# Two methods for computing machine-efficient polynomial approximants

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# Function evaluation on a machine

Problem : evaluation of a function $\varphi$ over $\mathbb{R}$ or a subset of $\mathbb{R}$.

We wish to only use additions, subtractions, multiplications (we should avoid divisions) $\Rightarrow$ use of polynomials.

The algorithms for evaluating elementary functions ($\exp, \ln, \cos, \sin, \arctan, \sqrt{\ }, \ldots$) use polynomial approximants.

# Evaluation of elementary functions

$\exp, \ln, \cos, \sin, \arctan, \sqrt{\ }, \ldots$

First step. Argument reduction (Payne & Hanek, Ng, Daumas *et al*) : evaluation of a function $\varphi$ over $\mathbb{R}$ or a subset of $\mathbb{R}$ is reduced to the evaluation of a function $f$ over $[a, b]$.

Second step. Polynomial approximation of $f$ :

- least square approximation ;
- minimax approximation.

# Minimax Approximation

Reminder. Let $g : [a, b] \to \mathbb{R}$, $||g||_{[a,b]} = \sup_{a \leq x \leq b} |g(x)|$.

We denote $\mathbb{R}_n[X] = \{p \in \mathbb{R}[X]; \deg p \leq n\}$.

Minimax approximation : let $f : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$, we search for $p \in \mathbb{R}_n[X]$ s.t.

$$||p - f||_{[a,b]} = \inf_{q \in \mathbb{R}_n[X]} ||q - f||_{[a,b]}.$$

An algorithm by Remez gives $p$.

Problem : we can't directly use minimax approx. in a computer since the coefficients of $p$ can't be represented on a finite number of bits.

Our context : the coefficients of the polynomials must be written on a finite (imposed) number of bits.

Let $m = (m_i)_{0 \leq i \leq n}$ a finite sequence of natural integers.

Let $q(x) = q_0 + q_1 x + \cdots + q_n x^n \in \mathbb{R}_n[x]$. Each $q_i$ must be an integer multiple of $2^{-m_i}$ : $q_i = a_i / 2^{m_i}$ with $a_i \in \mathbb{Z}$.

# Truncated Polynomials

Let $m = (m_i)_{0 \leq i \leq n}$ a finite sequence of natural integers. Let

$$\mathcal{P}_n^m = \{q = q_0 + q_1 x + \cdots + q_n x^n \in \mathbb{R}_n[X]; q_i \text{ integer multiple of } 2^{-m_i}, \forall i\}.$$

First idea. Remez $\longrightarrow p(x) = p_0 + p_1 x + \cdots + p_n x^n$. Every $p_i$ rounded to $\hat{p}_i$, the nearest integer multiple of $2^{-m_i} \longrightarrow \hat{p}(x) = \hat{p}_0 + \hat{p}_1 x + \cdots + \hat{p}_n x^n$.

Problem : $\hat{p}$ not necessarily the minimax approx. of $f$ among the polynomials of $\mathcal{P}_n^m$.

# Applications

Two targets :

- specific hardware implementations in low precision ($\sim 15$ bits). Reduce the cost (time and silicon area) keeping a correct accuracy ;
- single or double IEEE precision software implementations. Get very high accuracy keeping an acceptable cost (time and memory).

# Statement of the problem

Let $f : [a, b] \to \mathbb{R}, n \in \mathbb{N},\ m = (m_i)_{0 \leq i \leq n}$ a finite sequence of natural integers, $p(x) = p_0 + p_1 x + \cdots + p_n x^n$ the minimax approx. of $f$ over $[a, b]$ (Remez).

$$\mathcal{P}_n^m = \left\{ q(x) = \frac{a_0}{2^{m_0}} + \frac{a_1}{2^{m_1}} x + \cdots + \frac{a_n}{2^{m_n}} x^n ; a_i \in \mathbb{Z}, \forall i \right\}.$$

Every $p_i$ rounded to $\hat{p}_i$, the nearest integer multiple of $2^{-m_i}$ $\longrightarrow$ $\hat{p}(x) = \hat{p}_0 + \hat{p}_1 x + \cdots + \hat{p}_n x^n$.

Let

$$\varepsilon = ||f - p||_{[a,b]} \text{ and } \hat{\varepsilon} = ||f - \hat{p}||_{[a,b]}.$$

We compare $\varepsilon$ to $\hat{\varepsilon}$.

