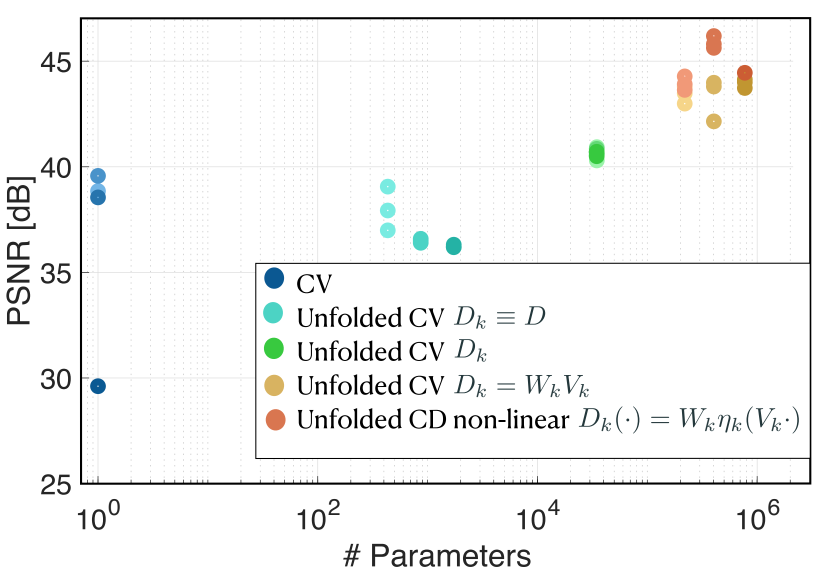


DeepPDNet for high-contrast recovery




45
40
35
30
25
PSNR [dB]
10^0
10^2
10^4
10^6
Parameters
CV
Unfolded CV $D_k \equiv D$
Unfolded CV D_k
Unfolded CV $D_k = W_k V_k$
Unfolded CD non-linear $D_k(\cdot) = W_k \eta_k(V_k \cdot)$

DeepPDNet for high-contrast recovery



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Conclusions

- ☛ PnP and unfolded: two frameworks to combine variational approaches and deep neural network.
- ☛ PnP: convergence guarantees but slow.
- ☛ Unfolded: fast but no convergence guarantees.
- ☛ Proximal unfolded schemes may help to design robust neural networks.
- ☛ Proximal unfolded NN schemes: good compromise between number of parameters and performance.
- ☛ More parameters + nonlinear D to achieve better performance with proximal unfolded schemes .