We choose $K \in [\varepsilon, \hat{\varepsilon}]$. We search for a truncated polynomial $p^\star \in \mathcal{P}_n^m$ s.t.

$$||f - p^\star||_{[a,b]} = \min_{q \in \mathcal{P}_n^m} ||f - q||_{[a,b]}$$

and

$$||f - p^\star||_{[a,b]} \leq K.$$

# Approach

We put $p^\star(x) = p_0^\star + p_1^\star x + \cdots + p_n^\star x^n$.

1. We find relations relations satisfied by the $p_i^\star \Rightarrow$ finite number of candidate polynomials.
2. If this number is small enough, we perform an exhaustive search : computation of the norms $||f - q||_{[a,b]}$, $q$ running among the candidate polynomials.

# First approach : Chebyshev polynomials

"Partial" method : it only works with intervals of the form $[0, a]$ or $[-a, a]$.

We work over $[0, a]$.

**Definition .** *Chebyshev polynomials can be defined either by the recurrence relation*

$$\begin{cases} T_0(x) & = & 1 \\ T_1(x) & = & x \\ T_n(x) & = & 2xT_{n-1}(x) - T_{n-2}(x); \end{cases}$$

*or by*

$$T_n(x) = \begin{cases} \cos\left(n \cos^{-1} x\right) & (|x| \leq 1) \\ \cosh\left(n \cosh^{-1} x\right) & (x > 1). \end{cases}$$

T. J. Rivlin, *Chebyshev polynomials*.

P. Borwein and T. Erdélyi, *Polynomials and Polynomials Inequalities*.

**Proposition .** *Let $a, b \in \mathbb{R}$, $a < b$. The monic degree-$n$ polynomial having the smallest $||\cdot||_{[a,b]}$ norm is*

$$\frac{(b-a)^n}{2^{2n-1}} T_n \left( \frac{2x-b-a}{b-a} \right).$$

Let $f:[0,a]\to\mathbb{R}$, $m_0,\ldots,m_n\in\mathbb{N}$, $p(x)=p_0+p_1x+\cdots+p_nx^n$ the minimax approx. of $f$ over $[0,a]$ (Remez),

$$\mathcal{P}_n^m=\left\{q(x)=\frac{a_0}{2^{m_0}}+\frac{a_1}{2^{m_1}}x+\cdots+\frac{a_n}{2^{m_n}}x^n; a_i\in\mathbb{Z},\forall i\right\}.$$

We determine bounds s. t. if the coefficients of $q\in\mathcal{P}_n^m$ are not within these bounds then

$$||f-q||_{[0,a]}>K \text{ i.e. } q\neq p^\star.$$

Idea : use $p$. We have

$$||f-q||_{[0,a]}\geq ||p-q||_{[0,a]}-||f-p||_{[0,a]}.$$

If $||p-q||_{[0,a]}>\varepsilon+K$, we are done.

We write the $i$-th coef. of $q$ as $p_i + \delta_i$, avec $\delta_i \neq 0$. We have

$$(q-p)(x) = \delta_i x^i + \sum_{\substack{0 \leq j \leq n, \\ j \neq i}} (q_j - p_j) x^j .$$

Hence, $||q-p||_{[0,a]}$ minimum implies

$$\left\| x^i + \frac{1}{\delta_i} \sum_{\substack{0 \leq j \leq n, \\ j \neq i}} (q_j - p_j) x^j \right\|_{[0,a]}$$

minimum.

We use $T_n^*(x) = T_n(2x-1)$.

We have $T_n^*(x) = T_{2n}(x^{1/2})$.

**Proposition .** *Let* $a \in ]0, +\infty[$*, we define*

$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n = T_n^*\left(\frac{x}{a}\right).$$

*Let* $k \in \mathbb{N}$*,* $0 \le k \le n$*, the polynomial*

$$\frac{1}{\alpha_k} T_n^*\left(\frac{x}{a}\right)$$

*has the smallest* $||\cdot||_{[0,a]}$ *norm among the polynomials of degree at most* $n$ *with a degree-*$k$ *coefficient equal to* $1$*. That norm is* $|1/\alpha_k|$*.*

Therefore

$$\left\| x^i + \frac{1}{\delta_i} \sum_{\substack{0 \le j \le n, \\ j \ne i}} (q_j - p_j) x^j \right\|_{[0,a]} \ge \frac{1}{|\alpha_i|},$$

where $\alpha_i$ is the $i$-th coef. of $T_n^*(x/a)$. From which, we have

$$||q - p||_{[0,a]} \ge \frac{|\delta_i|}{|\alpha_i|}.$$

Reminder : If $||q - p||_{[0,a]} > \varepsilon + K$, we have $q \ne p^\star$.

Hence, if there exists $i$, $0 \le i \le n$, s.t. $|\delta_i| > (\varepsilon + K)|\alpha_i|$ then

$$||q - p||_{[0,a]} > \varepsilon + K.$$

Reminder : $\delta_i = q_i - p_i$.

Thus, the $i$-th coef. of $p^\star$ must belong to

$$[p_i - (\varepsilon + K)|\alpha_i|, p_i + (\varepsilon + K)|\alpha_i|].$$

We put $\varepsilon = ||f - p||_{[0,a]}$ , $p^\star(x) = p_0^\star + p_1^\star x + \cdots + p_n^\star x^n$. For all $i$,

$$p_i - (\varepsilon + K)|\alpha_i| \leq p_i^\star \leq p_i + (\varepsilon + K)|\alpha_i|.$$

Reminder : $p_i^\star = a_i / 2^{m_i}$ with $a_i \in \mathbb{Z}$. We obtain, for all $i$

$$\underbrace{\lceil 2^{m_i}(p_i - (\varepsilon + K)|\alpha_i|)\rceil}_{c_i} \leq 2^{m_i} p_i^\star \leq \underbrace{\lfloor 2^{m_i}(p_i + (\varepsilon + K)|\alpha_i|)\rfloor}_{d_i}.$$

Therefore, we have $d_i - c_i + 1$ possible values for the rational integer $2^{m_i} p_i^\star$.

We have $A = \prod_{i=0}^{n}(d_i - c_i + 1)$ candidate polynomials. If $A$ small enough, exhaustive search : we compute the norms $||f - q||_{[0,a]}$, $q$ running among the candidate polynomials. Otherwise, second approach.

# Approximation of the function $\cos$ over $[0, \pi/4]$ by a degree-$3$ polynomial

```
>m := [12,10,6,4]:polstar(cos,Pi/4,3,m);

  "minimax = ", .9998864206 +
(.00469021603 + (-.5303088665 + .06304636099 x) x) x

      "Distance between f and p =", .0001135879209

                      1  3    17  2      5
          "hatp = ", -- x  - -- x  + ---- x + 1
                     16       32      1024


       "Distance between f and hatp =", .0006939707
```

```
>Do you want to continue (y;/n;)? y;
>Enter the value of parameter lambda: 1/2;

degree 0: 4 possible values between 2047/2048 and
          4097/4096
degree 1: 22 possible values between -3/512 and
          15/1024
degree 2: 5 possible values between -9/16 and
          -1/2
degree 3: 1 possible values between 1/16 and
          1/16
440 polynomials need be checked

>Do you want to try to refine the bounds (y;/n;)?n;
```

```
                     1  3    17  2     3       4095
"pstar = ",         -- x  - -- x  + --- x + ----
                    16      32      512     4096
```

```
"Distance between f and pstar =", .0002441406250
```

```
"Time elapsed (in seconds) =", 1.840
```

In this example, we gain $-\log_2(0.35) \approx 1.5$ bits of accuracy.

## Approximation of the exponential function over $[0, \log(1 + 1/2048)]$ by a degree-3 polynomial

```
>Digits:=30:
>m := [56,45,33,23]: polstar(exp,log(1.+1./2048),3,m);

"minimax = ", .999999999999999981509827946165 +
(1.00000000000121203815619648271
 + (.499999987586063030320493910112
 + .166707352549861488779274879363 x) x) x

                                                                  -16
  "Distance between f and p =", .1849017208895 10



          1398443  3   4294967189  2   35184372088875
"hatp =", ------- x  + ---------- x  + -------------- x
          8388608      8589934592      35184372088832
```

```
          72057594037927935
        + -----------------
          72057594037927936


    "Distance between f and hatp =",
                                                     -16
                .23624220969326235229443 10

>Do you want to continue (y;/n;)? y;
>Enter the value of parameter lambda: 1;

degree 0: 6 possible values between
18014398509481983/18014398509481984
          and 72057594037927937/72057594037927936
degree 1: 109 possible values between
35184372088821/35184372088832
          and 35184372088929/35184372088832
```

```
degree 2: 146 possible values between
          4294967117/8589934592
          and 2147483631/4294967296
degree 3: 194 possible values between 699173/4194304
          and 1398539/8388608
18 523 896 polynomials need be checked
```

# A more general and efficient approach : polytopes

First approach has several drawbacks :

- available only for domains of the form $[0, a]$ or $[-a, a]$ ;
- quickly not good enough. Not a surprise since
  - we use triangle inequality ;
  - the coefficients are handled independently.

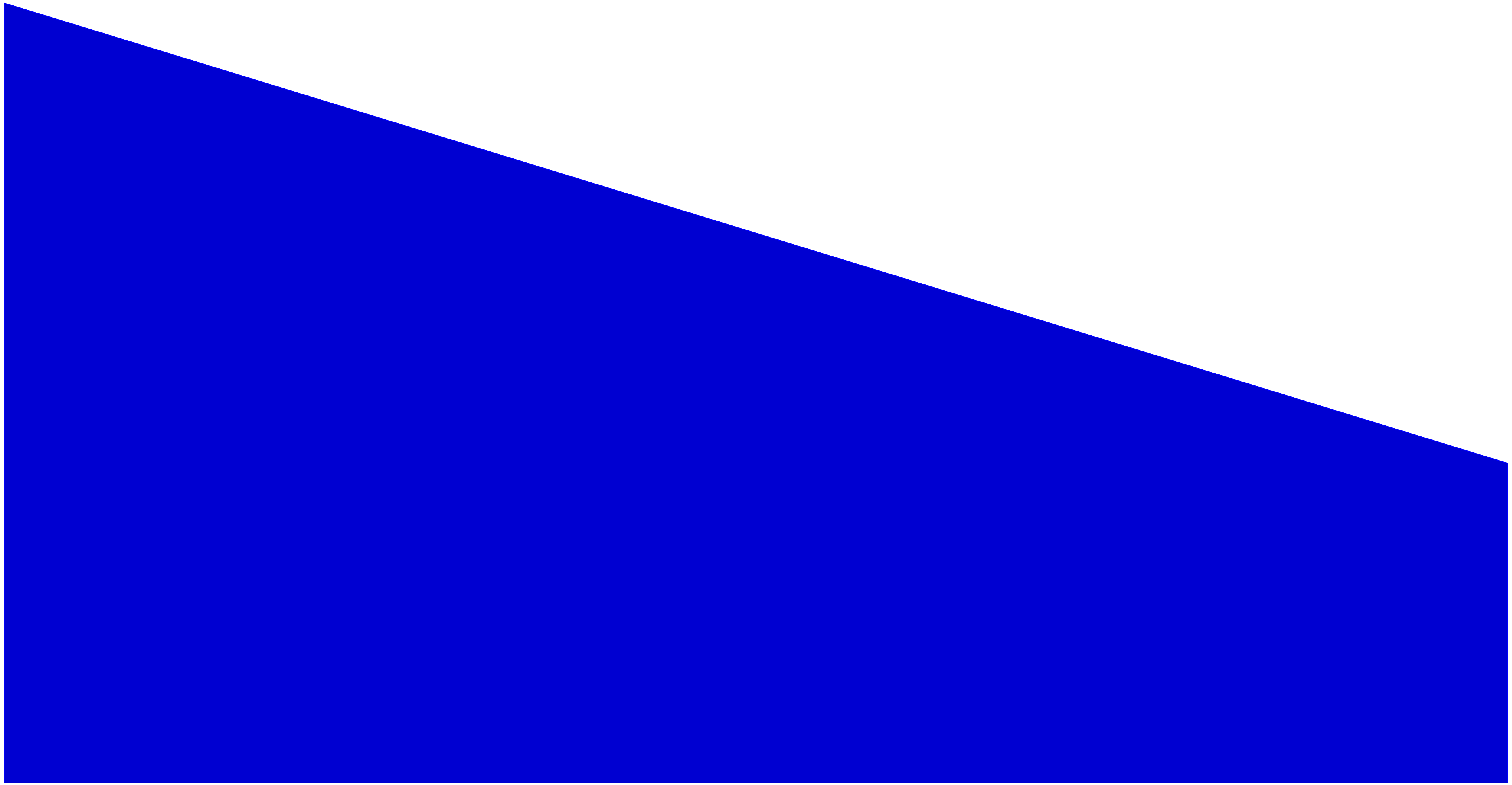
**Definitions .** *Let* $k \in \mathbb{N}$.

*A polyhedron is a subset* $\mathfrak{P}$ *of* $\mathbb{R}^k$ *s.t. there exists a matrix* $A \in \mathcal{M}_{m,k}(\mathbb{R})$ *and a vector* $b \in \mathbb{R}^m$ *(with* $m \geq 0$*) s. t.*

$$\mathfrak{P} = \{x \in \mathbb{R}^k | Ax \leq b\}.$$

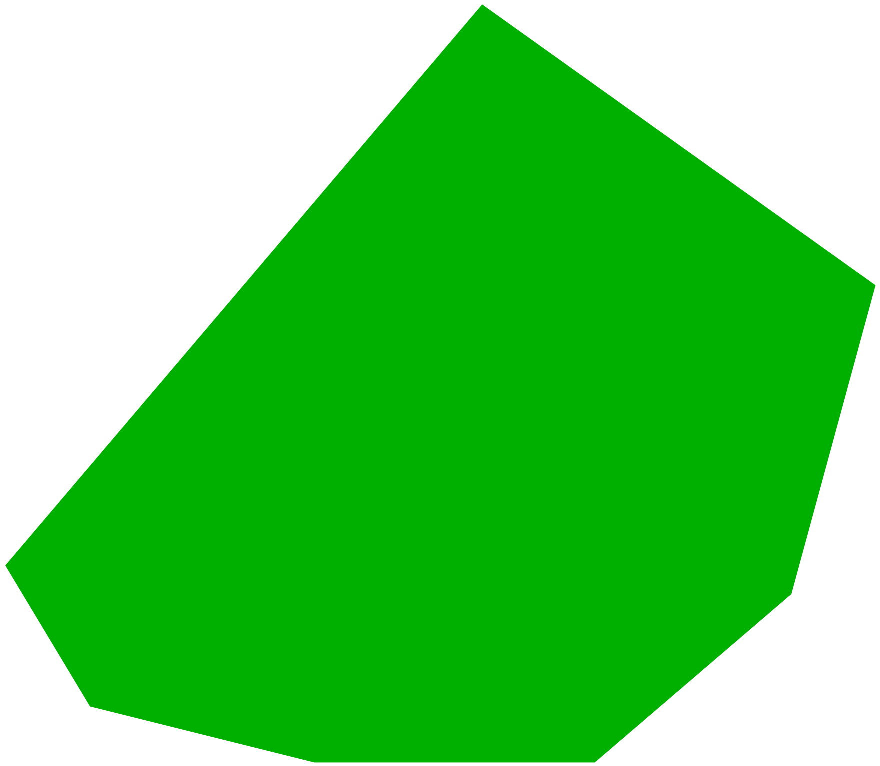
*A polytope is a bounded polyhedron.*

*A polyhedron (resp. polytope)* $\mathfrak{P}$ *is rational if it defined by a matrix and a vector with rational coefficients.*



An example of polyhedron : half-plane in $\mathbb{R}^2$.

An example of polyhedron : cone in $\mathbb{R}^2$.



An example of polytope.

# Reminder of the problem

We put

$$\varepsilon = ||f - p||_{[a,b]} \text{ and } \hat{\varepsilon} = ||f - \hat{p}||_{[a,b]}.$$

We compare $\varepsilon$ to $\hat{\varepsilon}$.

We choose $K \in [\varepsilon, \hat{\varepsilon}]$. We search for a truncated polynomial $p^\star \in \mathcal{P}_n^m$ s.t.

$$||f - p^\star||_{[a,b]} = \min_{q \in \mathcal{P}_n^m} ||f - q||_{[a,b]}$$

and

$$||f - p^\star||_{[a,b]} \leq K.$$

Let $p^\star(x) = p_0^\star + p_1^\star x + \cdots + p_n^\star x^n$.

Over $[0, a]$, we obtained for all $i$

$$\underbrace{\lceil 2^{m_i}(p_i - (\varepsilon + K)|\alpha_i|) \rceil}_{c_i} \leq 2^{m_i} p_i^\star \leq \underbrace{\lfloor 2^{m_i}(p_i + (\varepsilon + K)|\alpha_i|) \rfloor}_{d_i}.$$

They define a polytope which the integers $2^{m_i} p_i^\star$ belong to.

Idea : construct a polytope still containing the integers $2^{m_i} p_i^\star$ but with a smaller number of points with integer coordinates $\Rightarrow$ exhaustive research reduced.

Method works over any $[a, b]$.

We must have

$$f(x) - K \leq \sum_{i=0}^{n} p_i^\star x^i \leq f(x) + K \tag{1}$$

for all $x \in [a, b]$. We have $p_i^\star = a_i^\star / 2^{m_i}$ with $a_i^\star \in \mathbb{Z}$.

We plug into (1) $N$ rational numbers from $[a, b]$. Let $x = r/s$ with $r \in \mathbb{Z}, s \in \mathbb{N}$. We have

$$f\left(\frac{r}{s}\right) - K \leq \sum_{i=0}^{n} \frac{a_i^\star}{2^{m_i}} \left(\frac{r}{s}\right)^i \leq f\left(\frac{r}{s}\right) + K.$$

We chose $m(\frac{r}{s})$ and $M(\frac{r}{s}) \in \mathbb{Q}$ such that $m(\frac{r}{s}) \leq f\left(\frac{r}{s}\right) - K$ and $f\left(\frac{r}{s}\right) + K \leq M(\frac{r}{s})$, $m(\frac{r}{s})$ "close" to $f\left(\frac{r}{s}\right) - K$ and $M(\frac{r}{s})$ "close" to $f\left(\frac{r}{s}\right) + K$.

If $N \geq n+1 \Rightarrow$ we have a rational polytope whose the integers $a_i^\star = 2^{m_i} p_i^\star$ are elements.

If the number of integer points in the polytope is small enough, perform exhaustive research by running the points with integer coordinates of the polytope.

We can use C libraries (s. t. Polylib, CLooG or PIP) designed for efficiently scanning the integer points of polytopes.

**Remark .** *Gives only candidates (but forgets none of them).*

Method works over any $[a, b]$.

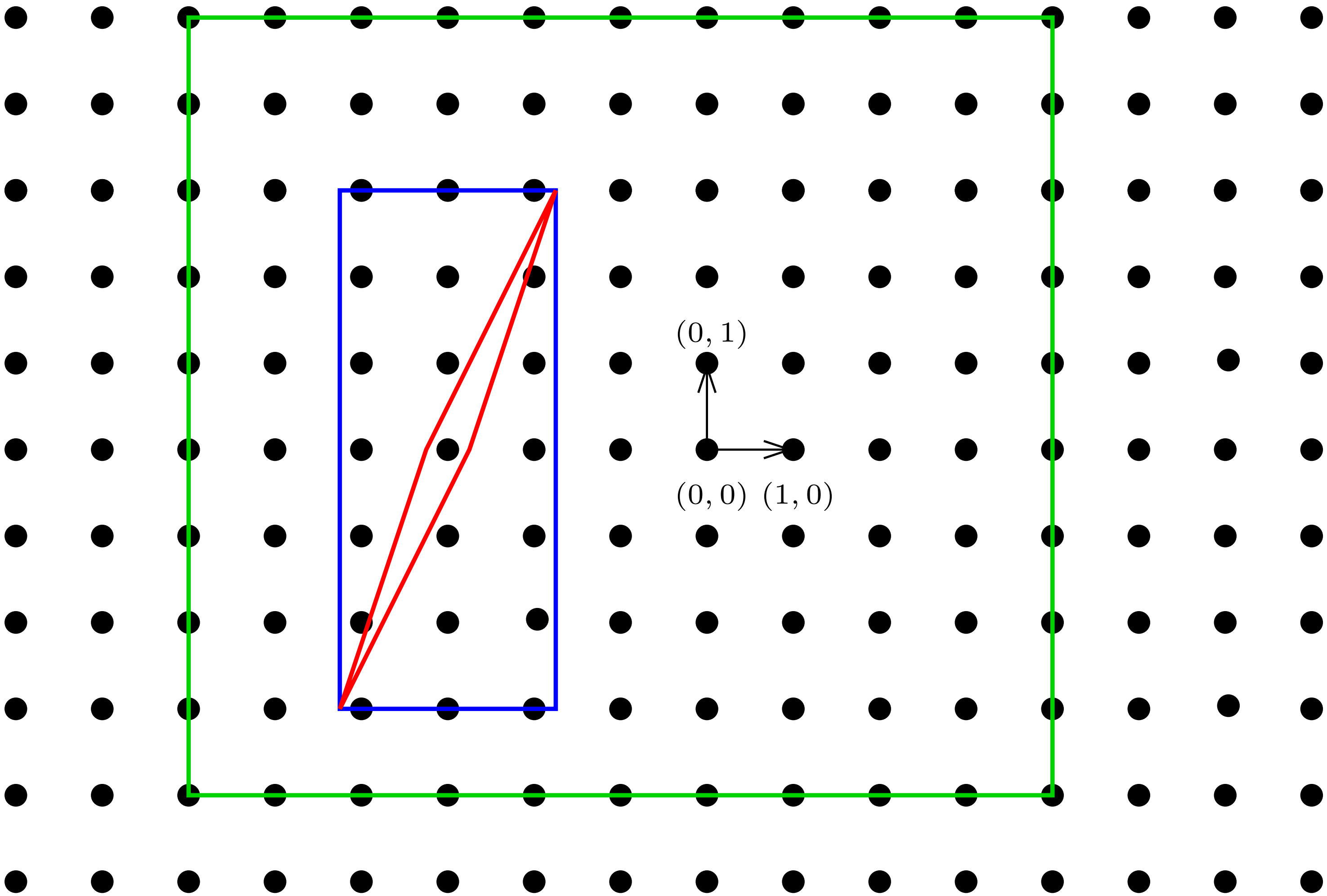
We must have

$$f(x) - K \leq \sum_{i=0}^{n} p_i^\star x^i \leq f(x) + K \qquad (2)$$

for all $x \in [a, b]$.

1. Chose $A$ and $B \in \mathbb{Q}$, $a \leq A \leq B \leq b$, $A$ “close” to $a$ and $B$ “close” to $b$. We define $x_k = A + \frac{k}{d}(B - A)$ where $d \in \mathbb{N}$ is chosen $\forall k$, $0 \leq k \leq d$.
2. We plug the $x_k$ into (2). We compute rational approx. of the $f(x_k) - K$ and $f(x_k) + K$.

   $d \geq n \Rightarrow$ we have a rational polytope which the integers $2^{m_i} p_i^\star$ belong to.
3. If the number of integer points of the polytope is “small enough”, perform exhaustive search by scanning the points with integer coord. of the polytope. To do so, we use C libraries (such as Polylib, CLooG ou PIP) designed for efficiently scanning the integer points of polytopes.




Approximation by a degree-$1$ polynomial. Green polytope : $110$ points of $\mathbb{Z}^2$. Blue polytope : $21$ points of $\mathbb{Z}^2$. Red polytope : $1$ point of $\mathbb{Z}^2$.

## Approximation of the exponential function over $[0, \log(1+1/2048)]$ by a degree-3 polynomial

```
>Digits:=30:
>m := [56,45,33,23]: polstar(exp,log(1.+1./2048),3,m);

"minimax = ", .999999999999999981509827946165 +
(1.00000000000121203815619648271
 + (.499999987586063030320493910112
 + .166707352549861488779274879363 x) x) x

                                   -16
  "Distance between f and p =", .1849017208895 10



          1398443  3   4294967189  2   35184372088875
"hatp =", ------- x  + ---------- x  + -------------- x
          8388608      8589934592      35184372088832
```

```
          72057594037927935
        + -----------------
          72057594037927936


    "Distance between f and hatp =",
                                                    -16
               .2362422096932623522944 3 10

>Do you want to continue (y;/n;)? y;
>Enter the value of parameter lambda: 1;

degree 0: 6 possible values between
18014398509481983/18014398509481984
          and 72057594037927937/72057594037927936
degree 1: 109 possible values between
35184372088821/35184372088832
          and 35184372088929/35184372088832
```

```
degree 2: 146 possible values between
          4294967117/8589934592
          and 2147483631/4294967296
degree 3: 194 possible values between 699173/4194304
          and 1398539/8388608
18 523 896 polynomials need be checked

>Do you want to try to refine the bounds (y;/n;)?y;
>Enter the value of parameter d: 25;

degree 0: 2 possible values between
72057594037927935/72057594037927936
          and 1
degree 1: 27 possible values between
          35184372088857/35184372088832
          and 35184372088883/35184372088832
degree 2: 32 possible values between
          536870897/1073741824
          and 4294967207/8589934592
```

```
degree 3: 44 possible values between 1398421/8388608
          and 21851/131072
76 032 polynomials need be checked

>Do you want to try to refine the bounds (y;/n;)?n;
>Do you want to change the value of Digits (y;/n;)?y;
>Enter the value of Digits: 21;
           1398443  3  2147483595  2  35184372088873
"pstar =", ------- x + ---------- x + -------------- x
           8388608     4294967296     35184372088832

           72057594037927935
         + -----------------
           72057594037927936

    "Distance between f and pstar =",
                                                  -16
                 .202462803670964701822285 10

         "Time elapsed (in seconds) =", 54721.961
```

In this example, we gain $-\log_2(0.85) \approx 0.22$ bits of accuracy.

# The polytope method is flexible !

We can add some constraints (fix values of some coef. for instance) or use other distances.

**Examples .**

- *If we restrict our search to odd truncated polynomials, we consider*

$$f(x_k) - K \leq \sum_{i=0}^{I} p_i^\star x_k^{2i+1} \leq f(x_k) + K, \quad k = 0, \ldots, d$$

*with $x_k \in \mathbb{Q} \cap [a, b]$, $d \geq I$. We compute rational approximations $m_k$ and $M_k$ of $f(x_k) - K$ and $f(x_k) + K$. We obtain a rational polytope $\mathfrak{P}$ of $\mathbb{R}^{k+1}$ whose we scan the points with integer coordinates.*

- *If we restrict our search to truncated polynomials whose constant term is* $1$*, we consider*

$$f(x_k) - K \leq 1 + \sum_{i=1}^{n} p_i^\star x_k^{2i+1} \leq f(x_k) + K, \quad k = 0, \dots, d$$

*with* $x_k \in \mathbb{Q} \cap [a, b]$, $d \geq n - 1$.

- *We can search for the best truncated polynomial for the relative error* $|| \cdot ||_{\textit{rel},[a,b]}$ *defined by*

$$||f - p||_{\textit{rel},[a,b]} = \sup_{a \leq x \leq b} \frac{1}{|f(x)|} |p(x) - f(x)|.$$

*Let $K \geq 0$, we search for a truncated polynomial $p^\star \in \mathcal{P}_n^m$ such that*

$$||f - p^\star||_{\textit{rel},[a,b]} = \min_{q \in \mathcal{P}_n^m} ||f - q||_{\textit{rel},[a,b]}$$

*and*

$$||f - p^\star||_{\textit{rel},[a,b]} \leq K.$$

*We consider*

$$-K|f(x)| - f(x) \leq \sum_{i=0}^{n} p_i^\star x^i \leq K|f(x)| + f(x)$$

*for at least $n+1$ rational values of $x \in [a,b]$.*

# Approximation of functions by sums of cosines

Let $f:[0,2\pi]\to\mathbb{R}$, $m=(m_i)_{0\le i\le 2n}$ a finite sequence of natural integers,

$$\mathcal{T}_n^m = \Big\{ t(x) = \frac{a_0}{2^{m_0}} + \frac{a_1}{2^{m_1}}\cos(x/2) + \frac{a_2}{2^{m_2}}\cos(x) + \cdots \\ + \frac{a_{2n-1}}{2^{m_{2n-1}}}\cos((n-1/2)x) + \frac{a_{2n}}{2^{m_{2n}}}\cos(nx)\,;\, a_i\in\mathbb{Z},\forall i \Big\}.$$

Let $K>0$. We search for a "sum of truncated cosines" $t^\star\in\mathcal{T}_n^m$ s.t.

$$||f-t^\star||_{[0,2\pi]} = \min_{t\in\mathcal{T}_n^m} ||f-t||_{[0,2\pi]}$$

and

$$||f-t^\star||_{[0,2\pi]} \le K.$$

Method working on $[0, 2\pi]$.

We must have

$$f(x) - K \leq \sum_{i=0}^{2n} t_i^\star \cos(ix/2) \leq f(x) + K \tag{3}$$

for all $x \in [0, 2\pi]$. We have $t_i^\star = a_i^\star / 2^{m_i}$ with $a_i^\star \in \mathbb{Z}$.

We want to construct rational polytope.

Trick : we plug into (3) $N$ values of $2 \arccos$ into rational points of $[-1, 1]$.

Indeed, let $x = r/s$ with $r \in \mathbb{Z}$, $s \in \mathbb{N}$, $|r| \leq s$. We have

$$f\left(2 \arccos\left(\frac{r}{s}\right)\right) - K \leq \sum_{i=0}^{2n} \frac{a_i^\star}{2^{m_i}} \underbrace{\cos\left(i \arccos\left(\frac{r}{s}\right)\right)}_{T_i(r/s) \in \mathbb{Q}!} \leq f\left(2 \arccos\left(\frac{r}{s}\right)\right) + K.$$

Let $x = r/s$ with $r \in \mathbb{Z},\ s \in \mathbb{N},\ |r| \leq s$. We have

$$f\left(2\arccos\left(\frac{r}{s}\right)\right) - K \leq \sum_{i=0}^{2n} \frac{T_i(r/s)}{2^{m_i}} a_i^\star \leq f\left(2\arccos\left(\frac{r}{s}\right)\right) + K.$$

Choose $m(\frac{r}{s})$ and $M(\frac{r}{s}) \in \mathbb{Q}$ such that $m(\frac{r}{s}) \leq f\left(2\arccos\left(\frac{r}{s}\right)\right) - K$ and $f\left(2\arccos\left(\frac{r}{s}\right)\right) + K \leq M(\frac{r}{s})$, $m(\frac{r}{s})$ "close" to $f\left(2\arccos\left(\frac{r}{s}\right)\right) - K$ and $M(\frac{r}{s})$ "close" to $f\left(2\arccos\left(\frac{r}{s}\right)\right) + K$.

If $N \geq 2n+1 \Rightarrow$ we have a rational polytope which the integers $a_i^\star = 2^{m_i} p_i^\star$ belong to.